时滞耦合神经环系统中周期振动的时空模式

宋永利

同济大学数学系，上海200092

摘要：在这篇文章中，我们研究一个由两个三元环形神经网络系统组成的耦合系统中时滞诱发的周期振动的时空模式。在耦合强度和子系统的内在反应函数的增益所组成的参数平面内，同相同步和反相同步存在的区域以及相关的耦合时滞区间被明确地确定。

关键词：耦合神经环；时滞；Hopf 分支；时空模式

中图分类号： O175

Spatio-temporal patterns of periodic oscillations in delay-coupled neural loops

SONG Yong-Li

Department of Mathematics, Tongji University, Shanghai 200092

Abstract: In this paper, we investigate the spatio-temporal patterns of Hopf bifurcating periodic oscillations induced by the coupling time delay in a pair of identical tri-neuron network coupled with time delay. The explicit intervals of delay and the regions in the plane of the coupling strength and the gain of the inherent response function for the existence of synchronized in-phase or anti-phase oscillation are obtained.

Key words: Coupled neural loops; Delay; Hopf bifurcation; Spatio-temporal pattern

0 Introduction

Research on Hopfield-type neural networks with delays, first introduced by Marcus and Westervelt [1], has shown that delays can modify dynamics in interesting ways. Since then delays have been inserted into various simple neural networks. Many authors have also investigated the dynamics of the neural networks of two or more neurons with delays, and have shown various types of dynamical behaviors like Hopf bifurcations, nonlinear waves and synchronization (see, for example, [2–12] and references therein). However, most of these work only considered the individual neural network but did not investigate the interactions between different neural networks.

As a matter of fact, coupled networks, which are combined by different subnetworks and each subnetwork has its own dynamical property, are ubiquitous and also common in many

基金项目： Supported by Doctoral Fund of Ministry of Education of China (No.200802471024)

作者简介： Song Yong-Li (1971-)，male，associate professor，major research direction：theory of functional differential equations and applications.

branches of science [13]. For instance, in order to describe the complicated interaction between billions of neurons in large neural networks, the neurons are often lumped into highly connected subnetworks and the brain organization can be viewed in gross sense as a number of local subnetworks coupled by long distance connections [14]. In [15], Campbell *et al.* studied the delayed neural network model coupled by a pair of Hopfield-like tri-neuron loops, in which they analyzed the roots of characteristic equation explicitly and specially investigated the local stability and bifurcation and observed some in-phase or anti-phase oscillations in numerical simulations. Recently, Hsu et al. [16] extended the results of Campbell *et al.* [15] to a delayed model consisting of a pair of loops each with n neurons. However, authors considered only the case when the delayed coupling exists between a single neuron of each loop in [15, 16]. In [17], we proposed a general system consisting of two three-neuron neural loops with three-way connections, i.e., the coupling exists between three neurons of each loop. To emphasize the interaction of neural loops, the delay is introduced in the coupling between the loops rather than in the connection within the individual loop. This network is modelled by the following system of nonlinear delay differential equations

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + bf(x_{i+1}(t)) + cg(y_i(t-\tau)), \\ \dot{y}_i(t) = -y_i(t) + bf(y_{i+1}(t)) + cg(x_i(t-\tau)), \end{cases} \quad i = 1, 2, 3 \ (\mathrm{mod}\ 3). \tag{1}$$

Here x_i, y_i represent the voltages of the corresponding neuron, $b \in \mathbb{R}$ is the gain of the inherent response function, assumed equal for each neuron, $c \in \mathbb{R}$ is coupling strength between two individual loops and $f, g : \mathbb{R} \to \mathbb{R}$ are synaptic (transfer) functions. The interactions are inhibitory if $b < 0$, and excitatory if $b > 0$. The coupling is inhibitory if $c < 0$, and excitatory if $c > 0$.

In [17], taking the coupling delay as the bifurcating parameter, conditional/absolute stability, stability switches and Hopf bifurcations induced by time delay have been investigated and stability switches are also found as the coupling time delay varies. In [18], the authors investigated the properties of Hopf bifurcations and the explicit conditions ensuring the stability and direction of Hopf bifurcations induced by the coupling time delay have been determined by applying the normal form theory and the center manifold theorem for functional differential equations. In the present paper, we are interested in studying spatio-temporal patterns of such bifurcating periodic oscillations. Spatio-temporal patterns involve the information of nonlinear oscillations relating space and time and how patterns are created and developed. Such area of research within coupled networks have been the focus of considerable recent interest (see [19, 20] and references therein). Synchronization phenomena in the coupled system is ubiquitous and of interest to researchers in different research fields. It was found that epilepsy, Parkinson's disease, Alzheimer's disease, and schizophrenia are associated with synchronized firings of neurons. These findings highlight the desire to explore the mechanism of synchronization so that we can develop efficient methods for preventing the formation of synchronization. Recently, based on extensive numerical simulations and explicit experimental verification, Prasad *et al.* [19, 20] found that the phase-flip bifurcation, where the coupled system alternate from a state of in-phase to anti-phase, is a general and important property of time-delay coupled nonlinear system. In this paper, we attempt to analytically investigate how the coupling time delay and the coupling strength affect spatio-temporal patterns of bifurcating periodic oscillations.

