On the existence of a unique periodic solution to a second-order p-Laplacian equation with deviating argument

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Abstract
In this paper, we study a kind of Lienard type p-Laplacian equation with deviating arguments as follows $(\varphi_p(x'(t)))' + f(x(t))'x(t) + g(t, x(t-\tau(t,|x|_\infty))) = e(t)$. A new result on the existence and uniqueness of periodic solutions of this equation is given by Mawhin continuation theorem and some new techniques.
Keywords: Periodic solutions; Existence and uniqueness; Mawhin continuation theory; p-Laplacian.

1 Introduction

In this present paper, we study the existence and uniqueness of periodic solutions of the following lienard type p-Laplacian non-autonomous equation with deviating arguments

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(t, x(t-\tau(t,|x|_\infty))) = e(t) \qquad (1.1)$$

or an equivalent system:

$$\begin{cases} x'(t) = |y(t)-\psi(x(t))|^{q-1}\operatorname{sgn}(y(t)-\psi(x(t))), \\ y'(t) = -g(t, x(t-\tau(t,|x|_\infty))) + e(t), \end{cases} \qquad (1.2)$$

Where $p,q>1, 1/p+1/q=1; \varphi_p : R \to R, \varphi_p(s) = |s|^{p-2}s$ is a one-dimensional p-Laplacian;

$\psi(x) = \int_0^x f(u)du, y(t) = \varphi_p(x'(t))\psi(x(t)); f, e \in C(R,R), g \in C(R^2,R), g(t,x), e(t)$ are T-periodic functions with respect to $t, T>0$.

As we know, Lienard equation appears in a number of physical modles and is always used to describe fluid mechanical and nonlinear elastic mechanical phenomena [1～5].

For example, In [1], Huang and Xiang studied the following type of Duffing equation with a single constant deviating argument

$$x''(t) + g(x(t-\tau)) = p(t). \qquad (1.3)$$

In [2], Ma studied a kind of delay Duffing equation of type

$$x''(t) + m^2 x(t) + g(x(t-\tau)) = p(t). \qquad (1.4)$$

In[3], Shipeng Lu and Weigao Ge considered Periodic solutions of a kind Lienard equation with a deviating argument

$$x''(t) + f(x(t))x'(t) + g(x(t-\tau(t))) = e(t). \qquad (1.5)$$

On the other hand, some other p-Laplacian equation also received much attention lately. In [6], Yong Wang, Xianzhi Dai and Xiaoxu Xia studied Lienard type p-Laplacian non-autonomous equation as follows

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(t, x(t)) = e(t). \qquad (1.6)$$

As far as we know, there exist much fewer results on the existence and unique of periodic solution of (1.1) The main difficulty lies in the first term $(\varphi_p(x'(t)))'$ of (1.1) (the p-Laplacian operator $\varphi_p : R \to R, \varphi_p(s) = |s|^{p-2} s$ is nonlinear when $p \neq 2$). For example,

In [7], Cheung and Ren first studied the following p-Laplacian Rayleigh delay equation

$$(\varphi_p(x'(t)))' + f(x(t)) + \beta g(x(t-\tau(t))) = e(t). \tag{1.7}$$

In [8], Peng and Zhu considered the Reyleigh type p-Laplacian delay equation

$$(\varphi_p(x'(t)))' + f(x'(t)) + g(x(t-\tau(t))) = e(t). \tag{1.8}$$

The main purpose in this work is to give some sufficient conditions for securing the existence of a unique T-periodic solution to (1.1) by using Mawhin continuation theorem[10,11,12,13,14,15] and some new techniques. Our results are new and extend some preciously known results.

