一类二阶非线性非自治混合型泛函微分方程的次调和周期解的存在性

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摘要： 本文通过引入下半连续凸泛函的次微分和共轭泛函，用临界点理论和算子理论方法，得出了二阶非线性非自治混合型泛函微分方程的多重次调和周期解。

关键词： 变分结构；临界点；算子方程；次微分；次调和周期解；二阶混合型非线性泛函微分方程。

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Existence of subharmonic periodic solutions to a class of second-order nonlinear non-autonomous mixed-type functional differential equations

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Abstract: In this paper, by introducing subdifferentiability of lower semicontinuous convex functions and the conjugate functions, as well as critical point theory and operator equations theory, we obtain multiple subharmonic periodic solutions to a class of second-order nonlinear non-autonomous mixed-type nonlinear functional differential equations

Key words: variational structure; critical points; operator equations; subdifferentiability; subharmonic periodic solutions; second-order mixed-type nonlinear functional differential equations.

1. INTRODUCTION

The existence of periodic solutions for differential system has received a great deal of attention in the last few decades.

Differing from ordinary differential equations and partial differential equations that do not contain delay variant, it is very difficult to study the existence of periodic solutions for functional differential equations. For this reason, many mathematicians developed a good many different approaches like the averaging method[1], the Massera-Yoshizawa theory[2,3], the Kaplan-York

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[4] method of coupled systems, the Grafton cone mapping method[5], the Nussbaum method of fixed point theory [6] and Mawhin [7] coincidence degree theory etc.

In this paper, by critical points and operator equations theory, we first study the existence of subharmonic periodic solutions to second-order nonlinear and nonautonomous mixed-type functional differential equations

$$\begin{cases} x''(t) + x''(t-2\tau) + f(t, x(t), x(t-\tau), x(t-2\tau)) = 0, \\ x(0) = 0. \end{cases} \tag{1.1}$$

Our basic assumptions are the following:

(A_1) $f(t, x_1, x_2, x_3) \in C(R^4, R)$, and $\frac{\partial f(t,x_1,x_2,x_3)}{\partial t} \neq 0$;

(A_2) there exists a continuously differentiable function $F(t, x_1, x_2) \in C^1(R^3, R)$ with such that

$$F_2'(t, x_1, x_2) + F_1'(t, x_2, x_3) = f(t, x_1, x_2, x_3),$$

where $F_2'(t, x_1, x_2)$ and $F_1'(t, x_2, x_3)$ denote $\frac{\partial F(t,x_1,x_2)}{\partial x_2}$ and $\frac{\partial F(t,x_2,x_3)}{\partial x_2}$, respectively;

(A_3) $F(t+\tau, x_1, x_2) = F(t, x_1, x_2)$ for all $x_1, x_2, \in R$.

2. Variational Structure

Fix $\gamma > 1, \tau > 0$, where γ is a positive integer number and consider

$$\begin{aligned} H_0^1[0, 2\gamma\tau] \quad &= \{x(t) \in L^2[0, 2\gamma\tau] \mid x'(t) \in L^2[0, 2\gamma\tau], x(t) \text{ is } 2\gamma\tau\text{-periodic function in } \ t \\ & \quad x(0) = 0, \ \text{ and } x(t) \text{ has compact support on } \ [0, 2\gamma\tau]\}. \end{aligned}$$

It is obvious that $H_0^1[0, 2\gamma\tau]$ is a Sobolev space by defining the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$ by

$$< x, y >_{H_0^1[0,2\gamma\tau]} = \int_0^{2\gamma\tau} x'(t)y'(t)dt, \quad \|x\|_{H_0^1[0,2\gamma\tau]} = (\int_0^{2\gamma\tau} |x'(t)|^2 dt \ )^{\frac{1}{2}}.$$

Let us consider the functional $I(x)$ defined on $H_0^1[0, 2\gamma\tau]$ by

$$I(x) = \int_0^{2\gamma\tau} [x'(t)x'(t-\tau) - F(t, x(t), x(t-\tau))]dt. \tag{2.1}$$

For all $x, y \in H_0^1[0, 2\gamma\tau]$ and $\varepsilon > 0$, we know that

$$\begin{aligned} < I'(x), y > \ &= \ \int_0^{2\gamma\tau} [-x''(t+\tau) - x''(t-\tau) \\ &- \ F'_{u_1}(t, x(t), x(t-\tau)) - F'_{u_2}(t, x(t+\tau), x(t))]y(t)dt. \end{aligned}$$

Therefore, the Euler equation corresponding to the functional $I(x)$ is

$$x''(t+\tau) + x''(t-\tau) + [F'_{u_1}(t, x(t), x(t-\tau)) + F'_{u_2}(t, x(t+\tau), x(t))] = 0. \tag{2.2}$$

Since $I(x)$ has neither a supremum nor an infimum, we do not seek critical points of the functional $I(x)$ by the extremum method. But we may use operator equation theory. First by the dual variational principle, we get new operator equations (see (4.2)) correlating with the equation (1.1). Then obtain solutions of the system (1.1) by seeking critical points of operator equations (4.2).

For this, in section 3, we introduce subdifferentiability of lower semicontinuous convex functions $\varphi(x(t), x(t-\tau))$ and its conjugate functions. In section 4, first we give the definition of the weak solutions of the equation (1.1), then we establish the new operator equations (4.2) correlating with the equation (1.1) by the conjugate functions of $F(t, x(t), x(t-\tau))$ and show that we can obtain solutions of the equation (1.1) from the solutions the operator equations (4.2). Finally, in section 5, by seeking critical points of the operator equations (4.2), we get the result that there exists multiple subharmonic periodic solutions of the system (1.1).

In this paper, our main tool is the Maintain pass theorem as follows:

Lemma 2.1 Let H be a real Banach space, $I(\cdot) \in C^1(H, R)$ satisfies the Palais-Smale condition, and the following conditions:

(1) There exist constants $\rho > 0$ and $a > 0$ such that $I(x) \geq a, \forall x \in \partial B_\rho$. where $B_\rho = \{x \in H : \|x\|_H < \rho\}$;

(2) $I(\theta) \leq 0$ and there exists $x_0 \overline{\in} B_\rho$ such that $I(x_0) \leq 0$. Then $c = \inf_{h\in\Gamma} \sup_{s\in[0,1]} I(h(s))$ is a positive critical value of I, where

$$\Gamma = \{h \in C([0,1], H) \mid h(0) = \theta, h(1) = x_0\}.$$

3. The subdifferentiability and the conjugate function of the lower semicontinuous convex function $\varphi(x(t), x(t-\tau))$

Let X be a space of all given $n \times \tau$-periodic functions in t and be a Banach space, where $n \in N$ is a positive integer number. Denote $\overline{R} = R \cup \{+\infty\}$. Let $\varphi : X^2 \to \overline{R}$ be a lower semicontinuous convex function. Generally, φ is not always differentiable, but we may generalize the definition of "derivative" as follows:

Definition 3.1 Let $(x_1^*, x_2^*) \in X^* \times X^*$. We say that (x_1^*, x_2^*) is a sub-gradient of φ at point $(x_0(t), (x_0(t-\tau)) \in X \times X$ if

$$\varphi(x_0(t), x_0(t-\tau)) + < x_1^*, x(t) - x_0(t) > + < x_2^*, x(t-\tau) - x_0(t-\tau) > \leq \varphi(x(t), x(t-\tau)).$$

For all $x_0(t) \in X$, the set of all sub-gradients of φ at point $(x_0(t), x_0(t-\tau))$ will be called the subdifferential of φ at point $(x_0(t), x_0(t-\tau))$, and will be denoted by $\partial\varphi(x_0(t), x_0(t-\tau))$.