1 Spatio-temporal patterns of bifurcating periodic solutions

To simply the analysis, the synaptic (transfer) functions $f, g : \mathbb{R} \to \mathbb{R}$ are assumed to be sufficiently smooth and without loss of generality we also assume the following condition is satisfied:

$$\text{(H1)}\ \ f, g, \in \mathrm{C}^1,\ f(0) = g(0) = 0,\ f'(0) = g'(0) = 1.$$

First we introduce the following results from our previous work [17].

Theorem 1. *Assuming that (H1) is satisfied, we have the following: If* $|1+b/2| < |c| < |1-b|$, *then system (1) undergoes Hopf bifurcations at the critical values* $\tau_k^{\pm}$ *such that* $\tau_j^+ \neq \tau_n^-$ *for any non-negative integer numbers* j, n*; if* $|c| > \max\{|1+b/2|, |1-b|\}$, *then system (1) undergoes Hopf bifurcations at the critical values* $\tau_k^{\pm}, \tau_k$ *such that no any two critical values of* τ_j^+, τ_n^- *and* τ_l *being equal for any non-negative integer numbers* j, n, l*; and if* $|1-b| < |c| < |1+b/2|$, *then system (1) undergoes Hopf bifurcations at* $\tau = \tau_k$. *Here* τ_k^+, τ_k^- *and* τ_k *are defined as follows:*

$$\tau_k^+ = \frac{1}{\omega_+}\left\{\arccos\left(\frac{-2-b}{2|c|}\right) + k\pi\right\}, \tag{2}$$

$$\tau_k^- = \begin{cases} \frac{1}{\omega_-}\left\{-\arccos\left(\frac{-2-b}{2|c|}\right) + (k+1)\pi\right\}, & |c| < \sqrt{1+b+b^2}, \\ \frac{1}{|\omega_-|}\left\{\arccos\left(\frac{-2-b}{2|c|}\right) + k\pi\right\}, & |c| > \sqrt{1+b+b^2} \end{cases} \tag{3}$$

and

$$\tau_k = \frac{1}{\omega}\left\{(k+1)\pi - \arccos\left(\frac{1-b}{|c|}\right)\right\}, \tag{4}$$

where

$$\omega_{\pm} = \frac{\sqrt{3}}{2}|b| \pm \sqrt{c^2 - \left(1+\frac{b}{2}\right)^2},\ \omega = \sqrt{c^2-(1-b)^2}. \tag{5}$$

For going ahead further, we should specify the corresponding relationship between the critical values and the eigenvalues. Set

$$v_j^{\pm} = (l_j, \pm l_j)^T, \tag{6}$$

where $j = 0, 1, 2$, $l_j = \left(1, \chi^k, \chi^{2k}\right)^T$ and $\chi = e^{\frac{2\pi}{3}i}$. Note that $v_2^+ = \overline{v_1^+}$ and $v_2^- = \overline{v_1^-}$. So, from [17], we also have the following results.

Theorem 2. *Assume that* $c > 0$, τ_k^+, τ_k^-, τ_k *and* v_j *are defined by (2), (3), (4) and (6), respectively.*

(*i*) *For the critical values of the coupling time delay* τ_k^+, *system (1) around the zero solution has a pair of purely imaginary eigenvalues* $\pm v_1^+ i$ *for* k *being odd and* $\pm v_1^- i$ *for* k *being even;*

(*ii*) *For the critical values of the coupling time delay* τ_k^-, *then when* $c < \sqrt{1+b+b^2}$, *system (1) around the zero solution has a pair of purely imaginary eigenvalues* $\pm v_1^+ i$ *for* k *being even and* $\pm v_1^- i$ *for* k *being odd, but when* $c > \sqrt{1+b+b^2}$, *system (1) around the zero solution has a pair of purely imaginary eigenvalues* $\pm v_1^+ i$ *for* k *being odd and* $\pm v_1^- i$ *for* k *being even;*

