具无穷时滞的分数阶抽象积分-微分方程 S-渐近ω-周期弱解的存在性

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摘要： 本文讨论了具无穷时滞的分数阶抽象积分-微分方程S-渐近ω-周期弱解的存在性问题，利用压缩映射原理得到了上述方程S-渐近ω-周期弱解的存在唯一性，并且给出一个实例来说明本文的主要结果。

关键词： S-渐近ω-周期弱解，分数积分-微分方程，解算子，不动点定理。
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Existence of S-asymptotically ω-periodic solutions for abstract fractional integro-differential equations with infinite delay

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Abstract: In this paper, existence of S-asymptotically ω-periodic mild solutions for abstract integro-differential equation with fractional order is considered. The main results are obtained by the contraction mapping principle. Moreover, an example is given to illustrate the main results.

Key words: S-asymptotically ω-periodic mild solutions; Fractional integral-differential equation; Solution operator; Fixed point theorems.

0 Introduction

The existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions is one of the most attracting topics in the qualitative theory of differential equations, due to its mathematical interest and applications. Some recent contributions on the existence of such solutions for

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abstract differential equations and fractional differential equations have been made, see [1-8] for details. The literature concerning the study of the existence of S-asymptotically ω-periodic solutions of ordinary differential equations described on finite dimensional spaces(see [9-13]). Recently Henríquez et al. [14], concerned a theory of S-asymptotically ω-periodic functions with values in Banach spaces. Fractional differential equations serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [15-23] and the references therein.

Motivated by above works, in this paper, we study the following fractional partial integro-differential neutral equation with infinite delay:

$$\frac{d}{dt}D(t,u_t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AD(s,u_s)ds + g(t,u_t), t \geq 0, \tag{1.1}$$

with $u_0 = \varphi \in \mathcal{B}$, where $1 < \alpha < 2, D(t,\varphi) = \varphi(0) + f(t,\varphi), A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X. the history $u_t : (-\infty, 0] \to X$ defined by $u_t(\theta) = u(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically. $f, g : [0,\infty) \times X \to X$ are appropriate function. The fractional derivative D_t^α is to be understood in Riemann-Liouville sense. To the best of the authors' knowledge, the existence of S-asymptotically ω-periodic mild solutions for abstract partial fractional integro-differential neutral equation with infinite delay is a subject that has not been treated in the literature.

The organization of this paper is as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, by using the method of the contraction mapping principle, we obtain the existence of S-asymptotically ω-periodic mild solutions of system (1.1). In Section 2, we examine sufficient conditions for the existence and uniqueness of S-asymptotically ω-periodic mild solutions for a concrete example.

1 Preliminaries

Definition 1. *[24] The fractional integral of order $\alpha > 0$ with the lower limit t_0 for a function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s)ds, t > t_0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$, where Γ is the Gamma function.

Definition 2. *[24] Riemann-Liouville derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : [t_0, \infty) \to R$ can be written as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t (t-s)^{-\alpha} f(s)ds, t > t_0, n-1 < \alpha < n.$$

More details of fractional differential equations, see [24] for details.

In the following, we give the definitions of sectorial linear operators and their associated solution operator.

A closed linear operator $(A, D(A))$ with dense domain $D(A)$ in a Banach space X is said to be sectorial of type ω and angle θ if there are constants $\omega \in R, \theta \in (0, \frac{\pi}{2}), M > 0$ such that its resolvent exists outside the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in C, |arg(-\lambda)| < \theta\}, \text{ and } \|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \lambda \in \omega + S_\theta.$$

Sectorial operators are well studied in the literature, usually for the case $\omega = 0$. For a recent reference including several examples and properties we refer the reader to [25]. Note that an operator A is sectorial of type ω if and only if $\omega I - A$ is sectorial of type 0.