By definition of subdifferentiability of the function φ, we may define its conjugate function φ^* by:

$$\varphi^*(x_1^*, x_2^*) = \sup\{< x_1^*, x(t) > + < x_2^*, x(t-\tau) > -\varphi(x(t), x(t-\tau))\},$$

where $< \cdot >$ denotes the duality relation of X^* and X. So, It is not difficult to obtain the following propositions.

Proposition 3.1 φ^* is a lower semicontinuous convex function (φ^* may have functional value $+\infty$, but not functional value $-\infty$).

Proposition 3.2 If $\varphi \le \psi$, then $\varphi^* \ge \psi^*$.

Proposition 3.3 (Yang inequality)

$$\varphi(x(t), x(t-\tau)) + \varphi^*(x_1^*, x_2^*) \ge < x_1^*, x(t) > + < x_2^*, x(t-\tau) > .$$

Proposition 3.4 $$\varphi(x(t), x(t-\tau)) + \varphi^*(x_1^*, x_2^*) = < x_1^*, x(t) > + < x_2^*, x(t-\tau) >$$
$$\Leftrightarrow (x_1^*, x_2^*) \in \partial\varphi(x(t), x(t-\tau)).$$

Proposition 3.5 φ^* does not always equals to $+\infty$.

Now that φ^* is a lower semicontinuous convex function that does not always equal $+\infty$, we may define its conjugate function φ^{**} by

$$\varphi^{**}(x_1^{**}, x_2^{**}) = \sup_{(x_1^*, x_2^*) \in X^* \times X^*} \{< x_1^{**}, x_1^* > + < x_2^{**}, x_2^* > - \varphi(x_1^*, x_2^*)\},$$

where $< \cdot >$ denotes the duality relation of X^{**} and X^*.

Theorem 3.1 Let φ be a lower semicontinuous convex function that does not always equal $+\infty$, then $\varphi^{**} = \varphi$.

Proof We divide our proof into two parts, first showing that $\varphi^{**} = \varphi$ holds when $\varphi > 0$ and then showing that $\varphi^{**} = \varphi$ holds for all lower semicontinuous convex functions φ that do not always equal to $+\infty$.

(i) $\varphi \ge 0$.

From the definition of φ^{**} and the Yang inequality, it is obvious that $\varphi^{**} \le \varphi$ holds.

Next, to prove $\varphi^{**} \ge \varphi$ holds, suppose to the contrary that there exist a point $(x_0(t), x_0(t-\tau)) \in X^2$, such that $\varphi^{**}(x_0(t), x_0(t-\tau)) < \varphi(x_0(t), x_0(t-\tau))$ holds.

Consider the two convex sets

$$A = eip\ \varphi \underline{\triangle} \{(x(t), x(t-\tau), \beta) \in X^2 \times R \mid \varphi(x(t), x(t-\tau)) < +\infty,\ \beta \ge \varphi(x(t), x(t-\tau))\}$$

$$B_0 = \{(x_0(t), x_0(t-\tau), \varphi^{**}((x_0(t), x_0(t-\tau)))\}.$$

By Hahn-Banach Theorem, we know that there exists $(g_1, g_2, k^*) \in X^* \times X^* \times R$ and $\alpha_1 \in R$ such that

$$< g_1, x(t) > + < g_2, x(t-\tau) > + k^*\beta > \alpha_1, \quad \forall (x(t), x(t-\tau), \beta) \in eip\ \varphi, \tag{3.1}$$

$$< g_1, x_0(t) > + < g_2, x_0(t-\tau) > + k^*\varphi^{**}((x_0(t), x_0(t-\tau)) < \alpha_1. \tag{3.2}$$

So, it follow that $k^* \ge 0$. Let $\varepsilon > 0$. Using $\varphi > 0$ and (3.1), one gets that

$$< g_1, x(t) > + < g_2, x(t-\tau) > + (k^* + \varepsilon)\varphi(x(t), x(t-\tau)) \ge \alpha_1\ \forall (x(t), x(t-\tau)) \in D(\varphi)$$

So, we have

$$\varphi^*(-\frac{g_1}{k^*+\varepsilon},-\frac{g_2}{k^*+\varepsilon}) \leq -\frac{\alpha_1}{k^*+\varepsilon}.$$

Then, by definition of φ^{**}, we obtain that

$$\begin{aligned}\varphi^{**}(x_0(t),x_0(t-\tau)) &\geq \ < -\frac{g_1}{k^*+\varepsilon}, x_0(t) > + < -\frac{g_2}{k^*+\varepsilon}, x_0(t-\tau) > -\varphi^*(-\frac{g_1}{k^*+\varepsilon},-\frac{g_2}{k^*+\varepsilon}) \\ &\geq \ < -\frac{g_1}{k^*+\varepsilon}, x_0(t) > + < -\frac{g_2}{k^*+\varepsilon}, x_0(t-\tau) > +\frac{\alpha_1}{k^*+\varepsilon}.\end{aligned}$$

That is to say,

$$< g_1, x_0(t) > + < g_2, x_0(t-\tau) > +(k^*+\varepsilon)\varphi^{**}((x_0(t),x_0(t-\tau)) \geq \alpha_1, \ \forall\varepsilon > 0,$$

which is a contradiction to (3.2).

(ii) For all φ, by Proposition 3.5, we know $D(\varphi^*) \neq \emptyset$. Choose $(x_{10}^*, x_{20}^*) \in D(\varphi^*)$, and define the function $\overline{\varphi}$ by

$$\overline{\varphi}(x(t),x(t-\tau)) = \varphi(x(t),x(t-\tau)) - < x_{10}^*, x(t) > - < x_{20}^*, x(t-\tau) > +\varphi^*(x_{10}^*, x_{20}^*).$$

Then $\overline{\varphi}$ is a lower semicontinuous convex function that not always $+\infty$ and satisfies $\overline{\varphi} \geq 0$. By the result of (i), one gets $\overline{\varphi}^{**} = \overline{\varphi}$. On the other hand, we have

$$\overline{\varphi}^*(x_1^*, x_2^*) = \varphi^*(x_1^*+x_{10}^*, x_2^*+x_{20}^*) - \varphi^*(x_{10}^*, x_{20}^*)$$

$$\overline{\varphi}^{**}(x(t),x(t-\tau)) = \overline{\varphi}(x(t),x(t-\tau)) - < x_{10}^*, x(t) > - < x_{20}^*, x(t-\tau) > +\varphi^*(x_{10}^*, x_{20}^*).$$

That is $\overline{\varphi}^{**} = \varphi$.