(*iii*) *For the critical values of the coupling time delay* τ_k, *system (1) around the zero solution has a pair of purely imaginary eigenvalues* $\pm v_0^+ i$ *for* k *being odd and* $\pm v_0^- i$ *for* k *being even.*

Theorem 3. *Assume that* $c < 0$, τ_k^+, τ_k^-, τ_k *and* v_j *are defined by (2), (3), (4) and (6), respectively.*

(*i*) *For the critical values of the coupling time delay* τ_k^+, *system (1) around the zero solution has a pair of purely imaginary eigenvalues* $\pm v_1^+ i$ *for* k *being even and* $\pm v_1^- i$ *for* k *being odd;*

(*ii*) *For the critical values of the coupling time delay* τ_k^-, *then when* $c < \sqrt{1+b+b^2}$, *system (1) around the zero solution has a pair of purely imaginary eigenvalues* $\pm v_1^+ i$ *for* k *being odd and* $\pm v_1^- i$ *for* k *being even, but when* $c > \sqrt{1+b+b^2}$, *system (1) around the zero solution has a pair of purely imaginary eigenvalues* $\pm v_1^+ i$ *for* k *being even and* $\pm v_1^- i$ *for* k *being odd;*

(*iii*) *For the critical values of the coupling time delay* τ_k, *system (1) around the zero solution has a pair of purely imaginary eigenvalues* $\pm v_0^+ i$ *for* k *being even and* $\pm v_0^- i$ *for* k *being odd.*

In this section, we further investigate the spatio-temporal patterns of bifurcating periodic solutions. For simplification of notations, throughout this section, we also assume that the characteristic equation has a pair of simple purely imaginary roots $\pm i\omega_*$ at the critical value τ_k^*. But keep in mind that $\pm i\omega_*$ and τ_k^* have different expressions in different bifurcation regions (see Lemma 1 and Fig.1). We refer to [21] for explanation of the terminology used in this section. To investigate the spatio-temporal patterns of bifurcating periodic solutions, we first have a discussion of the isotropy subgroup of $Z_3 \times Z_2 \times S^1$ and its fixed point subspace on the eigenspace spanned by the eigenvectors associated with a Hopf bifurcation. Then, we determine the existence of the Hopf bifurcating periodic solutions with spatio-temporal symmetries is obtained.

The symmetry of a system is important in determining the patterns of oscillation. We first explore the symmetry in system (1). Let $G : C \to \mathbb{R}^n$ and Γ be a compact Li group. It follows from [21, 22] that the system $\dot{u}(t) = G(u_t)$ is said to be Γ-equivariant if $G(\rho u_t) = \rho G(u_t)$ for all $\rho \in \Gamma$. Let $\Gamma = Z_3 \times Z_2$ be a group of 6 elements generated by ρ_1 and ρ_2, that is $Z_3 \times Z_2 = \langle \rho_1, \rho_2 \rangle$, where $\rho_1^3 = e, \rho_2^2 = e$ and e is the identity map. The action of ρ_1 and ρ_2 on $\mathbb{R}^6$ is given by

$$\rho_i(x, y)^T = (\rho_i x, \rho_i y),\ i = 1, 2, \tag{7}$$

for all $x, y \in \mathbb{R}^3$, where

$$(\rho_1 x)_i = x_{i+1},\ (\rho_1 y)_i = y_{i+1},\ (\rho_2 x)_i = y_i,\ (\rho_2 y)_i = x_i,\ \ i = 1, 2, 3 \pmod 3 \tag{8}$$

Letting $G(u_t)$ be the vector field of system (1), it is easy to very that $G(\rho_i u_t) = \rho_i G(u_t)$. Then the following lemma follows immediately.

Theorem 4. *System (1) is equivariant with respect to the group $Z_3 \times Z_2$.*

It is well known [23] that a linear functional differential equation generates a strongly continuous semigroup of linear operators with infinitesimal generator $A(\tau)$ given by

$$\begin{aligned} &A(\tau)\varphi = \dot{\varphi},\ \ \varphi \in \mathrm{Dom}(A), \\ &\mathrm{Dom}(A) = \{\varphi \in C, \varphi(0) = L(\tau)\varphi\}, \end{aligned}$$

