分数阶中立泛函微分方程权伪概周期解的存在性

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摘要：本文讨论了具无穷时滞的分数阶中立泛函微分方程权伪概周期解的存在性问题，利用压缩映射原理得到了上述方程权伪概周期解的存在唯一性，并且给出一个实例来说明本文的主要结果。

关键词：权伪概周期解，分数阶中立泛函微分方程，Banach压缩映射原理。

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Weighted pseudo almost periodic solutions of a fractional neutral functional differential equation

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Abstract: In this paper, by applying the properties of weighted pseudo almost periodic functions and Banach contraction mapping theorem, a set of sufficient conditions for the existence of weighted pseudo almost periodic solutions of a fractional neutral functional differential equations is obtained. In addition, we list a specific example of application of theorem to explain our proof.

Key words: Weighted pseudo almost periodic solutions; fractional neutral functional differential equations; Existence Banach contraction mapping theorem.

0 Introduction

In the recent years, the existence, uniqueness, stability of almost periodic, almost automorphic, asymptotically almost periodic, pseudo almost periodic solutions are rather attractive topics in the theory of almost periodic functions(see [1-7] for details). For instance, in [8], the author introduced a new class of almost periodic functions. This new concept is called weighted pseudo-almost periodicity. The theory of weighted pseudo-almost periodic is based on notions of pseudo-almost periodic functions which are introduced by Zhang[9-11], on the

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basis of definition of pseudo-almost periodic functions[12-14], with the help of a weighted measure $d\mu(x) = \rho(x)dx$, where $\rho : R \to (0, \infty)$ is a locally integrable function over R, which is commonly called weight. In [15], the authors considered some sufficient conditions of existence and uniqueness of a weighted pseudo-almost periodic mild solution to the following semi-linear fractional differential equation

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)),\ t \in R, \tag{0.1}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X and $f : R \times X \to X$ is a weighted pseudo-almost function satisfying suitable conditions in X. More details of weighted pseudo-almost periodicity see[16-19] for details.

Motivated by [15], we considered some sufficient conditions of existence and uniqueness of a weighted pseudo-almost periodic mild solution to the following neutral functional fractional differential equations

$$D_t^\alpha (u(t) - F_1(t, B_1u(t))) = A(u(t) - F_1(t, B_1u(t))) + D_t^{\alpha-1} F_2(t, u(t), B_2u(t)),\ t \in R, \tag{0.2}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X, B_1, B_2 are bounded linear operators. $F_1 : R \times X \to X$ and $F_2 : R \times X \times X \to X$ are all weighted pseudo-almost functions satisfying suitable conditions in X.

This paper is organized as follows. In Section 1, we recall some basic definitions, lemmas and preliminary facts which will be used throughout this paper. Our main results are proved in Sections 2. In Section 3, we list a specific example of application of theorem to explain our proof.

1 Preliminaries

Throughout the work, let $(Z, |\cdot|)$ and $(W, |\cdot|)$ be two Banach spaces. The notation $B(Z, W)$ represents the Banach space of all bounded linear operators from Z into W endowed with the supremum norm defined by $\|x\| = \sup_{t\in R}\{|x(t)|\}$. And when $Z = W$, we abbreviate it to $B(Z)$.

1.1 Weighted pseudo-almost periodic solutions

Now we denote U as the collection of functions $\rho : R \to (0, +\infty)$ satisfying: (i) ρ is piecewise continuous; (ii) ρ has the property which is a locally integrable function on R.

For any $r > 0$, each $\rho \in U$, set $m(r, \rho) = \int_{-r}^{r} \rho(x)dx$. Let

$$U_\infty = \{\rho \in U : \lim_{r\to\infty} m(r, \rho) = \infty\};$$
$$U_B = \{\rho \in U : \rho \text{ is bounded and } \inf_{x\in R} \rho(x) > 0\}.$$

It is obviously that $U_B \subseteq U_\infty \subseteq U$.

We assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are two Banach spaces, $BC(R\times Y, X)$ is the Banach space of all linear bounded continuous functions from $R \times Y$ into X (respectively, the class of jointly bounded continuous functions $f : R\times Y \to X$). The space $BC(R, X)$ equipped with its natural norm $\|u\| = \sup_{t\in R} \|u(t)\|$. Furthermore, $C(R, Y)$ is the Banach space of all continuous functions from R into Y (respectively, $C(R \times Y, X)$ the class of jointly bounded continuous functions $f : R \times Y \to X$).

Definition 1. (1) *A strongly continuous function $f : R \to X$ is is called almost periodic in $t \in R$ if for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that $\|f(t + \tau) - f(t)\| \le \varepsilon$ for each $t \in R$. The number τ is called an ε-translation number of f and the collection of all functions will be denoted $AP(X)$.*

(2) *A continuous function $f : R \times Y \to X$ is called almost periodic in $t \in R$ uniformly in $y \in Y$ if for every $\varepsilon > 0$ and any compact $K \subset Y$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that $\|f(t + \tau, y) - f(t, y)\| \le \varepsilon$ for each $t \in R$ and $y \in K$. The collection of those functions is denoted by $AP(Y, X)$.*

Let $\rho \in U_\infty$. Define

$$PAP_0(X, \rho) = \{f \in BC(R, X) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(s)\|_X \rho(s)ds = 0\}.$$

Obviously, when the weight function $\rho(x)$=1 for each $x \in R$, $PAP_0(X, \rho)$ is said to be the so-called ergodic space $PAP_0(X)$([9-11]), which is defined by

$$PAP_0(X) = \{f \in BC(R, X) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(s)\|_X ds = 0\}$$

Similarly, we define $PAP_0(Y, X, \rho)$ as the collection of jointly continuous functions $f : R\times Y \to X$, $f(\cdot, y)$ is bounded, for any $y \in Y$, such that

$$PAP_0(Y, X, \rho) = \{f \in BC(R \times Y, X) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(s, y)\|\rho(s)ds = 0\}$$

Definition 2. [8] *Let $\rho \in U_\infty$. A function $f \in BC(R, X)$ is called weighted pseudo almost periodic if $f = g+\phi$, where $g \in AP(X)$ and $\phi \in PAP_0(X, \rho)$. The collection of these functions are denoted by $PAP(X, \rho)$. The functions g and ϕ are respectively called the almost periodic and the weighted ergodic perturbation components of f.*

Definition 3. [8] *Let $\rho \in U_\infty$. A function $f \in BC(R \times Y, X)$ is called weighted pseudo almost periodic in $t \in R$ uniformly in $\phi \in Y$ if $f = g+\phi$, where $g \in AP(Y, X)$ and $\phi \in PAP_0(Y, X, \rho)$. The collection of these functions are denoted by $PAP(Y, X, \rho)$.*

Lemma 1. [8] *Let $\rho \in U_\infty$, $f \in PAP(Y, X, \rho)$ satisfy the Lipschitz condition*

$\|f(t, u) - f(t, v)\| \le L\|u - v\|_Y$, *for any* $u, v \in Y, t \in R$

If $h(\cdot) \in PAP(Y, \rho)$, then $f(\cdot, h(\cdot)) \in PAP(X, \rho)$.

Similar to the proof of Lemma 1, we easily have

Lemma 2. *Let $\rho \in U_\infty$, $f \in PAP(Y \times Y, X, \rho)$ satisfy the Lipschitz condition*

$$\|f(t,x,u) - f(t,y,v)\| \le L(\|x-y\|_Y + \|u-v\|_Y),\ \text{for any } x,y,u,v \in Y, t \in R$$

If $h_1(\cdot), h_2(\cdot) \in PAP(Y,\rho)$, then $f(\cdot, h_1(\cdot), h_2(\cdot)) \in PAP(X,\rho)$.

Lemma 3. [8] *Fix $\rho \in U_\infty$. The decomposition of weighted pseudo almost periodic function $f = g + \phi$, where $g \in AP(Y,X)$ and $\phi \in PAP_0(Y,X,\rho)$, is unique.*

More details of weighted pseudo almost periodic functions, see [8,14] for details.