Corollary 3.1 Let φ be a lower semicontinuous convex function that does not always equal to $+\infty$. Then $(x_1^*, x_2^*) \in \partial\varphi(x(t),x(t-\tau))$ if only if

$$(x(t),x(t-\tau)) \in \partial\varphi^*(x_1^*, x_2^*).$$

4. The weak solutions of the equation (1.1)

Definite operator $A = \frac{d^2}{dt^2}$. By (2.2) and

$$< u(t), A(\omega(t)) > = \int_0^{2\gamma\tau} u(t)\omega''(t)dt = u(t)(\omega'(t))|_0^{2\gamma\tau} - \int_0^{2\gamma\tau} \omega'(t)u'(t)dt = < Au(t), \omega(t) >$$

as well as

$$< u(t), A(\omega(t-\tau)) > = < Au(t+\tau), \omega(t) > \quad \text{and} \quad < u(t-\tau), A(\omega(t)) > = < A(u(t-\tau)), \omega(t) >,$$

we may define a weak solution of the equation (1.1) as follows:

Definition 4.1 For $u \in L^p[0, 2\gamma\tau]$, we say that u is a weak solution of the equation (1.1), if

$$\begin{aligned}&< u(t), A(\omega(t-\tau)) > + < u(t-\tau), A(\omega(t)) > + < \omega(t), F_1'(t,u(t),u(t-\tau)) > \\ &\quad + < \omega(t-\tau), F_2'(t,u(t),u(t-\tau)) > = 0,\end{aligned}$$

for all $\omega(t) \in D(A) \cap L^p[0, 2\gamma\tau]$, where

$$< u(t), \upsilon(t) >= \int_0^{2\gamma\tau} u(t)\upsilon(t)dt,$$

when $u(t) \in L^p[0, 2\gamma\tau], \upsilon(t) \in L^q[0, 2\gamma\tau]$, where $2 < p < +\infty, \frac{1}{p} + \frac{1}{q} = 1$.

Our object is that define the conjugate functions of $F(t, x(t), x(t-\tau))$ using the definition of definition subdifferentiability of lower semicontinuous convex functions, and making use of the dual variational structure. So we add the conditions on the function $F(t, x(t), x(t-\tau))$ as follows:

(A_4) $u = (u_1, u_2) \to F(t, u_1, u_2)$ is a continuously differentiable and strictly convex function, and satisfies

$$F(t, 0, 0) = 0, \quad F_1'(t, 0, 0) = F_2'(t, 0, 0) = 0, \qquad \forall t \in [0, 2\gamma\tau];$$

(A_5) for $\alpha_2 = \frac{1}{p}$, there exist constants $M, C > 0$, such that when $|u| = \sqrt{u_1^2 + u_2^2} \geq C$ we have,

$$F(t, u_1, u_2) \leq \alpha_2[F_1'(t, u_1, u_2)u_1 + F_2'(t, u_1, u_2)u_2],$$
$$F(t, u_1, u_2) \leq M|u|^{\frac{1}{\alpha_2}};$$

(A_6)

$$\lim_{|u|\to 0} \frac{F(t, u_1, u_2)}{|u|^2} = 0.$$

So, we set up the conjugate functions of the function $F((t, x(t), x(t-\tau))$ by

$$H(t, \omega(t), \omega(t-\tau)) = \sup_{x(t)\in L^p[0,2\gamma\tau]} \{< \omega(t), x(t) > + < \omega(t-\tau), x(t-\tau) > -F(t, x(t), x(t-\tau))\},$$

where $t \in [0, 2\gamma\tau]$.

Then H is a continuously differentiable and strictly convex function. By duality principle (Corollary 3.1), we get that

$$(\omega(t), \omega(t-\tau) = (F_1'(t, x(t), x(t-\tau)), F_2'(t, x(t), x(t-\tau)))$$
$$\Leftrightarrow (H_1'(t, \omega(t), \omega(t-\tau)), H_2'(t, \omega(t), \omega(t-\tau))) = (x(t), x(t-\tau)), \qquad (4.1)$$

where $H_1'(t, \omega(t), \omega(t-\tau))$ and $H_2'(t, \omega(t), \omega(t-\tau))$ denote $\frac{\partial H(t,\omega(t),\omega(t-\tau))}{\partial \omega(t)}$ and $\frac{\partial H(t,\omega(t),\omega(t-\tau))}{\partial \omega(t-\tau))}$, respectively.

Let $R(A)$ denote value field of the operator A. Then $R(A)$ is a closed set. Let P be the orthogonal projection operator of $R(A)$ and $\widehat{K} = A^{-1}P$. Then it is not difficult to see that $\widehat{K}$ maps continuous continuation into a compact operator of $L^q[0, 2\gamma\tau] \to L^q[0, 2\gamma\tau]$.

Let

$$\begin{aligned} E &= \{(\upsilon(t), \upsilon(t-\tau)) \in (L^q[0, 2\gamma\tau])^2 \ \upsilon(0) = 0 \mid \ < \phi(t), \upsilon(t) >=< \phi(t), \upsilon(t-\tau) > \\ &= < \phi(t-\tau), \upsilon(t) >= 0, \quad \forall \phi(t) \in \Re(A) \cap L^p[0, 2\gamma\tau], \ \phi(0) = 0\}, \end{aligned}$$

where $\Re(A) = \{u \in D(A) \mid A((u(t) + u(t-2\tau) = 0\}$.

We want $\{(v(t), v(t-\tau)), (\chi(t), \chi(t-\tau)))\}$ to satisfy

$$\begin{cases} \chi(t) = \widehat{K}(v(t-\tau)) + H_1'(t, v(t), v(t-\tau)), \\ \chi(t-\tau) = \widehat{K}(v(t)) + H_2'(t, v(t), v(t-\tau)), \end{cases} \tag{4.2}$$

where $(v(t), v(t-\tau)) \in E, \chi(t) \in \Re(A) \cap L^p[0, 2\gamma\tau]$, that is $(\chi(t), \chi(t-\tau)) \in E^{\perp}$.

If $\{(v(t), v(t-\tau)), (\chi(t), \chi(t-\tau))\}$ is a solution of (4.2), then when let $u(t) = H_1'(t, v(t), v(t-\tau)), u(t-\tau) = H_2'(t, v(t), v(t-\tau))$, by the duality principle and (4.2), we can get $u(t)$ is a weak solution of the equation (1.1).

5. Seeking the solutions of the operator equation (4.2) via critical point theory

Theorem 5.1 Under the assumptions $(A_1) \sim (A_6)$, the problem (1.1) has at least one nontrivial weak $2\gamma\tau$-periodic solution.