with $L(\tau)$ being defined by the linearization of system (1) at the zero solution. Moreover, the spectrum $\sigma(A(\tau))$ of $A(\tau)$ consists of roots of the characteristic equation of system (1) around the zero solution. It follows from [23] that the eigenspace, denoted by $U_{i\omega_*}(A(\tau_k^*))$, of $A(\tau_k^*)$ for $\pm i\omega_*$ is spanned by the eigenvectors $\mathrm{Re}\{e^{i\omega_*\theta}v\}, \mathrm{Im}\{e^{i\omega_*\theta}v\}$, where v is from v_j according to the associated critical values of the coupling time delay, i.e.,

$$U_{i\omega_*}(A(\tau_k)) = \{x_1\varepsilon_1(\theta) + x_2\varepsilon_2(\theta),\ x_1, x_2 \in \mathbb{R}\},$$

where

$$\varepsilon_1(\theta) = \cos(\omega_*\theta)\mathrm{Re}(v) - \sin(\omega_*\theta)\mathrm{Im}(v),\ \ \varepsilon_2(\theta) = \sin(\omega_*\theta)\mathrm{Re}(v) + \cos(\omega_*\theta)\mathrm{Im}(v).$$

Denote by P_ω the Banach space of all continuous $\omega-$periodic mappings from $\mathbb{R}$ into $\mathbb{R}^6$, equipped with the supremum norm. Let $\omega = \frac{2\pi}{\omega_*}$ and denote by SP_ω the subspace of P_ω consisting of all $\omega-$periodic solutions of (1) at $\tau = \tau_k^*$. Then

$$SP_\omega = \{\eta_1\varepsilon_1(t) + \eta_2\varepsilon_2(t),\ \eta_1, \eta_2 \in \mathbb{R}\}.$$

For the circle group S^1, the action of $\Gamma \times S^1$ on the subspace SP_ω can be defined by the shifting arguments as follows

$$\left(\rho_i, e^{i\theta}\right) u(t) = \rho_i u(t+\theta),\ (\rho_i, \theta) \in \Gamma \times S^1,\ u(t) \in SP_\omega,\ i = 1, 2.$$

Clearly, for any $\theta \in (0, \omega)$,

$$\Sigma^\theta_{\rho_i} = \left\{ \left(\rho_i, e^{i \frac{2\pi}{\omega} \theta} \right) \right\},\ \ i = 1, 2,$$

is a subgroup of $\Gamma \times S^1$. We first determine the fixed point set

$$\mathrm{Fix}(\Sigma^\theta_{\rho_i}, SP_\omega) = \left\{ u(t) \in SP_\omega;\ \left(\rho_i, e^{i\theta}\right) u = u \ \text{ for all } \ \left(\rho_i, e^{i\theta}\right) \in \Sigma^\theta \right\}$$

which is directly related to the different types of periodic solutions.

Theorem 5. *(i)*

$$Fix(\Sigma^\theta_{\rho_1}, SP_\omega) = \begin{cases} SP_\omega, & \text{if } \theta = \frac{3-j}{3}\omega, \\ \{0\}, & \text{if } \theta \neq \frac{3-j}{3}\omega, \end{cases}$$

where $j = 0, 1$ corresponds to the subscript of $v_k^\pm$ and then

$$dim\ Fix(\Sigma^\theta_{\rho_1}, SP_\omega) = \begin{cases} 2, & \text{if } \theta = \frac{3-j}{3}\omega, \\ 0, & \text{if } \theta \neq \frac{3-j}{3}\omega. \end{cases}$$

(ii)

$$Fix(\Sigma^\theta_{\rho_2}, SP_\omega) = \begin{cases} SP_\omega, & \text{either } \theta = m\omega \text{ for } v_k^+ \text{ or } \theta = \left(m + \frac{1}{2}\right)\omega \text{ for } v_k^-, \\ \{0\}, & \text{otherwise}, \end{cases}$$

where $m \in \mathbb{Z}$, and then

$$dim\ Fix(\Sigma^\theta_{\rho_2}, SP_\omega) = \begin{cases} 2, & \text{either } \theta = m\omega \text{ for } v_k^+ \text{ or } \theta = \left(m + \frac{1}{2}\right)\omega \text{ for } v_k^-, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma. Note that

$$\rho_1(\mathrm{Re}(v_j^\pm)) = \cos\left(\frac{2j\pi}{3}\right)\mathrm{Re}(v_j^\pm) + \sin\left(\frac{2j\pi}{3}\right)\mathrm{Im}(v_j^\pm),$$

$$\rho_1(\mathrm{Im}(v_j^\pm)) = -\sin\left(\frac{2j\pi}{3}\right)\mathrm{Re}(v_j^\pm) + \cos\left(\frac{2j\pi}{3}\right)\mathrm{Im}(v_j^\pm),$$

$$\rho_2(\mathrm{Re}(v_j^+)) = \mathrm{Re}(v_j^+),\ \rho_2(\mathrm{Re}(v_j^-)) = -\mathrm{Re}(v_j^-),$$

and

$$\rho_2(\mathrm{Im}(v_j^+)) = \mathrm{Im}(v_j^+),\ \rho_2(\mathrm{Im}(v_j^-)) = -\mathrm{Im}(v_j^-).$$