Definition 3. [26] *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A is the generator of a solution operator if there are $\omega \in R$ and a strongly continuous function $S_\alpha : R^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subseteq \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, Re\lambda > \omega, x \in X.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

If A is sectorial of type ω with $0 \leq \theta \leq \pi(1 - \frac{\alpha}{2})$, then A is the generator of a solution operator given by

$$S_\alpha(t) := \frac{1}{2\pi i} \int_\gamma e^{-\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda, \tag{2.1}$$

where γ is a suitable path lying outside the sector $\omega + S_\theta$ (cf. [26]). In [27], Cuesta has proved that if A is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $0 \leq \theta \leq \pi(1 - \frac{\alpha}{2})$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\|_{B(X)} \leq \frac{CM}{1 + |\omega| t^\alpha}, \tag{2.2}$$

for $t \geq 0$. In the border case $\alpha = 1$, this is analogous to saying that A is the generator of a exponentially stable C_0-semigroup. The main difference is that in the case $\alpha > 1$ the solution family $S_\alpha(t)$ decays like $t^{-\alpha}$. Cuesta's result proves that $S_\alpha(t)$ is integrable. We also note that

$$\int_0^\infty \frac{1}{1 + |\omega| s^\alpha} ds = \frac{\omega^{-\frac{1}{\alpha}\pi}}{\alpha \sin \frac{\pi}{\alpha}}, \text{for } 1 < \alpha < 2 \qquad (2.3).$$

therefore $S_\alpha(t)$ is integrable. The concept of a solution operator is closely related to the concept of a resolvent family (see [28, Chapter I]). For the scalar case, where there is a large bibliography, we refer the reader to the monograph [29], and references therein. Because of the uniqueness of the Laplace transform, in the border case $\alpha = 1$ the family $S_\alpha(t)$ corresponds to a C_0-semigroup, whereas in the case $\alpha = 2$ a solution operator corresponds to the concept of a cosine family; see [30]. We note that solution operators, as well as resolvent families, are a particular case of (a,k)-regularized families introduced in [31]. According to [31] a solution operator $S_\alpha(t)$ corresponds to $(1,\frac{t^{\alpha-1}}{\Gamma(\alpha)})$-regularized family.

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced in [32]. More precisely, $\mathcal{B}$ is a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ such that the next axioms hold.

(A) If $x:(-\infty,\sigma+a)\to X, a>0, \sigma\in R$ is continuous on $[\sigma,\sigma+a)$ and $x_\sigma\in\mathcal{B}$, then for every $t\in[\sigma,\sigma+a)$, the followings hold:

(i) $x_t\in\mathcal{B}$,

(ii) $\|x_t\|_X\le H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_B\le K(t-\sigma)\sup\{\|x(s)\|:\sigma\le s\le t\}+M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H>0$ is a constant; $K,M:[0,\infty)\to[1,\infty), K$ is continuous, M is locally bounded and H,K,M ar independent of $x(\cdot)$.

$(A1)$ For the function $x(\cdot)$ appearing in condition (A), the function $t\to x_t$ is continuous from $[\sigma,\sigma+a)$ into $\mathcal{B}$.

(B) The space $\mathcal{B}$ is complete.

(C) If $(\varphi^n)_{n\in N}$ is a uniformly bounded sequence in $BC((-\infty,0],X)$ given by functions with compact support and $\varphi^n\to\varphi$ in the compact open topology, then $\varphi\in\mathcal{B}$ and $\|\varphi^n-\varphi\|_{\mathcal{B}}\to 0$ as $n\to\infty$.

Let the space $\mathcal{B}_0=\{\varphi\in\mathcal{B}:\varphi(0)=0\}$ and the operator $S(t):\mathcal{B}\to\mathcal{B}$ defined by

$$S(t)\phi(\theta)=\begin{cases}\phi(0), & \theta\in[-t,0],\\ \phi(t+\theta), & \theta\in(-\infty,-t].\end{cases}$$

It is well known that $\{S(t)_{t\ge 0}\}$ is a C_0-semigroup [32].

Definition 4. *The phase space $\mathcal{B}$ is called a fading memory space if $\|S\varphi\|_{\mathcal{B}}\to 0$ as $t\to\infty$ for every $\varphi\in\mathcal{B}_0$. We said that $\mathcal{B}$ is a uniform fading memory space if $\|S\|_{L(\mathcal{B})}\to 0$ as $t\to\infty$.*

Remark 1. *Since $\mathcal{B}$ satisfies axiom (C), the space $C_b((-\infty,0],X)$ consisting of all continuous and bounded functions $\psi:(-\infty,0]\to X$, is continuously included in $\mathcal{B}$. Thus, there exists a constant $L\ge 0$ such that $\|\psi\|_{\mathcal{B}}\le L\|\psi\|_\infty$, for every $\psi\in C_b((-\infty,0],X)$*[32, *Proposition* 7.1.1]. *Moreover, if $\mathcal{B}$ is a fading memory space, then K,M are bounded functions* [32, *Proposition* 7.1.5].