1.2 Sectorial linear operators and their associated solution operator

A closed linear operator A with dense domain $D(A)$ in a Banach space X is said to be sectorial of type ω and angle θ if there are constants $\omega \in R, \theta \in (0, \frac{\pi}{2}), M > 0$ such that its resolvent exists outside the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in C, |arg(-\lambda)| < \theta\} \text{ and } \|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - \omega|}, \lambda \in \omega + S_\theta. \tag{1.1}$$

Sectorial operators are well studied in the literature, usually for the case $\omega = 0$. For a recent reference including several examples and properties we refer the reader to [20]. Note that an operator A is sectorial of type ω if and only if $\omega I - A$ is sectorial of type 0.

Definition 4. [21] *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A is the generator of a solution operator if there are $\omega \in R$ and a strongly continuous function $S_\alpha : R^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subseteq \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, Re\lambda > \omega, x \in X.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

If A is sectorial of type ω with $0 \le \theta \le \pi(1 - \frac{\alpha}{2})$, then A is the generator of a solution operator given by

$$S_\alpha(t) := \frac{1}{2\pi i} \int_\gamma e^{-\lambda t} \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda,$$

where γ is a suitable path lying outside the sector $\omega + S_\theta$ (cf. [21]). In [22] has proved that if A is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $0 \le \theta \le \pi(1 - \frac{\alpha}{2})$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\|_{B(X)} \le \frac{CM}{1 + |\omega| t^\alpha}, \text{ for } t \ge 0. \tag{1.2}$$

In the border case $\alpha = 1$, this is analogous to saying that A is the generator of a exponentially stable C_0-semigroup. The main difference is that in the case $\alpha > 1$ the solution family $S_\alpha(t)$ decays like $t^{-\alpha}$. Cuesta′s result proves that $S_\alpha(t)$ is integrable. We also note that

$$\int_0^\infty \frac{1}{1 + |\omega| s^\alpha} ds = \frac{\omega^{-\frac{1}{\alpha}\pi}}{\alpha \sin \frac{\pi}{\alpha}}, \text{for } 1 < \alpha < 2, \qquad (1.3).$$

therefore $S_\alpha(t)$ is integrable. The concept of a solution operator is closely related to the concept of a resolvent family (see [23 , Chapter I]). For the scalar case, where there is a large bibliography, we refer the reader to the monograph [24], and references therein. Because of the uniqueness of the Laplace transform, in the border case $\alpha = 1$ the family $S_\alpha(t)$ corresponds to a C_0-semigroup, whereas in the case $\alpha = 2$ a solution operator corresponds to the concept of a cosine family; see [25]. We note that solution operators, as well as resolvent families, are a particular case of (a, k)-regularized families introduced in [26]. According to [26] a solution operator $S_\alpha(t)$ corresponds to $(1, \frac{t^{\alpha-1}}{\Gamma(\alpha)})$-regularized family.

2 Main results

Following [15], we give this definition.

Definition 5. *Assume that A generates an integrable solution operator $E_\alpha(t)$. A continuous function $u : R \to X$ satisfying the integral equation*

$$u(t) = F_1(t, B_1u(t)) + \int_{-\infty}^{t} E_\alpha(t-s)F_2(s, u(s), B_2u(s))ds,\ t \in R$$

is called a mild solution on R to (0.2).

Throughout the rest of the paper, we now require the following assumptions:

(H_1) The functions $F_1 \in PAP(Y, X, \rho)$. There exist positive numbers L_f such that

$$\|F_1(t, x_1) - F_1(t, x_2)\| \le L_{F_1}\|x_1 - x_2\|_Y$$

for each all $t \in R$ and $x_i \in X, i = 1, 2$ satisfying $|x_i| \le K$.

(H_2) The function $F_2(t, x, y) : R \times Y \times Y \to X$ is weighted pseudo almost periodic in $t \in R$ uniformly if $x, y \in X$. There exist positive numbers L_g such that

$$\|F_2(s, x_1, y_1) - F_2(s, x_2, y_2)\| \le L_{F_2}(\|x_1 - x_2\|_Y + \|y_1 - y_2\|_Y),$$

for each all $t \in R$ and $x_i, y_i \in X, i = 1, 2$ satisfying $|x_i|, |y_i| \le K$. Setting $F_2 = f_1 + f_2$ with $f_1 \in AP(Y \times Y, X)$ and $f_2 \in PAP_0(Y \times Y, X, \rho)$, we suppose that $f_2(\cdot, u(\cdot), B_2u(\cdot)) \in L^1(R)$ for $u \in PAP(X, \rho)$, where B_2 is the bounded linear operator.