2 Main Lemmas

Set $C_T^1 = \{x \in C^1(R,R), x(t+T) = x(t)\}$, Which is a Banach space endowed with the norm $\|\bullet\|$ defined by $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$,and

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, |x|_k = (\int_0^T |x(t)|^k)^{1/k}.$$

Let

$$g_t(t, x(t-\tau(t,|x|_\infty))) = \frac{\partial g(t,x)}{\partial t}, g_x(t, x(t-\tau(t,|x|_\infty))) = \frac{\partial g(t,x)}{\partial x}, \bar{e} = \frac{1}{T}\int_0^T e(t)dt .$$

The following conditions will be used later:

(H1) there exists a constant $d \geq 0$ such that $x(g(t,x) - \bar{e}) < 0$ for all $|x| > d$ and $t \in R$.

(H2) $g \in C^1(R^2, R), \tau(t,|x|_\infty) \equiv \tau, \tau \in R,$ and $g_x(t,x) < 0,$ for all $t, x \in R$.

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = h(t, x, x'), x(0) = x(T), x'(0) = x'(T), \tag{2.1}$$

where $h \in C(R^3, R)$ is T-periodic in the first variable. The following continuation theorem can be induced directly from the theory in [16], and is citied as Lemma 1 in [19].

Lemma 1.(*Mawhin [16]*). Let $B = \{x \in C_T^1 : \|x\| < r\}$ for some $r > 0$. Suppose the following two conditions hold:

(i) For each $\lambda \in (0,1)$ the problem $(\varphi_p(x'(t)))' = \lambda h(t,x,x')$ has no solution on ∂B .

(ii)The continuous function F defined on R by $F(a) = \frac{1}{T}\int_0^T h(t,a,0)dt$ is such that

$F(-r)F(r) < 0$.

Then the periodic boundary value problem (2.1) has at least one T-periodic solution on $\bar{B}$

Consider the homotopic equation of (1.1):

$$(\varphi_p(x'(t)))' + \lambda f(x(t))'x(t) + \lambda g(t, x(t-\tau(t,|x|_\infty))) = \lambda e(t), \lambda \in (0,1) \tag{2.2}$$

We have the following lemma,

Lemma 2. Suppose (H1) holds. Then the set of T-periodic solutions of (2.2) are bounded in C_T^1 .

Proof. Let $S \subset C_T^1$ be the set of T-periodic solutions of (2.2). If $S = \varnothing$, the proof is ended.

Suppose $S \neq \varnothing$, and let $x \in S$. Noticing that $x(0)=x(T), x'(0)=x'(T)$ and $\varphi_p(0)=0$, it follows from (2.2) that

$$\int_0^T g(t, x(t-\tau(t,|x|_\infty)))dt = T\overline{e} .$$

Which implies that there exists $t_0 \in [0,T]$ such that

$$g(t_0, x(t_0-\tau(t_0,|x|_\infty))) = \overline{e}$$

By (H1), we have

$$|x(t_0-\tau(t_0,|x|_\infty))| \le d \qquad (2.3)$$

Let

$$t_0-\tau(t_0,|x|_\infty) = kT+\xi$$

where k is an integer and $\xi \in [0,T]$.

Then, for any $t \in [t_0, t_0+T]$

$$|x(t)| = \left|x(\xi) + \int_\xi^t x'(s)ds\right|$$

$$\le d + \int_0^T |x'(s)|ds$$

which leads to

$$|x|_\infty \le d + |x'|_1 \qquad (2.4)$$

Define $E_1 = \{t : t \in [0,T], |x(t)| > d\}, E_2 = \{t : t \in [0,T], |x(t)| \le d\}$. Multiplying two sides of (2.2) with $x(t)$ and integrating from 0 to T, by (H1) we have

$$\int_0^T |x'(t)|^p dt = -\int_0^T (\varphi_p(x'(t)))'x(t)dt$$

$$= \lambda\int_0^T (g(t,x(T))-\overline{e})x(t)dt - \lambda\int_0^T (e(t)-\overline{e})x(t)dt$$

$$= \lambda\int_{E_1} (g(t,x(T))-\overline{e})x(t)dt + \lambda\int_{E_2} (g(t,x(T))-\overline{e})x(t)dt$$

$$-\lambda\int_0^T (e(t)-\overline{e})x(t)dt$$

$$\le \lambda\int_{E_2} (g(t,x(T))-\overline{e})x(t)dt - \lambda\int_0^T (e(t)-\overline{e})x(t)dt$$

$$\le (\max_{t\in[0,T],|x|<d} (g(t,x)-\overline{e}) + |e-\overline{e}|_\infty)T|x|_\infty$$