Consequently,

$$\begin{aligned}
&\rho_1(\eta_1\varepsilon_1(t)+\eta_2\varepsilon_2(t))\\
=\;&\eta_1\left[\cos\left(\tfrac{2\pi}{\omega}t\right)\rho_1(\mathrm{Re}(v_j^{\pm}))-\sin\left(\tfrac{2\pi}{\omega}t\right)\rho_1(\mathrm{Im}(v_j^{\pm}))\right]\\
&+\eta_2\left[\sin\left(\tfrac{2\pi}{\omega}t\right)\rho_1(\mathrm{Re}(v_j^{\pm}))+\cos\left(\tfrac{2\pi}{\omega}t\right)\rho_1(\mathrm{Im}(v_j^{\pm}))\right]\\
=\;&\left(\eta_1\cos\left(\tfrac{2j\pi}{3}\right)-\eta_2\sin\left(\tfrac{2j\pi}{3}\right)\right)\varepsilon_1(t)+\left(\eta_1\sin\left(\tfrac{2j\pi}{3}\right)+\eta_2\cos\left(\tfrac{2j\pi}{3}\right)\right)\varepsilon_2(t),
\end{aligned}$$

$$\begin{aligned}
\rho_2(\eta_1\varepsilon_1(t)+\eta_2\varepsilon_2(t)) &= \eta_1\left[\cos\left(\tfrac{2\pi}{\omega}t\right)\rho_2(\mathrm{Re}(v))-\sin\left(\tfrac{2\pi}{\omega}t\right)\rho_2(\mathrm{Im}(v))\right]\\
&\quad+\eta_2\left[\sin\left(\tfrac{2\pi}{\omega}t\right)\rho_2(\mathrm{Re}(v))+\cos\left(\tfrac{2\pi}{\omega}t\right)\rho_2(\mathrm{Im}(v))\right]\\
&= \begin{cases}\eta_1\varepsilon_1(t)+\eta_2\varepsilon_2(t), & \text{for } v=v_j^+,\\ -\eta_1\varepsilon_1(t)-\eta_2\varepsilon_2(t), & \text{for } v=v_j^-,\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\eta_1\varepsilon_1(t+\theta)+\eta_2\varepsilon_2(t+\theta)\\
=\;&\eta_1\left[\cos\left(\tfrac{2\pi}{\omega}t\right)\cos\left(\tfrac{2\pi}{\omega}\theta\right)-\sin\left(\tfrac{2\pi}{\omega}t\right)\sin\left(\tfrac{2\pi}{\omega}\theta\right)\right]\mathrm{Re}(v)\\
&-\eta_1\left[\sin\left(\tfrac{2\pi}{\omega}t\right)\cos\left(\tfrac{2\pi}{\omega}\theta\right)+\cos\left(\tfrac{2\pi}{\omega}t\right)\sin\left(\tfrac{2\pi}{\omega}\theta\right)\right]\mathrm{Im}(v)\\
&+\eta_2\left[\sin\left(\tfrac{2\pi}{\omega}t\right)\cos\left(\tfrac{2\pi}{\omega}\theta\right)+\cos\left(\tfrac{2\pi}{\omega}t\right)\sin\left(\tfrac{2\pi}{\omega}\theta\right)\right]\mathrm{Re}(v)\\
&+\eta_2\left[\cos\left(\tfrac{2\pi}{\omega}t\right)\cos\left(\tfrac{2\pi}{\omega}\theta\right)-\sin\left(\tfrac{2\pi}{\omega}t\right)\sin\left(\tfrac{2\pi}{\omega}\theta\right)\right]\mathrm{Im}(v)\\
=\;&\eta_1\cos\left(\tfrac{2\pi}{\omega}\theta\right)\left[\cos(\tfrac{2\pi}{\omega}t)\,\mathrm{Re}(v)-\sin\left(\tfrac{2\pi}{\omega}t\right)\mathrm{Im}(v)\right]\\
&-\eta_1\sin\left(\tfrac{2\pi}{\omega}\theta\right)\left[\sin(\tfrac{2\pi}{\omega}t)\,\mathrm{Re}(v)+\cos\left(\tfrac{2\pi}{\omega}t\right)\mathrm{Im}(v)\right]\\
&+\eta_2\cos\left(\tfrac{2\pi}{\omega}\theta\right)\left[\sin(\tfrac{2\pi}{\omega}t)\,\mathrm{Re}(v)+\cos\left(\tfrac{2\pi}{\omega}t\right)\mathrm{Im}(v)\right]\\
&+\eta_2\sin\left(\tfrac{2\pi}{\omega}\theta\right)\left[\cos(\tfrac{2\pi}{\omega}t)\,\mathrm{Re}(v)-\sin\left(\tfrac{2\pi}{\omega}t\right)\mathrm{Im}(v)\right]\\
=\;&\left[\eta_1\cos\left(\tfrac{2\pi}{\omega}\theta\right)+\eta_2\sin\left(\tfrac{2\pi}{\omega}\theta\right)\right]\varepsilon_1(t)+\left[-\eta_1\sin\left(\tfrac{2\pi}{\omega}\theta\right)+\eta_2\cos\left(\tfrac{2\pi}{\omega}\theta\right)\right]\varepsilon_2(t).
\end{aligned}$$