Let $v = (v(t), v(t-\tau))$, and

$$K\begin{pmatrix} v(t) \\ v(t-\tau) \end{pmatrix} = \begin{pmatrix} 0 & \widehat{K} \\ \widehat{K} & 0 \end{pmatrix}\begin{pmatrix} v(t) \\ v(t-\tau) \end{pmatrix} = \begin{pmatrix} \widehat{K}v(t-\tau) \\ \widehat{K}v(t) \end{pmatrix}.$$

It is not difficult to verify that $< K(v), \psi > = < v, K(\psi) > = < \overline{K}v(t-\tau), \psi(t) > + < \overline{K}v(t), \psi(t-\tau) >$, where $\psi = (\psi(t), \psi(t-\tau))$, that is, the operator K is symmetric operator.

We may get the solutions of the equations (4.2) by seeking critical points of the functional $J(v)$ defined by

$$\begin{aligned} J(v) &= \tfrac{1}{2} < K(v), v > + \int_0^{2\gamma\tau} H(t, v)dt \\ &= \tfrac{1}{2} < \widehat{K}v(t-\tau), v(t) > + \tfrac{1}{2} < \widehat{K}v(t), v(t-\tau) > + \int_0^{2\gamma\tau} H(t, v(t), v(t-\tau))dt. \end{aligned} \tag{5.1}$$

Because J may be regarded as the restriction to E of the function $\widehat{J}$ defined on $L^q[0, 2\gamma\tau] \times L^q[0, 2\gamma\tau]$ and having identical components. Moreover

$$\widehat{J}'(v) = K(v) + H'(v).$$

Since

$$< \widehat{J}'(v) - J'(v), z > = 0 \qquad \forall v \in E,\ z = (z(t), z(t-\tau)) \in E,$$

then there exists $\chi(t) \in \Re(A)$ and $\chi_v = (\chi_v(t), \chi_v(t-\tau)) \in E^{\perp}$ such that

$$\widehat{J}'(v) - J'(v) = \chi_v.$$

So, if v^* is a critical point of $J'(v^*) = 0$ on E, then there exists $\chi^*_{v^*} = (\chi^*_{v^*}(t), \chi^*_{v^*}(t-\tau)) \in E^{\perp}$ such that

$$K(v^*) + H'(v^*) = \chi^*_{v^*}.$$

So, $\{\upsilon^*, \chi^*_{\upsilon^*}\}$ is a solution of the equation (4.2), that is, $\{(\upsilon^*(t), \upsilon^*(t-\tau)), (\chi^*_{\upsilon^*}(t), \chi^*_{\upsilon^*}(t-\tau))\}$ is a solution of the equation (4.2).

Lemma 5.1 The following two conditions are equivalent:

(1) $F(t, u_1, u_2) \le \alpha_2[F_1'(t, u_1, u_2)u_1 + F_2'(t, u_1, u_2)u_2], \quad \forall t \in [0, 2\gamma\tau]$, when $|u| = \sqrt{u_1^2 + u_2^2} \ge C$.

(2) $F(t, \beta u_1, \beta u_2) \ge \beta^{\frac{1}{\alpha_2}} F(t, u_1, u_2) > 0, \quad \forall \beta \ge 1, t \in [0, 2\gamma\tau], |u| \ge C$.

Proof For all $\forall u = (u_1, u_2), |u| \ge C$, let $\Phi(\beta) = F(t, \beta u_1, \beta u_2), \Psi(\beta) = \beta^{\frac{1}{\alpha_2}} F(t, u_1, u_2)$.

(2) $\Rightarrow$ (1) By $\Phi(\beta) \ge \Psi(\beta)$, $\forall \beta \ge 1$ and $\Phi(1) = \Psi(1)$, it is easy to see $\Phi'(1) \ge \Psi'(1)$, that is,

$$F_1'(t, u_1, u_2)u_1 + F_2'(t, u_1, u_2)u_2 \ge \frac{1}{\alpha_2} F(t, u_1, u_2).$$

(1) $\Rightarrow$ (2) By

$$\begin{aligned}\Phi'(\beta) &= F_1'(t, \beta u_1, \beta u_2)u_1 + F_2'(t, \beta u_1, \beta u_2)u_2 \\ &= \frac{1}{\beta}[F_1'(t, \beta u_1, \beta u_2)\beta u_1 + F_2'(t, \beta u_1, \beta u_2)\beta u_2] \ge \frac{1}{\alpha_2 \beta}\Phi(\beta),\end{aligned}$$

it follows that

$$F(t, \beta u_1, \beta u_2) \ge \beta^{\frac{1}{\alpha_2}} F(t, u_1, u_2) > 0, \quad \forall \beta \ge 1, t \in [0, 2\gamma\tau].$$

Lemma 5.2 Let $F(t, u_1, u_2)$ satisfy the assumptions (A_4) and (A_5). Then there exist constants $m > 0$ and $M > 0$, such that

$$F(t, u_1, u_2) \ge m(\sqrt{u_1^2 + u_2^2}\,)^{\frac{1}{\alpha_2}} \quad \forall t \in [0, 2\gamma\tau], \quad \text{when} \quad |u| \ge C,$$

$$|F'(t, u_1, u_2)| \le (2^{\frac{1}{\alpha_2}} M - m)(\sqrt{u_1^2 + u_2^2}\,)^{\frac{1}{\alpha_2} - 1}, \quad \text{when} \quad |u| \ge C,$$

where $|F'(t, u_1, u_2)| = \sqrt{|F_1'(t, u_1, u_2)|^2 + |F_2'(t, u_1, u_2)|^2}$.

Proof Let

$$m = \min_{(u_1, u_2) \in \partial B_C} \frac{F(t, u_1, u_2)}{C^{\frac{1}{\alpha_2}}},$$

where B_C denotes the ball of radius centered at the origin in C. By (A_4), one knows $m > 0$. On the other hand, by Lemma 5.1 and (A_5), we get

$$\begin{aligned}F(t, u_1, u_2) &\ge F\Big(t, \frac{Cu_1}{\sqrt{u_1^2 + u_2^2}}, \frac{Cu_2}{\sqrt{u_1^2 + u_2^2}}\Big)\Big(\frac{\sqrt{u_1^2 + u_2^2}}{C}\,\Big)^{\frac{1}{\alpha_2}} \\ &\ge m(\sqrt{u_1^2 + u_2^2}\,)^{\frac{1}{\alpha_2}}.\end{aligned}$$

By convexity of the function F, one has

$$F(t, u_1, u_2) + F_1'(t, u_1, u_2)(z_1 - u_1) + F_2'(t, u_1, u_2)(z_2 - u_2) \le F(t, z_1, z_2).$$

Let $z=(z_1,z_2)$ run all over the ball $B_{|u|}(z)$ of radius centered at $u=(u_1,u_2)$ in $|u|$, and choose the maximum of $F_1'(t,u_1,u_2)(z_1-u_1)+F_2'(t,u_1,u_2)(z_2-u_2)$. Then it is not difficult to see that

$$|F'(t,u_1,u_2)|\sqrt{u_1^2+u_2^2}\le M(\sqrt{z_1^2+z_2^2}\,)^{\frac{1}{\alpha_2}}-m(\sqrt{u_1^2+u_2^2}\,)^{\frac{1}{\alpha_2}}.$$

By $z\le 2|u|$ we get that

$$|F'(t,u_1,u_2)|\le(2^{\frac{1}{\alpha_2}}M-m)(\sqrt{u_1^2+u_2^2}\,)^{\frac{1}{\alpha_2}-1}.$$