In this work $C_b([0,\infty),X)$ denotes the space consisting of the continuous and bounded functions from $[0,\infty)$ into X, endowed with the norm of the uniform convergence which is denoted by $\|\cdot\|_\infty$. Let us recall the notion of S-asymptotic ω-periodicity which will come into play later on.

Definition 5. [14] *A function $f\in C_b([0,\infty),X)$ is called S-asymptotically periodic if there exists $\omega>0$ such that $\lim_{t\to\infty}(f(t+\omega)-f(t))=0$. In this case, we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic.*

In this work the notation $SAP_\omega(X)$ stands for the subspace of $C_b([0,\infty),X)$ consisting of the S-asymptotically ω-periodic functions. We note that $SAP_\omega(X)$ is a Banach space (see [14], Proposition 3.5)

Definition 6. [14] *A continuous function $f \in [0,\infty)\times X \to X$ is called uniformly S-asymptotically ω-periodic on bounded subset K of X, the set $\{f(t,x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t\to\infty}(f(t+\omega,x) - f(t,x)) = 0$ uniformly in $x \in K$.*

Definition 7. [14] *A continuous function $f : [0,\infty)\times X \to X$ is called asymptotically uniformly continuous on bounded subsets if for every $\varepsilon > 0$ and every bounded subset K of X, there exist $t_{\varepsilon,K} \geq 0$ and $\delta_{\varepsilon,K} > 0$ such that $\|f(t,x) - f(t,y)\| \leq \varepsilon$ for all $t \geq t_{\varepsilon,K}$ and all $x, y \in K$ with $\|x - y\| \leq \delta_{\varepsilon,K}$.*

Definition 8. [14] *Let $f : [0,\infty) \times X \to X$ be uniformly S-asymptotically periodic on bounded subset and asymptotically uniformly continuous on bounded seta and let $u : [0,\infty) \to X$ be S-asymptotically periodic function. Then then function $v(t) = f(t,u(t))$ is S-asymptotically periodic.*

Definition 9. [14] *A function $f \in C_b([0,\infty),X)$ is called asymptotically asymptotically ω-periodic if there exists an ω-periodic function g and $\phi \in C_0([0,\infty),X)$ such that $f = g + \phi$(here $C_0([0,\infty),X)$ denotes the subspace of $C_b([0,\infty),X)$ such that $\lim_{t\to\infty}\|x(t)\| = 0$).*

Lemma 1. [33] *Let $f : [0,\infty) \times Z \to W$ be S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets and let $u \in SAP_\omega(X)$. Then $\lim_{t\to\infty}(f(t+\omega, u(t+\omega)) - f(t,u(t))) = 0$.*

Lemma 2. [33] *Assume that $\mathcal{B}$ is a fading memory space. Let $u : R \to X$ be a function with $u_0 \in \mathcal{B}$ and $u|_{[0,\infty)} \in SAP_\omega(X)$. Then the function $t \to u_t \in SAP_\omega(\mathcal{B})$.*

2 Main results

Definition 10. *Suppose A generates an integrable solution operator $S_\alpha(t)$. A function $u \in C_b([0,\infty),X)$ is said to be an S-asymptotically ω-periodic mild solutions to (1.1) if $u(\cdot)$ is S-asymptotically ω-periodic such that*

$$u(t) = S_\alpha(\varphi(0), f(0,\varphi(0))) - f(t,u_t) + \int_{-\infty}^{t} S_\alpha(t-s)g(s,u_s)ds, t \geq 0.$$

Lemma 3. [34] *Assume that A is sectorial of type $\mu < 0$. Let $u \in SAP_\omega(X)$ and let $v_\alpha : [0,\infty) \to X$ be the function defined by $v_\alpha(t) = \int_0^t S_\alpha(t-s)u(s)ds$. Then $v_\alpha \in SAP_\omega(X)$.*

Following the method of [34], we will the following theorems.

Theorem 1. *Assume that A is sectorial of type $\omega < 0$ and that $\mathcal{B}$ is a fading memory space. Let $f, g : [0,\infty) \times \mathcal{B} \to X$ be functions uniformly S-asymptotically ω-periodic on bounded sets and there are two positive constants L_f, L_g such that*

$$\|f(t,x) - f(t,y)\| \le L_f \|x - y\|_{\mathcal{B}},$$
$$\|g(t,x) - g(t,y)\| \le L_g \|x - y\|_{\mathcal{B}}, \text{ for all } x, y \in \mathcal{B}, t \ge 0.$$

If $L(L_f + CML_g \frac{|\mu|^{-\frac{1}{\alpha}}\pi}{\alpha \sin(\frac{\pi}{\alpha})}) < 1$. Then (1.1) has a unique S-asymptotically ω-periodic mild solution.