Next, we look for conditions guaranteeing the following equality

$$\rho_i(\eta_1\varepsilon_1(t)+\eta_2\varepsilon_2(t))=\eta_1\varepsilon_1(t+\theta)+\eta_2\varepsilon_2(t+\theta), \tag{9}$$

to be satisfied. For the case of $i=1$, we must have

$$\eta_1\cos\left(\frac{2j\pi}{3}\right)-\eta_2\sin\left(\frac{2j\pi}{3}\right)=\eta_1\cos\left(\frac{2\pi}{\omega}\theta\right)+\eta_2\sin\left(\frac{2\pi}{\omega}\theta\right)$$

and

$$\eta_1\sin\left(\frac{2j\pi}{3}\right)+\eta_2\cos\left(\frac{2j\pi}{3}\right)=-\eta_1\sin\left(\frac{2\pi}{\omega}\theta\right)+\eta_2\cos\left(\frac{2\pi}{\omega}\theta\right).$$

Thus, (9) with $i=1$ holds if and only if

$$\begin{aligned}
&\theta=\tfrac{3-j}{3}\omega,\;\; \eta_1,\eta_2\in\mathbb{R},\;\text{or}\\
&\theta\neq\tfrac{3-j}{3}\omega,\; \eta_1=\eta_2=0.
\end{aligned}$$

For the case of $i=2$, we must have, for v_j^+,

$$\eta_1 \cos\left(\frac{2\pi}{\omega}\theta\right) + \eta_2 \sin\left(\frac{2\pi}{\omega}\theta\right) = \eta_1, \quad -\eta_1 \sin\left(\frac{2\pi}{\omega}\theta\right) + \eta_2 \cos\left(\frac{2\pi}{\omega}\theta\right) = \eta_2$$

and, for v_j^-,

$$x_1 \cos\left(\frac{2\pi}{\omega}\theta\right) + x_2 \sin\left(\frac{2\pi}{\omega}\theta\right) = -\eta_1, \quad -x_1 \sin\left(\frac{2\pi}{\omega}\theta\right) + x_2 \cos\left(\frac{2\pi}{\omega}\theta\right) = -\eta_2.$$

So, (9) with $i=2$ holds if and only if

$$\begin{aligned} &\theta = m\omega \ \text{ for } v_j^+, \text{ or} \\ &\theta = \left(m+\tfrac{1}{2}\right)\omega \ \text{ for } v_j^-, \text{ or} \\ &\eta_1 = \eta_2 = 0, \ \text{ otherwise.} \end{aligned}$$

where $m \in \mathbb{Z}$. This completes the proof. □

Lemmas 1-5 allow us to apply the symmetric Hopf bifurcation theorem for delay differential equations due to Wu [21] to obtain the following results.

csoarticle 1. *Assume that $c>0$, τ_k^+, τ_k^- and τ_k are defined by (2), (3) and (4), respectively, and p is the corresponding period of the bifurcating periodic solution.*

(i) The bifurcating periodic solutions of system (1), occurring at the critical values τ_k^+ with k being odd, τ_k^- with k being even and $c<\sqrt{1+b+b^2}$, or τ_k^- with k being odd and $c>\sqrt{1+b+b^2}$, or τ_k with k being odd, satisfy

$$x_{i-1}(t) = x_i\left(t - \frac{j}{3}p\right), \; y_{i-1}(t) = y_i\left(t - \frac{j}{3}p\right), \tag{10}$$

and

$$x_i(t) = y_i(t), \tag{11}$$

where $i=1,2,3 \pmod 3$, τ_k corresponds to $j=0$, τ_k^+ and τ_k^- correspond to $j=1$.