Lemma 5.3 $H\in C^1(R^3,R)$ is a strictly convex function and satisfies

$$H_1'(t,0,0)=H_2'(t,0,0)=0,\qquad H(t,0,0)=0,\qquad \forall t\in[0,2\gamma\tau],$$

$$\frac{C_{\alpha_2}}{M}|\omega|^{\frac{1}{1-\alpha_2}}-C_1\le H(t,\omega(t),\omega(t-\tau))\le\frac{C_{\alpha_2}}{m}|\omega|^{\frac{1}{1-\alpha_2}}+C_2,\tag{5.2}$$

$$C_{\alpha_2}'|\omega|^{\frac{\alpha_2}{1-\alpha_2}}-C_4\le|H'(t,\omega(t),\omega(t-\tau))|\le C_{\alpha_2}(\frac{2^{\frac{1}{1-\alpha_2}}}{m}-\frac{1}{M})|\omega|^{\frac{\alpha_2}{1-\alpha_2}}+C_3,\tag{5.3}$$

where $C_1,\cdots,C_4$ are constants, $C_{\alpha_2},C_{\alpha_2}'$ are constants depending on α_2, and $|\omega|=\sqrt{\omega^2(t)+\omega^2(t-\tau)}$, $|H'(t,\omega(t),\omega(t-\tau))|=\sqrt{|H_1'(t,\omega(t),\omega(t-\tau))|^2+|H_2'(t,\omega(t),\omega(t-\tau))|^2}$. Moreover, the function H satisfies

$$\lim_{|\omega|\to 0}\frac{H(t,\omega(t),\omega(t-\tau))}{|\omega|^2}=\infty.\tag{5.4}$$

Proof By Corollary 3.1 and $F_1'(t,0,0)=F_2'(t,0,0)=0\Longrightarrow H_1'(t,0,0)=H_2'(t,0,0)=0,\ \ \forall t\in[0,2\gamma\tau]$. And by the definition of H, we know $H(t,0,0)=0$.

Now we show that (5.2) holds. By (A_5) one gets $F(t,u_1,u_2)\le M|u|^{\frac{1}{\alpha_2}}+C_1,\ \forall u=(u_1,u_2)\in R^2$. So, by Proposition 3.2 and Example 4.1, it is easy to see

$$H(t,\omega(t),\omega(t-\tau))\ge\frac{C_{\alpha_2}}{M}|\omega|^{\frac{1}{1-\alpha_2}}-C_1,$$

where $C_{\alpha_2}=2^{\frac{1}{1-\alpha_2}}/M^{\frac{\alpha_2}{1-\alpha_2}-1}(\ \alpha_2^{\frac{\alpha_2}{1-\alpha_2}}-\alpha_2^{\frac{1}{1-\alpha_2}}\ )$.

Similar arguments in the proof of Lemma 5.2, we know that there exists a constant C_2 such that

$$F(t,u_1,u_2)\ge m|u|^{\frac{1}{\alpha_2}}-C_2.$$

So, it follows that

$$H(t,\omega(t),\omega(t-\tau))\le\frac{C_{\alpha_2}}{m}|\omega|^{\frac{1}{1-\alpha_2}}+C_2.$$

Then next we show that (5.3) holds. Again as in the proof of Lemma 5.2, we can estimate H' by

$$|H'(t,\omega(t),\omega(t-\tau))|\le C_{\alpha_2}(\frac{2^{\frac{1}{1-\alpha_2}}}{m}-\frac{1}{M})|\omega|^{\frac{\alpha_2}{1-\alpha_2}}+C_3,$$

where $C_3 = \max\{C_1 + C_2, \sup_{|\omega|<1} |H'(t,\omega(t),\omega(t-\tau))|\}$. By Lemma 5.2 again and the duality principle

$$(u_1,u_2) = (H_1'(t,\omega(t),\omega(t-\tau)), H_2'(t,\omega(t),\omega(t-\tau)))$$
$$\Leftrightarrow (\omega(t),\omega(t-\tau)) = (F_1'(t,u_1,u_2), F_2'(t,u_1,u_2)),$$

when $|H'(t,\omega(t),\omega(t-\tau))| \geq C$, we have

$$|\omega| \leq (2^{\frac{1}{\alpha_2}}M - m)|H'(t,\omega(t),\omega(t-\tau))|^{\frac{1}{\alpha_2}-1}.$$

And since there exists a constant M_C such that when $|u| = \sqrt{u_1^2+u_2^2} = |H'(t,\omega(t),\omega(t-\tau)| \leq C$, we have

$$|\omega| = |F'(t,u_1,u_2)| \leq M_C.$$

Choose

$$C'_{\alpha_2} = (2^{\frac{1}{\alpha_2}}M - m)^{\frac{\alpha_2}{\alpha_2-1}}, \qquad C_4 = C'_{\alpha_2} M_C^{\frac{\alpha_2}{1-\alpha_2}}.$$

Then it is not difficult to see

$$|H'(t,\omega(t),\omega(t-\tau))| \geq C'_{\alpha_2}|\omega|^{\frac{\alpha_2}{1-\alpha_2}} - C_4.$$

Finally, we show that (5.4) holds. By (A_6), for all $\varepsilon > 0$, there exists $\delta > 0$ so that when $|u| = \sqrt{u_1^2+u_2^2} < \delta$, we have

$$F(t,u_1,u_2) \leq \varepsilon\sqrt{u_1^2+u_2^2}.$$

Now, for all $K > 0$, choose $\varepsilon = \frac{1}{4K}$ and let $\eta = 2\varepsilon\delta(\varepsilon)$. Then when $\sqrt{\omega^2(t)+\omega^2(t-\tau)} < \eta$, and we get

$$H(t,\omega(t),\omega(t-\tau)) \geq \frac{1}{4\varepsilon}(\omega^2(t)+\omega^2(t-\tau)) = K|\omega|^2.$$

That is,

$$\lim_{|\omega|\to 0} \frac{H(t,\omega(t),\omega(t-\tau))}{|\omega|^2} = \infty.$$

Lemma 5.4 There exist constants C_δ and C'_δ depending on δ, such that

$$H(t,\omega(t),\omega(t-\tau)) \geq \begin{cases} C_\delta|\omega|^2, & \text{when } |\omega| \leq \delta, \\ C'_\delta|\omega|^q, & \text{when } |\omega| \geq \delta, \end{cases}$$

and when $\delta \to +0$, $C_\delta \to +\infty$.

Proof By (5.4), we know

$$\lim_{|\omega|\to 0} \frac{H(t,\omega(t),\omega(t-\tau))}{|\omega|^2} = \infty.$$

So, when $\delta \to +0$, one gets $C_\delta \triangleq \inf\{H(t,\omega(t),\omega(t-\tau))/|\omega|^2 : \ |\omega| \leq \delta\} \to +\infty$. That is

$$H(t,\omega(t),\omega(t-\tau)) \geq C_\delta|\omega|^2, \tag{5.5}$$

when $|\omega| \le \delta$.