Proof. We firstly define a set $SAP_{\omega,0}(X) = \{x \in SAP_\omega(X) : x(0) = 0\}$. It is easy to see that $SAP_{\omega,0}(X)$ is a closed subspace of $SAP_\omega(X)$. Secondly, we identify the elements $x \in SAP_{\omega,0}(X)$ with its extension to R given by $x(\theta) = 0, \theta \le 0$. Moreover, we denote by $y(\cdot)$ the function defined by $y_0 = \varphi$ and $y(t) = S_\alpha \varphi(0)$ for $t \ge 0$. Since $\sup_{t\ge 0} \|S_\alpha(t)\|_{B(X)} < \infty$, we have $y \in C_b([0,\infty), X)$. On the other hand, (2.2) implies that $y|_{[0,\infty)} \in SAP_\omega(X)$. So by Lemma 2, we have $y_t \in SAP_\omega(\mathcal{B})$. In the following, we define the map T_α on $SAP_{\omega,0}(X)$ by $(T_\alpha x)_0 = 0$ and

$$(T_\alpha x)(t) = -f(t, x_t + y_t) + v_\alpha(t), \ t \ge 0, \tag{3.1}$$

where $v_\alpha(t) = \int_0^t S_\alpha(t-s) g(s, x_s + y_s) ds$. Taking into account that $\mathcal{B}$ is a fading memory space and Lemma 2, we have that the function $g(s, x_s + y_s) \in SAP_\omega(\mathcal{B})$. In view of f being asymptotically uniformly continuous on bounded sets. By Lemma 1, we conclude that the function $g(s, x_s + y_s) \in SAP_\omega(X)$. From Lemma 3, we infer that v_α is a map from $SAP_{\omega,0}(X)$ to $SAP_{\omega,0}(X)$. We also infer that v_α is a map from $SAP_\omega(X)$ to $SAP_\omega(X)$. Furthermore, $g(s, x_s + y_s)$ is a map from $SAP_{\omega,0}(X)$ to $SAP_{\omega,0}(X)$. Moreover, we have the estimation

$$\begin{aligned} \|(T_\alpha x)(t) - (T_\alpha z)(t)\| &\le L_f \|x_t - z_t\|_{\mathcal{B}} + CML_g \int_0^t \frac{\|x_s - z_s\|_{\mathcal{B}}}{1 + |\mu|(t-s)^\alpha} ds \\ &\le LL_f \|x - z\|_\infty + CMLL_g \int_0^t \frac{\|x_s - z_s\|_\infty}{1 + |\mu| s^\alpha} ds \\ &\le L(L_f + CML_g \frac{|\mu|^{-\frac{1}{\alpha}}\pi}{\alpha \sin(\frac{\pi}{\alpha})}) \|x - z\|_\infty \end{aligned}$$

It is not difficult to verify that the maps T_α is a contraction mapping. By the contraction mapping principle, we see that T_α has a unique fixed point $x \in SAP_{\omega,0}(X)$. Define $u(t) = y(t) + x(t)$ for $t \ge 0$, we can confirm that $u \in SAP_\omega(X)$ is a unique S-asymptotically ω-periodic mild solution of (1.1).

Theorem 2. *Assume that A is sectorial of type $\omega < 0$ and that $\mathcal{B}$ is a fading memory space. Let $g : [0,\infty) \times \mathcal{B} \to X$ be functions uniformly S-asymptotically ω-periodic on bounded sets and there is a positive constant L_g such that*

$$\|g(t,x) - g(t,y)\| \le L_g \|x - y\|_{\mathcal{B}}, \text{ for all } x, y \in \mathcal{B}, t \ge 0.$$

Let $f : [0,\infty)\times\mathcal{B}\to X$ be functions uniformly S-asymptotically ω-periodic on bounded sets and satisfies the Lipschitz condition

$$\|f(t,x)-f(t,y)\|\le L_f(t)\|x-y\|_{\mathcal{B}},\ \textit{for all } x,y\in\mathcal{B}, t\ge 0,$$

where $L_f(t)\in L^1(R)$. If $L(L_f+C_\alpha\|L_g\|_1)<1$. Then (1.1) has a unique S-asymptotically ω-periodic mild solution.