Let $M_0 = (\max_{t\in[0,T],|x|<d} (g(t,x)-\overline{e}) + |e-\overline{e}|_\infty)T$. Then we obtain

$$|x'|_p \le M_0^{1/p}|x|_\infty^{1/p}. \qquad (2.5)$$

Let $q>1$ such that $1/p+1/q=1$. Then by the holder inequality we have

$$|x'|_1 \le |x'|_p |1|_q = T^{1/q}|x'|_p \qquad (2.6)$$

By (2.4),(2.5) and (2.6), we can get

$$|x'|_1 \le T^{1/q}M_0^{1/p}(d+|x'|_1)^{1/p}.$$

Which yields that there exists $M_1>0$ such that $|x'|_1 \le M_1$ since $p>1$, and this together with (2.4) implies that

$$|x|_\infty \le d+M_1. \qquad (2.7)$$

Meanwhile, there exists $\bar{t}_0 \in [0,T]$ such that $x'(\bar{t}_0)=0$ since $x(0)=x(T)$. Then by (2.2) we have, for $t\in[\bar{t}_0,\bar{t}_0+T]$,

$$|\varphi_p(x'(t))| = \left|\int_{\bar{t}_0}^{t}(\varphi_p(x'(s)))'ds\right|$$

$$=\lambda\left|\int_{\bar{t}_0}^{t}(f(x(s))x'(s)+g(s,x(s-\tau(s,|x|_\infty)))+e(s))ds\right|$$

$$\le \int_0^T(|f(x(s))||x'(s)|+|g(s,x(s-\tau(s,|x|_\infty)))|+|e(s)|)ds$$

$$\le FM_1+(G+|e|_\infty)T$$

where $F=\max\{|f(x)|:|x|\le d+M_1\}, G=\max\{|g(t,x)|:t\in[0,T],|x|\le d+M_1\}$. So we obtain

$$|x'|_\infty = \max_{t\in[0,T]}\left\{|\varphi_p(x'(t))|^{1/(p-1)}\right\} \le (FM_1+(G+|e|_\infty)T)^{1/(p-1)}.$$

Let $M=\max\{d+M_1,(FM_1+(G+|e|_\infty)T)^{1/(p-1)}\}$. Then $\|x\|\le M$. This completes the proof.

Lemma 3. Suppose (H2) holds, if. Then (1.1) has at most one T-periodic solutions.

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1). Then, from (1.2), we obtain

$$\begin{cases} x_i'(t)=|y_i(t)-\psi(x_i(t))|^{q-1}\operatorname{sgn}(y_i(t)-\psi(x_i(t))), \\ y_i'(t)=-g(t,x_i(t-\tau))+e(t), i=1,2. \end{cases} \qquad (2.8)$$

Setting

$$v(t)=x_1(t)-x_2(t), \quad u(t)=y_1(t)-y_2(t). \qquad (2.9)$$

It follows from (2.8) that

$$\begin{cases} v'(t)=|y_1(t)-\psi(x_1(t))|^{q-1}\operatorname{sgn}(y_1(t)-\psi(x_1(t)))-|y_2(t)-\psi(x_2(t))|^{q-1}\operatorname{sgn}(y_2(t)-\psi(x_2(t))), \\ u'(t)=-[g(t,x_1(t-\tau))-g(t,x_2(t-\tau)]. \end{cases} \qquad (2.10)$$

Now, we prove that

$u(t)\le 0$ for all $t\in R$.