We next show that the second part of the inequality holds.

For all $\omega_0 = (\omega_0(t), \omega_0(t-\tau))$, $|\omega_0| = 1$, let $\phi_{\omega_0}(\beta) = H(t, \beta\omega_0(t), \beta\omega_0(t-\tau))$. Then

$$\phi'_{\omega_0}(\beta) = H'_1(t, \beta\omega_0(t), \beta\omega_0(t-\tau))\omega_0(t) + H'_2(t, \beta\omega_0(t), \beta\omega_0(t-\tau))\omega_0(t-\tau).$$

Since ϕ_{ω_0} is a convex function, for all $\beta > 0$, we have

$$H'_1(t, \beta\omega_0(t), \beta\omega_0(t-\tau))\omega_0(t) + H'_2(t, \beta\omega_0(t), \beta\omega_0(t-\tau))\omega_0(t-\tau) \ge \frac{1}{\beta}\phi_{\omega_0}(\beta).$$

So, by (5.5), one gets

$$H'_1(t, \delta\omega_0(t), \delta\omega_0(t-\tau))\omega_0(t) + H'_2(t, \delta\omega_0(t), \delta\omega_0(t-\tau))\omega_0(t-\tau) \ge C_\delta \cdot \delta.$$

By convexity of H again, it is easy to see

$$\begin{aligned} H(t, s\omega_0(t), s\omega_0(t-\tau)) &\ge H'_1(t, \delta\omega_0(t), \delta\omega_0(t-\tau))(s-\delta)\omega_0(t) \\ &+ H'_2(t, \delta\omega_0(t), \delta\omega_0(t-\tau))(s-\delta)\omega_0(t-\tau) + H(t, \delta\omega_0(t), \delta\omega_0(t-\tau)) \\ &\ge C_\delta \cdot \delta(s-\delta) + C_\delta\delta^2 = C_\delta\delta s, \qquad \forall s > 0. \end{aligned}$$

So, we get

$$H(t, \omega(t), \omega(t-\tau)) \ge C_\delta \cdot \delta|\omega|. \tag{5.6}$$

And by Lemma 5.3, we get that there exists $T > 0$ such that

$$H(t, \omega(t), \omega(t-\tau)) \ge \frac{C_{\alpha_2}}{2M}|\omega|^q. \tag{5.7}$$

Let $C'_\delta = \min\{\frac{C_{\alpha_2}}{2M}, T^{1-q}\delta C_\delta\}$. By (5.5), (5.6) and (5.7) we obtain

$$H(t, \omega(t), \omega(t-\tau)) \ge \begin{cases} C_\delta|\omega|^2, & |\omega| \le \delta, \\ C'_\delta|\omega|^q, & |\omega| \ge \delta. \end{cases}$$

Lemma 5.5 Let $\upsilon_m = (\upsilon_m(t), \upsilon_m(t-\tau)) \rightharpoonup \upsilon = (\upsilon(t), \upsilon(t-\tau))$ (weakly convergent sequence on $L^q([0, 2\gamma\tau]))^2$) and satisfy

$$\int_0^{2\gamma\tau} H(t, \upsilon_m)dt \to \int_0^{2\gamma\tau} H(t, \upsilon)dt.$$

Then

$$\int_0^{2\gamma\tau} H(t, \upsilon_m - \upsilon)dt \to 0.$$

Proof (I) First, we show that the terms in $\{H(t, \upsilon_m)\}$ have equicontinuous integrals, that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall\, m, \int_\Omega H(t, \upsilon_m)dt < \varepsilon \qquad \text{when} \quad \mu(\Omega) < \delta.$$

Since H is convex, we have

$$H_1'(t,\upsilon)(\upsilon(t)-\upsilon_m(t))+H_2'(t,\upsilon)(\upsilon(t-\tau)-\upsilon_m(t-\tau))\leq H(t,\upsilon_m)-H(t,\upsilon).$$

So, from $\upsilon_m \rightharpoonup \upsilon$ and the above equality, one gets

$$\int_0^{2\gamma\tau} H(t,\upsilon)dt \leq \varliminf_{m\to\infty}\int_0^{2\gamma\tau} H(t,\upsilon_m)dt. \tag{5.8}$$

And by $H\geq 0$ and the assumption $\int_0^{2\gamma\tau} H(t,\upsilon)dt \to \int_0^{2\gamma\tau} H(t,\upsilon_m)dt$, it is not difficult to see

$$\lim_{m\to\infty}\int_\Omega H(t,\upsilon_m)dt=\int_\Omega H(t,\upsilon)dt, \qquad \text{for all measurable sets } \ \Omega. \tag{5.9}$$

Suppose to the contrary that $\{H(t,\upsilon_m)\}$ does not have equicontinuous integrals, that is, there exists $\varepsilon_0>0$ and the functions $\upsilon_{m_k}=(\upsilon_{m_k}(t),\upsilon_{m_k}(t-\tau))$ as well as measurable sets Ω_k, such that

$$\int_\Omega H(t,\pm\upsilon)dt<\varepsilon_0, \qquad \text{for all measurable sets } \Omega \ \text{ and } \mu(\Omega)<\delta, \tag{5.10}$$

holds, but

$$\int_{\Omega_k} H(t,\upsilon_{m_k})dt\geq\varepsilon_0, \quad \mu(\Omega_k)<\frac{\delta}{2^k}$$

also holds. Then choose $\Omega_0=\bigcup_{k=1}^{\infty}\Omega_k$. It is not difficult to obtain $\mu(\Omega_0)<\delta$ and

$$\int_{\Omega_0} H(t,\upsilon_{m_k})dt\geq\int_{\Omega_k} H(t,\upsilon_{m_k})dt\geq\varepsilon_0,$$

which is a contradiction to (5.8) and (5.9).

(II) For all $b>0$, we divide $[0,2\gamma\tau]$ into the following three subsets:

$$Q_1=\{t\in[0,2\gamma\tau]\mid |\upsilon|=\sqrt{\upsilon^2(t)+\upsilon^2(t-\tau)}>b\},$$

$$Q_2^m=\{t\in[0,2\gamma\tau]\mid |\upsilon|\leq b, |\upsilon_m-\upsilon|\geq\delta\},$$

$$Q_3^m=\{t\in[0,2\gamma\tau]\mid |\upsilon|\leq b, |\upsilon_m-\upsilon|<\delta\},$$

where $|\upsilon_m-\upsilon|=\sqrt{(\upsilon_m(t)-\upsilon(t))^2+(\upsilon_m(t-\tau)-\upsilon(t-\tau))^2}$. By inequality (5.2), we know that there exist constants K and L such that

$$H(t,2z(t),2z(t-\tau))\leq KH(t,z(t),z(t-\tau))+L, \forall(z(t),z(t-\tau))\in R^2.$$

And by convexity of the function H on Q_1, we obtain

$$H(t,\upsilon_m-\upsilon)\leq\frac{1}{2}[H(t,2\upsilon_m)+H(t,-2\upsilon)]\leq\frac{K}{2}(H(t,\upsilon_m)+H(t,-\upsilon))+L.$$