(ii) The bifurcating periodic solutions of system (1), occurring at the critical values τ_k^+ with k being even, τ_k^- with k being odd and $c<\sqrt{1+b+b^2}$, or τ_k^- with k being even and $c>\sqrt{1+b+b^2}$, or τ_k with k being even, satisfy (10)

$$x_i(t) = y_i\left(t + \frac{1}{2}p\right), \quad i = 1,2,3 \pmod 3, \tag{12}$$

csoarticle 2. *Assume that $c<0$, τ_k^+, τ_k^- and τ_k are defined by (2), (3) and (4), respectively, and p is the corresponding period of the bifurcating periodic solution.*

(i) The bifurcating periodic solutions of system (1), occurring at the critical values τ_k^+ with k being even, τ_k^- with k being odd and $c>-\sqrt{1+b+b^2}$, or τ_k^- with k being even and $c<-\sqrt{1+b+b^2}$, or τ_k with k being even, satisfy (10) and (11);

(ii) The bifurcating periodic solutions of system (1), occurring at the critical values τ_k^+ with k being odd, τ_k^- with k being even and $c > -\sqrt{1+b+b^2}$, or τ_k^- with k being odd and $c < -\sqrt{1+b+b^2}$, or τ_k with k being odd, satisfy (10) and (12).

Theorems 1 and 2 have shown that there are different spatio-temporal patterns for three types of critical values of the coupling time delay. According to (2), (3),(4), and Lemma 1, the distribution of critical values of the coupling time delay can be plotted in Fig.1, which is very helpful for us to understand the above results. Regions D_{i1} and $D_{i2}, i = 1, 2, 3,$ are bounded by four solid straight lines. In each region, the in-phase and anti-phase oscillations between two loops coexist, but the spatio-temporal patterns within each loop are different. In regions D_{11} and D_{12}, the oscillations within each loop are phase-locked with one third period. In regions D_{21} and D_{22}, within each loop the phase-locked oscillations with one third period and synchronized oscillations coexist. In regions D_{31} and D_{32}, the oscillations within each loop are synchronous. In each three-neuron loop, there are discrete travelling and the two loops are either synchronized or half a period out of phase with each other. Theorems 1 and 2 also have show that the coupling time delay does not affect the spatio-temporal patterns of the individual neural loop but it has the significant impact on the spatio-temporal patterns between the two loops. For instance, as the increasing of the coupling time delay the oscillation patterns between the two loops can be from anti-phase motion to in-phase motion in the region D_{31} but vice versa in the region D_{31}.

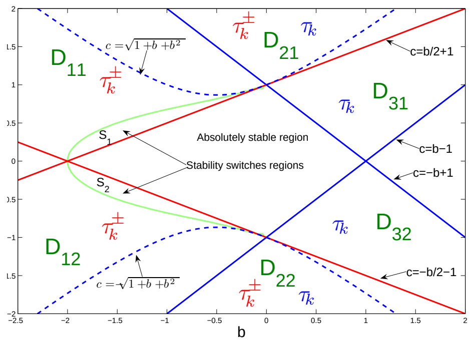

图 1: Distribution of critical values of the coupling time delay for Hopf bifurcations and regions of different spatio-temporal patterns. Regions D_{i1} and $D_{i2}, i = 1, 2, 3,$ are bounded by four solid straight lines. In each region, the in-phase and anti-phase oscillations between two loops coexist, but the spatio-temporal patterns within each loop are different. In regions D_{11} and D_{12}, the oscillations within each loop are phase-locked with one third period. In regions D_{21} and D_{22}, within each loop the phase-locked oscillations with one third period and synchronized oscillations coexist. In regions D_{31} and D_{32}, the oscillations within each loop are synchronous.