(H_3) Let V_∞ be the collection of all continuous weights $\rho : R \to (0, \infty)$ such that $\sup\limits_{s \in R}\left[\frac{\rho(s+\tau)}{\rho(s)}\right] < \infty$, $\sup\limits_{s>0}\left[\frac{\rho(s+\tau)}{\rho(s)}\right] < \infty$ for every $\tau \in R$. We easily know if $\rho \in V_\infty$ then the space $PAP(X, \rho)$ is translation invariant.

Lemma 4. [22] *Let $A : D(A) \subset X \to X$ be a sectorial operator in a complex Banach space X, satisfying the hypothesis in* (1.1). *Then there exists $C(\theta, \alpha) > 0$ depending solely on θ and α, such that*

$$\|E_\alpha(t)\| \le \frac{MC(\theta, \alpha)}{1 + |\omega|t^\alpha}. \tag{2.1}$$

Lemma 5. *Let $\rho \in V_\infty$. Assume that A is a sectorial operator of type $\omega < 0$. If $f : R \to X$ is a weighted pseudo-almost periodic function and Γf is given by $(\Gamma f)(t) = \int_{-\infty}^{t} E_\alpha(t-s)F_2(s,u(s),B_2u(s))ds,\ t \in R$. Then $\Gamma f \in PAP(X,\rho)$.*

Proof. Let $u \in PAP(X,\rho)$, we easily have that the functions $B_iu(t), i = 1,2$ are weighted pseudo-almost periodic. Under assumptions $(H_1),(H_2)$, by using Lemma 1, it follows that $F_1(t,B_1u(t)) \in PAP(X,\rho)$ and by using Lemma 2, it follows that $F_2(t,u(t),B_2u(t)) \in PAP(X,\rho)$. Now write $F_2(t,u(t),B_2u(t)) = f_1(t,u(t),B_2u(t)) + f_2(t,u(t),B_2u(t))$, where $f_1 \in AP(X)$ and $f_2 \in PAP_0(X,\rho)$. Let $G_1u(t) = \int_{-\infty}^{t} E_\alpha(t-s)f_1(s,u(s),B_2u(s))ds$, $G_2u(t) = \int_{-\infty}^{t} E_\alpha(t-s)f_2(s,u(s),B_2u(s))ds$. To complete the proof, one must prove that $G_1u(t) \in AP(X), G_2u(t) \in PAP_0(X,\rho)$. Since $f_1(s,u(s),B_2u(s)) \in AP(X)$, therefor for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\gamma \in R$, there is a $\tau \in [\gamma, \gamma+\delta]$ with

$$|f_1(t+\tau,u(t+\tau),B_2u(t+\tau)) - f_1(t,u(t),B_2u(t))| < \varepsilon$$

for each $t \in R$. By setting $s = t-\tau$, we get that $G_1u(t+\tau) = \int_{-\infty}^{t} E_\alpha(t-s)f_1(s+\tau,u(s+\tau),B_2u(s+\tau))ds$. Considering $G_1u(t+\tau) - G_1u(t)$, it easily follows that

$$\begin{aligned}\|G_1u(t+\tau) - G_1u(t)\| &= \int_{-\infty}^{t} \|E_\alpha(t-s)\||g_1(s+\tau,u(h_2(s+\tau)) - g_1(s,u(h_2(s))|ds \\ &< \varepsilon \int_{-\infty}^{t} \frac{MC(\theta,\alpha)}{1+|\omega|(t-s)^\alpha}ds = \frac{MC(\theta,\alpha)|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha \sin(\pi/\alpha)}\varepsilon.\end{aligned}$$

Thus $G_1u \in AP(X)$.