So by (I), we may choose a constant b big enough and fixed such that $\mu(Q_1)$ is small enough so that

$$\int_{Q_1} H(t,\upsilon_m-\upsilon)dt\leq\frac{K}{2}\int_{Q_1}(H(t,\upsilon_m)+H(t,-\upsilon))dt+L\mu(Q_1)<\frac{\varepsilon}{3}. \tag{5.11}$$

For the fixed constant b, choose δ enough small and fix it, so that

$$\int_{Q_3^m} H(t, \upsilon_m - \upsilon)dt < \frac{\varepsilon}{3}. \quad (5.12)$$

For the fixed constants b and δ, let

$$\kappa = \inf_{|\omega - z| \geq \delta, |z| \leq b} [H(t,\omega) - H(t,z) - H_1'(t,\omega)(\omega(t) - z(t)) - H_2'(t,\omega)(\omega(t-\tau) - z(t-\tau))].$$

Then $\kappa > 0$. Now

$$\begin{aligned}\kappa\mu(Q_2^m) &\leq \int_{Q_2^m} [H(t,\upsilon_m) - H(t,\upsilon) - H_1'(t,\upsilon)(\upsilon_m(t) - \upsilon(t)) - H_2'(t,\upsilon)(\upsilon_m(t-\tau) - \upsilon(t-\tau))]dt \\ &\leq \int_0^{2\gamma\tau} [H(t,\upsilon_m) - H(t,\upsilon) - H_1'(t,\upsilon)(\upsilon_m(t) - \upsilon(t)) - H_2'(t,\upsilon)(\upsilon_m(t-\tau) - \upsilon(t-\tau))]dt.\end{aligned}$$

Then we have $\mu(Q_2^m) \to 0$ when $m \to \infty$. Hence, it is easy to see that $\int_{Q_2^m} H(t, \upsilon_m) \longrightarrow 0$.

Repeating the above argument on Q_1, we know that there exists an n_0 such that when $m > n_0$, we have

$$\int_{Q_2^m} H(t, \upsilon_m - \upsilon)dt < \frac{\varepsilon}{3}. \quad (5.13)$$

From (5.11),(5.12) and (5.13), we get

$$\lim_{m\to\infty} \int_0^{2\gamma\tau} H(t, \upsilon_m - \upsilon)dt = 0.$$

Corollary 5.1 $\upsilon_m = (\upsilon_m(t), \upsilon_m(t-\tau)) \to \upsilon = (\upsilon(t), \upsilon(t-\tau))$ ($L^q([0, 2\gamma\tau]) \times L^q([0, 2\gamma\tau])$) if only if

$$\int_0^{2\gamma\tau} H(t, \upsilon_m - \upsilon)dt = 0.$$

Proof (⇒). $\upsilon_m \to \upsilon$ contain $\upsilon_m \rightharpoonup \upsilon$ (weakly). And by inequality (5.2) and continuity of the composition operator, one gets $H(t, \upsilon_m) \to H(t, \upsilon)$ $(L^1([0, 2\gamma\tau]))$, that is to say $\int_0^{2\gamma\tau} H(t, \upsilon_m)dt \to \int_0^{2\gamma\tau} H(t, \upsilon)dt$. So, by Lemma 5.5, we get the conclusion.

(⇐). By Lemma 5.4, there exists constants B_1 and $B_2 > 0$ such that

$$\int_0^{2\gamma\tau} H(t,\upsilon)dt \geq B_1 \int_{|\upsilon|\geq\delta} |\upsilon|^q dt + B_2 \int_{|\upsilon|<\delta} |\upsilon|^2 dt \geq C_\delta \min\{\int_0^{2\gamma\tau} |\upsilon|^q dt, (\int_0^{2\gamma\tau} |\upsilon|^q dt)^{\frac{2}{q}}\}.$$

Choose δ small enough. Then $C_\delta > 0$ is a constant and it is not difficult to see the conclusion is correct.

We next use the Maintain pass theorem to prove Theorem 5.1.

We divide our proof into three parts.

(i) We show that J satisfies the P. S. condition in E. Let $\{\upsilon_n = (\upsilon_n(t), \upsilon_n(t-\tau))\} \subset E$ and let the constants C_1, C_2 satisfy

$$C_1 \leq J(\upsilon_n) \leq C_2 \quad (5.14)$$

and

$$J'(\upsilon_n) \to \theta. \tag{5.15}$$

That is to say, we want to show that $\{\upsilon_n\}$ has a convergence subsequence in E.

First, we show that $\{\upsilon_n\}$ is bounded. In fact, by

$$z_m = K\upsilon_m + H'(t, \upsilon_m) - \chi_m \to \theta$$

and

$$C_1 \leq \frac{1}{2} < K\upsilon_m, \upsilon_m > + \int_0^{2\gamma\tau} H(t, \upsilon_m)dt \leq C_2,$$

where $z_m = (z_m(t), z_m(t-\tau)), \upsilon_m = (\upsilon_m(t), \upsilon_m(t-\tau)), \chi_m = (\chi_m(t), \chi_m(t-\tau))$. We know that there exists $n(\varepsilon) > 0$ for all $\varepsilon > 0$ such that the following inequality holds when $m \geq m(\varepsilon)$:

$$\int_0^{2\gamma\tau} H(t, \upsilon_m)dt - \frac{1}{2}[H_1'(t, \upsilon_m)\upsilon_m(t) + H_2'(t, \upsilon_m)\upsilon_m(t-\tau)]$$

$$\leq C_2 + \frac{\varepsilon}{2}(\|\upsilon_m(t)\|_{L^q} + \|\upsilon_m(t-\tau)\|_{L^q}) = C_2 + \varepsilon\|\upsilon_m(t)\|_{L^q}. \tag{5.16}$$

On the other hand, by Lemma 5.1 and Lemma 5.2, one gets that there exist constants α_2, C_3, C_4 and C_5 such that

$$\begin{array}{rcl} H(t,\omega) & - & \frac{1}{2}H_1'(t,\omega)\omega(t) - \frac{1}{2}H_2'(t,\omega)\omega(t-\tau)) \\ & \geq & (\frac{1}{2\alpha_2} - 1)F(t, z(t), z(t-\tau)) - C_3 \\ & \geq & m|z|^{\frac{1}{\alpha_2}}(\frac{1}{2\alpha_2} - 1) - C_4 \\ & \geq & |\omega|^q - C_5, \end{array} \tag{5.17}$$

where $\omega(t) = F_1'(t, z(t), z(t-\tau)), \omega(t-\tau) = F_2'(t, z(t), z(t-\tau)); z(t) = H_1'(t, \omega(t), \omega(t-\tau))$, $z(t-\tau) = H_2'(t, \omega(t), \omega(t-\tau)); |\omega| = \sqrt{\omega^2(t) + \omega^2(t-\tau)}; |z| = \sqrt{z^2(t) + z^2(t-\tau)}$.