2 Conclusions

The synchronization of coupled networks can be either in-phase state or anti-phase state [24]. Recently, the phase-flip bifurcation has been considered as a general and important property of time-delay coupled systems. In this paper, we analytically investigate the influence of the coupling delay and strength on such phenomenon. We found that there are different synchronized in-phase and anti-phase oscillation in the plane of the coupling strength and the gain of the inherent response function. A remarkable finding is that the spatio-temporal patterns between the two loops depend not only on the parity of the critical value k of the coupling time delay, but also on the parameter region where the bifurcation occurs, while the spatio-temporal patterns within each loop only depend on the parameter region where the bifurcation occurs and are independent of the parity of critical values(see Theorems 1 and 2 and Fig.1). For each neural loop, there are two types of the spatio-temporal patterns: one is phase-locked with one third period (in regions D_{11}, D_{12}, D_{21}, D_{22}) and the other is synchronous (in regions D_{21}, D_{22}, D_{31} and D_{32}). For the coupled neural loops, in each regions D_{i1} and $D_{i2}, i = 1, 2, 3$, there are also two types of the spatio-temporal patterns: one is in-phase and the other is anti-phase.

参考文献（References）

[1] C. M. Marcus, R. M. Westervelt, Stability of analog neural networks with delay, Phys. Rev. A 39 (1989) 347-359.

[2] J. Bélair, Stability in a model of a delayed neural network, J. Dynamics and Differential Equations 5 (1993) 607-623.

[3] S.A. Campbell, I. Ncube, J. Wu, Multistability and stable asynchronous periodic oscillations in a multiple-delayed neural system, Physica D 214 (2006) 101-119.

[4] S.A. Campbell, Y. Yuan, S.D. Bungay, Equivariant Hopf bifurcation in a ring of identical cells with delayed coupling, Nonlinearlity 18 (2005) 2827-2846.

[5] D. Fan and J. Wei, Hopf bifurcation analysis in a tri-neuron network with time delay, Nonlinear Anal. Real World Appl. 9 (2008) 9-25.

[6] S. Guo, L. Huang, Hopf Bifurcating Periodic Orbits in a Ring of Neurons with Delays, Physica D 183 (2003) 19-44.

[7] L. Huang, J. Wu, Nonlinear waves in networks of neurons with delayed feedback: Pattern formation and continuation, SIAM J. Math. Anal. 34(4) (2003) 836-860.

[8] Y. Song, M. Han, J. Wei, Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays, Physica D, 200 (2005), 185-204.

[9] J. Wei, S. Ruan, Stability and bifurcation in a neural network model with two delays, Physica D 130 (1999) 255-272.

[10] J. Wu, T. Faria, Y. S. Huang, Synchronization and stable phase-locking in a network of neurons with memory, Math. Comput. Model. 30 (1999) 117-138.

[11] Y. Yuan, Dynamics in a delayed-neural network, Chaos Solitons Fractals, 33 (2007) 443-454.

[12] Y. Yuan, S.A. Campbell, Stability and synchronization of a ring of identical cells with delayed coupling, J. Dyn. Differ. Equ. 16 (2004) 709-744.

[13] S. Boccaletti, L. M. Pecora, A. Pelaez, A unifying framework for synchronization of coupled dynamical systems, Phys. Rev E 63 (2001) 066219.

[14] E. R. Kandel, J. H. Schwartz, T. M. Jessell, Principles of neural science, McGraw-Hill, New York, 2000.

[15] S.A. Campbell, R. Edwards, P. Van den Driessche, Delayed coupling between two neural network loops, SIAM J. Appl. Math. 65(1) (2004) 316-335.

[16] C. H. Hsu, T. S. Yang, Periodic oscillations arising and death in delay-coupled neural loops, Internat. J. Bifur. Chaos, 17 (2007) 4015-4032.

[17] Y. Peng, Y. Song, Stability switches and Hopf bifurcations in a pair of identical tri-neuron network loops, Physics Letters A 373 (2009) 1744.

[18] S. Yuan, P. Li, Stability and direction of Hopf bifurcations in a pair of identical tri-neuron network loops, Nonlinear Dynamics, 61 (2010) 569-578.

[19] A. Prasad, S. K. Dana, R. Karnatak, J. Kurths, B. Blasius, and R. Ramaswamy, Universal occurrence of the phase-flip bifurcation in time-delay coupled systems Chaos,18 (2008) 023111.

[20] A. Prasad, J. Kurths, S. K. Dana, R. Ramaswamy, Phase-flip bifurcation induced by time delay, Phys. Rev. E 74 (2006) 0352041.

[21] J. Wu, Symmetric functional-differential equations and neural networks with memory, Trans. Amer. Math. Soc. 350 (1998) 4799-4838.

[22] M. Golubitsky, I. Stewart, D. Schaeffer, Singularities and Groups in Bifurcation Theory, Singularities and Groups in Bifurcation Theory, Volume II, Springer-Verlag, New York, 1988.

[23] J. K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.

[24] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization, A Universal Concept in Nonlinear Science, Cambridge University Press, Cambridge, 2001.