The next step is to show $G_2u(t) \in PAP_0(X,\rho)$. It is clear that $G_2u(t)$is bounded continuous. Thus it remains to show $G := \lim\limits_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} |G_2u(t)|\rho(t)dt = 0$. Clearly,

$$\begin{aligned}G &\le \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \int_{-\infty}^{t} \|E_\alpha(t-s)\|\|f_2(s,u(s),B_2u(s))\|ds\rho(t)dt \\ &\le CM \lim_{r\to\infty} \int_0^\infty \frac{1}{1+|\omega|s^\alpha} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f_2(t,u(t),B_2u(t))\|\rho(t)dtds.\end{aligned}$$

By using that the space $PAP(X,\rho)$ is translation invariant, it follows that $f_2(t-s,u(t-s),B_2u(t-s)) \in PAP(X,\rho)$ for each $s \in R$. Let $\Phi(r) = \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f_2(s,u(s),B_2u(s))\|\rho(s)ds$, hence $\Phi(r) \to 0$ as $r \to \infty$. Since $\Phi(r)$ is bounded($\|\Phi(r)\| \le \|f_2\|$) and $\frac{1}{1+|\omega|s^\alpha}$ is integrable in $[0,\infty)$, by using the Lebesgue's dominated convergence theorem it follows that $\lim_{r\to\infty} \lim\limits_{r\to\infty} \int_0^\infty \frac{1}{1+|\omega|s^\alpha} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|f_2(s,u(s),B_2u(s))\|\rho(t)dtds = 0$, thus $G_2u(t) \in PAP_0(X,\rho)$.

To prove our main results, we define a mapping Φ by

$$(\Phi u)(t) = F_1(t,B_1u(t)) + \int_{-\infty}^{t} E_\alpha(t-s)F_2(s,u(s),B_2u(s))ds,\ t \in R. \tag{2.2}$$

Lemma 6. *Let $\rho \in V_\infty$. Under assumptions $(H_1)-(H_3)$, the function $(\Phi u)(t)$ maps $PAP(X,\rho)$ into itself.*

Proof. From Lemma 1, Lemma 5, we have $F_1(t,B_1u(t)), \int_{-\infty}^{t} E_\alpha(t-s)F_2(s,u(s),B_2u(s))ds \in PAP(X,\rho)$. In all $(\Phi u)(t) = F_1(t,B_1u(t)) + \int_{-\infty}^{t} E_\alpha(t-s)F_2(s,u(s),B_2u(s))ds$ belong to $PAP(X,\rho)$.)

Theorem 1. *Let $\rho \in V_\infty$. Assume the conditions $(H_1)-(H_2)$ are satisfied. If there exists a positive number r such that*

$$L_{F_1}r + \sup_{t\in R}|F_1(t,0)| + \frac{MC(\theta,\alpha)|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha\sin(\pi/\alpha)}(L_{F_2}r + \sup_{t\in R}|F_2(t,0,0)|) < r. \tag{2.3}$$

Then Eq.(0.2) has one unique weighted pseudo almost periodic solution.

Proof. Let $B_r = \{u \in PAP(X,\rho), \|u\| \le r\}$, where r satisfies (2.3). Now we prove that $(\Phi u)(t)$ is a contraction from B_r into B_r. If $u \in B_r$, we have

$$\begin{aligned}
\|\Phi u\| &= \sup_{t\in R}\{|F_1(t,B_1u(t)) + \int_{-\infty}^t E_\alpha(t-s)F_2(s,u(s),B_2u(s))ds|\} \\
&\le \sup_{t\in R}\{|F_1(t,B_1u(t)) - F_1(t,0)| + |F_1(t,0)| \\
&\quad + \int_{-\infty}^t \|E_\alpha(t-s)\|(|F_2(s,u(s),B_2u(s)) - F_2(s,0,0)| + |F_2(s,0,0)|)ds\} \\
&\le L_{F_1}\|u\| + \sup_{t\in R}|F_1(t,0)| + (L_{F_2}\|u\| + \sup_{t\in R}|g(t,0)|)\int_{-\infty}^t \tfrac{MC(\theta,\alpha)}{1+|\omega|(t-s)^\alpha}ds \\
&\le L_{F_1}\|u\| + \sup_{t\in R}|F_1(t,0)| + \tfrac{MC(\theta,\alpha)|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha\sin(\pi/\alpha)}(L_{F_2}\|u\| + \sup_{t\in R}|F_2(t,0,0)|) \\
&\le L_{F_1}r + \sup_{t\in R}|F_1(t,0)| + \tfrac{MC(\theta,\alpha)|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha\sin(\pi/\alpha)}(L_{F_2}r + \sup_{t\in R}|F_2(t,0,0)|) < r,
\end{aligned}$$