So, by (5.16) and (5.17), it is easy to see

$$\|\upsilon_m(t)\|_{L^q[0,2\gamma\tau]} = \|\upsilon_m(t-\tau)\|_{L^q[0,2\gamma\tau]} \leq C_6. \quad \text{(constant)}$$

That is, $\{\upsilon_n\}$ is bounded. We next will show that $\{\upsilon_n\}$ has a convergence subsequence. Since $L^q[0, 2\gamma\tau]$ is a reflexive Banach space, there exists a subsequence of $\{\upsilon_n\}$ which is weakly convergent in $L^q[0, 2\gamma\tau]$. We denote it by $\{\upsilon_{m_k}\}$, that is to say, $\upsilon_{m_k}(t) \rightharpoonup \upsilon^*(t)$, $\upsilon_{m_k}(t-\tau) \rightharpoonup \upsilon^*(t-\tau)$. On the one hand, by convexity of the function H, we get

$$H(t, \upsilon^*(t), \upsilon^*(t-\tau)) + H_1'(t, \upsilon^*(t), \upsilon^*(t-\tau))(\upsilon_{m_k}(t) - \upsilon^*(t))$$
$$+H_2'(t, \upsilon^*(t), \upsilon^*(t-\tau))(\upsilon_{m_k}(t-\tau) - \upsilon^*(t-\tau)) \leq H(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-\tau)).$$

So, we have

$$\int_0^{2\gamma\tau} H(t, \upsilon^*(t), \upsilon^*(t-\tau))dt \leq \varliminf_{k\to\infty} \int_0^{2\gamma\tau} H(t, \upsilon_{m_k}(t), \upsilon_{m_k}(t-\tau))dt. \tag{5.18}$$

On the other hand, by convexity of the function H again, we obtain

$$H(t,\upsilon^*(t),\upsilon^*(t-\tau)) \geq H(t,\upsilon_{m_k}(t),\upsilon_{m_k}(t-\tau)) + H_1'(t,\upsilon_{m_k}(t),\upsilon_{m_k}(t-\tau))(\upsilon^*(t)-\upsilon_{m_k}(t))$$
$$+H_2'(t,\upsilon_{m_k}(t),\upsilon_{m_k}(t-\tau))(\upsilon^*(t-\tau)-\upsilon_{m_k}(t-\tau))$$
$$= H(t,\upsilon_{m_k}(t),\upsilon_{m_k}(t-\tau)) + (-K\upsilon_{m_k}+z_{m_k}+\chi_{m_k})\cdot(\upsilon^*-\upsilon_{m_k}).$$

Since the operators A and K are compact and $(z_{m_k}(t), z_{m_k}(t-\tau)) \to \theta$, we know

$$\overline{\lim_{k\to\infty}} \int_0^{2\gamma\tau} H(t,\upsilon_{m_k}(t),\upsilon_{m_k}(t-\tau))dt \leq \int_0^{2\gamma\tau} H(t,\upsilon^*(t),\upsilon^*(t-\tau))dt. \tag{5.19}$$

By (5.18), (5.19), and making use of Lemma 5.5 and Corollary 5.1, it is not difficult to see

$$(\upsilon_{m_k}(t),\upsilon_{m_k}(t-2\tau)) \to (\upsilon^*(t),\upsilon^*(t-\tau)).$$

(ii) We will show that there exist constant $\rho, r > 0$, such that

$$J|_{\partial\Omega_r} \geq \rho > 0, \tag{5.20}$$

where $\partial\Omega_r = \{(\upsilon(t),\upsilon(t-\tau)) \in L^q[0,2\gamma\tau]\times L^q[0,2\gamma\tau] \mid \ \|\upsilon(t)\|_{L^q[0,2\gamma\tau]} = \|\upsilon(t-\tau)\|_{L^q[0,2\gamma\tau]} = r\}$.

Let $\beta = \|\widehat{K}\|_{\mathcal{L}(L^p,L^q)}$, and choose $\delta > 0$ such that the constant C_δ big enough and choose r small enough so that, by Lemma 5.4, when $\|\upsilon(t)\|_{L^q} = r$, there exists the constant $C_7 > 0$ satisfying

$$C_\delta \int_{|\upsilon|<\delta} |\upsilon(t)|^2 dt - 4\beta\Big(\int_{|\upsilon|<\delta} |\upsilon(t)|^q dt\ \Big)^{\frac{2}{q}} \geq C_7\Big(\int_{|\upsilon|<\delta} |\upsilon(t)|^q dt\ \Big)^{\frac{2}{q}}, \tag{5.21}$$

$$C_\delta' \int_{|\upsilon|\geq\delta} |\upsilon(t)|^q dt - 4\beta\Big(\int_{|\upsilon|\geq\delta} |\upsilon(t)|^q dt\ \Big)^{\frac{2}{q}} \geq C_7\Big(\int_{|\upsilon|\geq\delta} |\upsilon(t)|^q dt\ \Big)^{\frac{2}{q}}, \tag{5.22}$$

where $|\upsilon| = \sqrt{\upsilon^2(t)+\upsilon^2(t-\tau)}$.

By (5.21), (5.22) and the inequality

$$a^c + b^c \leq (a+b)^c \leq 2^c(a^c+b^c),$$

where $a, b > 0$, and $c > 1$, we get

$$\begin{aligned}
J(\upsilon) \ &\geq\ -\frac{\beta}{2}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} - \frac{\beta}{2}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} \\
&+\ C_\delta\int_{|\upsilon|<\delta}|\upsilon(t)|^2dt + C_\delta\int_{|\upsilon|<\delta}|\upsilon(t-\tau)|^2dt + C_\delta'\int_{|\upsilon|\geq\delta}\big(\sqrt{\upsilon^2(t)+\upsilon^2(t-\tau)}\ \big)^q dt \\
&\geq\ -\frac{\beta}{2}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} - \frac{\beta}{2}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} + C_\delta\int_{|\upsilon|<\delta}|\upsilon(t)|^2dt \\
&+\ C_\delta\int_{|\upsilon|<\delta}|\upsilon(t-\tau)|^2dt + C_\delta'\int_{|\upsilon|\geq\delta}|\upsilon(t)|^q dt \\
&\geq\ C_7\Big[\Big(\int_{|\upsilon|<\delta}|\upsilon(t)|^q dt\ \Big)^{\frac{2}{q}} + \Big(\int_{|\upsilon|\geq\delta}|\upsilon(t)|^q dt\ \Big)^{\frac{2}{q}}\Big] + C_\delta\int_{|\upsilon|<\delta}|\upsilon(t-\tau)|^2dt \\
&\geq\ \frac{C_7}{2^{\frac{2}{q}}}\|\upsilon(t)\|^2_{L^q[0,2\gamma\tau]} = \frac{C_7}{2^{\frac{2}{q}}}r^2.
\end{aligned}$$

Then choose $\rho = \frac{C_7}{2^{\frac{2}{q}}} r^2$. That is the conclusion that we want to prove.

(iii) It is obvious $J(\theta) = 0$. and $J(v)$ is an even function in v.

From (i),(ii),(iii) and the Maintain pass theorem, we obtain the problem (4.2) has at least one nontrivial $2\gamma\tau$-periodic solution, that is Theorem 5.1 holds.

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