Proof. Using the notation introduced in the proof of Theorem 1, we consider the map T_α defined on $SAP_{\omega,0}(X)$. It follows from our assumptions that T_α is well defined. Let x,z be in $SAP_{\omega,0}(X)$ and define $C_\alpha=\sup_{t\ge 0}\|S_\alpha(t)\|_{B(X)}$. We have

$$\begin{aligned}\|(T_\alpha x)(t)-(T_\alpha z)(t)\| &\le L_f\|x_t-z_t\|_{\mathcal{B}}+\int_0^t\|S_\alpha(t-s)(g(s,x_s+y_s)-g(s,x_s+z_s))\|_X ds\\ &\le LL_f\|x-z\|_\infty+C_\alpha\int_0^t L(s)\|x_s-z_s\|_{\mathcal{B}}ds\\ &\le L(L_f+C_\alpha\|L_g\|_1)\|x-z\|_\infty\end{aligned}$$

It is not difficult to verify that the maps T_α is a contraction mapping. By the contraction mapping principle, we see that T_α has a unique fixed point $x\in SAP_{\omega,0}(X)$. Define $u(t)=y(t)+x(t)$ for $t\ge 0$, we can confirm that $u\in SAP_\omega(X)$ is a unique S-asymptotically ω-periodic mild solution of (1.1).

3 Application

Consider the following fractional differential equation

$$\begin{aligned}&\frac{\partial}{\partial t}\Big[u(t,\xi)+\int_{-\infty}^t a_1(t-s)u(s,\xi)ds\Big]=J_t^{\alpha-1}\Big(\frac{\partial^2}{\partial\xi^2}-v\Big)\Big[u(t,\xi)+\int_{-\infty}^t a_1(t-s)u(s,\xi)ds\Big]d\xi\\ &+\int_{-\infty}^t a_2(t-s)u(s,\xi)ds,\\ &u(t,0)=u(t,\pi)=0,\ t\ge 0,\ u(0,\xi)=\varphi(0,\xi),\end{aligned}\tag{4.1}$$

for $(t,\xi)\in[0,\infty)\times[0,\pi], v>0$. Choose the space $X=L^2([0,\pi])$. In what follows, $A: D(A)\subseteq X\to X$ is the operator given by $Ax=x''-vx$ with the domain $D(A)=\{x\in X: x''\in X, x(0)=x(\pi)=0\}$. It is well known that $\Delta x=x''$ is the infinitesimal generator of an analytic semigroup on $L^2([0,\pi])$. Hence, A is a sectorial of type $\mu=-v<0$. Moreover, we have identified $\varphi(0,\xi)=\varphi(0,\xi)\in X$. The functions $a_1,a_2:[0,\infty)\to R$ are continuous functions with $L_f=\sqrt{\int_{-\infty}^0 a_1^2(-s)ds}<\infty, L_g=\sqrt{\int_{-\infty}^0 a_2^2(-s)ds}<\infty$. Setting $f,g:[0,\infty)\times\mathcal{B}\to X$ by

$$\begin{aligned}&f(t,\varphi)(\xi)=\int_{-\infty}^t a_1(t-s)\varphi(s)(\xi)ds,\ D(t,\varphi)=\varphi(0)(\xi)+f(t,\varphi)(\xi),\\ &g(t,\varphi)(\xi)=\int_{-\infty}^t a_2(t-s)\varphi(s)(\xi)ds,\ J_t^{\alpha-1}f(t)=\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds.\end{aligned}$$

On can rewrite (4.1) as the form (1.1). Moreover, f,g are bounded linear operators satisfying

$$\|f(t,x)-f(t,y)\|\le L_f\|x-y\|_{\mathcal{B}},\ \|g(t,x)-g(t,y)\|\le L_g\|x-y\|_{\mathcal{B}}.$$

Theorem 3. *Under the assumptions of Theorem 2, (4.1) has a unique S-asymptotically ω-periodic mild solution whenever L_f, L_g are small enough.*

4 Conclusion

In this paper, existence of S-asymptotically ω-periodic solutions for an abstract fractional integro-differential neutral equations with infinite delay is considered. By using contraction mapping principle, some sufficient conditions for the existence of S-asymptotically ω-periodic mild solutions of system (1.1) are obtained. In the end, we examine sufficient conditions for the existence and uniqueness of S-asymptotically ω-periodic mild solutions for a concrete example.

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