which implies that $\|\Phi u\| \le r$ and so that $\Phi(B_r) \subseteq B_r$.

Moreover, for $u, v \in B_r$ we get

$$\begin{aligned}
\|\Phi u - \Phi v\| &\le \|F_1(t,B_1u(t)) - F_1(t,B_1v(t))\| \\
&\quad + \int_{-\infty}^t \|E_\alpha(t-s)\|\|F_2(s,u(s),B_2u(s)) - F_2(s,v(s),B_2v(s))\|ds \\
&\le L_{F_1}\|u-v\| + L_{F_2}\|u-v\|\int_{-\infty}^t \tfrac{MC(\theta,\alpha)}{1+|\omega|(t-s)^\alpha}ds \\
&= (L_{F_1} + L_{F_2}\tfrac{MC(\theta,\alpha)|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha\sin(\pi/\alpha)})\|u-v\|.
\end{aligned}$$

Since (2.3), by using Banach contraction mapping theorem, we have that (0.2) has a unique weighted pseudo almost periodic solution.

3 Application

Consider the following fractional differential equation

$$\begin{aligned}
&\partial_t^\alpha(u(t,x) - qu(t-\tau,x)) = \tfrac{\partial^2}{\partial x^2}u(t,x) - \mu u(t,x) + \partial_t^{\alpha-1}(u(t,x) + \beta u(t-\tau,x)),\ t \in R, x \in [0,\pi], \\
&u(t-\tau,0) = u(t-\tau,\pi) = 0,\ t \in R,
\end{aligned} \tag{3.1}$$

where $\tau > 0, 0 \le q < 1, 0 \le \beta < 1$. Let $X = (L^2[0,\pi]; \|\cdot\|_2)$. We define $A : D(A) \subseteq X \to X$ given by $Au = u'' - \mu u (\mu > 0)$ with the domain $D(A) = \{u(\cdot) : u'' \in X, u(0) = u(\pi) = 0\}$. Then A is the infinitesimal generator of an analytic semigroup on $L^2([0,\pi])$. Hence A is sectorial of type $\omega = -\mu < 0$. Let $u(t) = u(t,\cdot)$, Eq.(3.1) can be formulated by the inhomogeneous problem

(0.2). We take $F_1(t, B_1\phi) = q\phi(t-\tau)$, $F_2(t, \phi, B_2\phi) = \phi(t) + \beta(t-\tau)$. Then (3.1) takes the following form

$$\partial_t^{\alpha}(u(t) - qu(t-\tau)) = Au(t) + \partial_t^{\alpha-1}(u(t) + \beta u(t-\tau)),$$

According to Theorem 1 , we have the following result.

Theorem 2. *Assume the conditions* $(H_1)-(H_3)$ *are satisfied. If* $q + \frac{MC(\theta,\alpha)|\omega|^{-\frac{1}{\alpha}}\pi}{\alpha\sin(\pi/\alpha)}\beta < 1$. *Then Eq.*(3.1) *has one weighted pseudo almost periodic solution.*

4 Conclusion

In this paper, existence of weighted pseudo almost periodic solutions of a fractional neutral functional differential equations is considered. By applying the properties of weighted pseudo almost periodic functions and Banach contraction mapping theorem, a set of sufficient conditions for the existence of weighted pseudo almost periodic solutions of a fractional neutral functional differential equations is obtained. In the end, we examine sufficient conditions for the existence and uniqueness of weighted pseudo almost periodic solutions for a concrete example.

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