Contrarily, in view of $u\in C^2[0,T]$ and $u(t)=u(t+T)$ for all $t\in R$, we obtain

$$\max_{t\in R} u(t)>0.$$

Then, there exists $t^*\in R$ (for convenience, we can choose $t^*\in[0,T]$) such that

$$u(t^*)=\max_{t\in[0,T]}u(t)=\max_{t\in R}u(t)>0,$$

which implies that

$$u'(t^*) = -[g(t^*, x_1(t^* - \tau)) - g(t, x_2(t^* - \tau))] = 0 \quad (2.11)$$

$$u''(t^*) = -[g_t(t, x_1(t - \tau)) + g_x(t, x_1(t-))x_1'(t)]\Big|_{t=t^*}$$

$$+[g_t(t, x_2(t - \tau)) + g_x(t, x_2(t - \tau))x_2'(t)]\Big|_{t=t^*}$$

$$= -[g_t(t^*, x_1(t^* - \tau)) + g_x(t^*, x_1(t^* - \tau))x_1'(t^*)]$$

$$+[g_t(t, x_2(t^* - \tau)) + g_x(t^*, x_2(t^* - \tau))x_2'(t^*)]$$

$$\le 0 \quad (2.12)$$

Since $g_x(t, x) < 0$ for all $t \in R$, from (2.10) and (2.11), we get

$$x_1(t^* - \tau) = x_2(t^* - \tau)$$

$$g(t^*, x_1(t^* - \tau)) = g(t^*, x_2(t^* - \tau))$$

and

$$u''(t^*) = -g_x(t^*, x_1(t^* - \tau))[x_1'(t^*) - x_2'(t^*)]$$

$$= -g_x(t^*, \tau)[|y_1(t) - \psi(x_1(t))|^{q-1} \operatorname{sgn}(y_1(t) - \psi(x_1(t)))$$

$$-|y_2(t) - \psi(x_2(t))|^{q-1} \operatorname{sgn}(y_2(t) - \psi(x_2(t)))]$$

$$= -g_x(t^*, \tau)[|y_1(t) - \psi(x_1(t))|^{q-1} \operatorname{sgn}(y_1(t) - \psi(x_1(t)))$$

$$-|y_2(t) - \psi(x_1(t))|^{q-1} \operatorname{sgn}(y_2(t) - \psi(x_1(t)))] \quad (2.13)$$

In view of

$$-g_x(t^*, \tau) > 0, u(t^*) = y_1(t^*) - y_2(t^*) > 0 \quad (2.14)$$

It follows from (2.13) that

$$u''(t^*) = -g_x(t^*, \tau)[|y_1(t) - \psi(x_1(t))|^{q-1} \operatorname{sgn}(y_1(t) - \psi(x_1(t)))$$

$$-|y_2(t) - \psi(x_1(t))|^{q-1} \operatorname{sgn}(y_2(t) - \psi(x_1(t)))]$$

$$> 0, \quad (2.15)$$

which contradicts (2.12). This contradiction implies that

$u(t) = y_1(t) - y_2(t) \le 0$ fo r all $t \in R$

By using a similar argument, we can also show that

$y_2(t) - y_1(t) \le 0$ fo r all $t \in R$

Therefore, we obtain

$y_1(t) \equiv y_2(t)$ fo r all $t \in R$

Then, from (2.10), we get

$$g(t, x_1(t - \tau)) = g(t, x_2(t - \tau))$$

again from $g_x(t, x) < 0$, w hich implies that

$x_1(t) \equiv x_2(t)$ fo r all $t \in R$

Hence

$$(\varphi_p(x'(t)))' + f(x(t))'x(t) + g(t, x(t-\tau)) = e(t)$$

has at most one T-periodic solution. The proof of lemma 3 is now completed.

3 Main Results

Now we are in the position to give our main results.

Theorem 1. Suppose (H1) hold. Then (1.1) has at least a T-periodic solution.

Proof. Set

$$h(t, x(t), x(t-\tau(t,|x|_\infty)), x'(t)) = -f(x(t))'x(t) - g(t, x(t-\tau(t,|x|_\infty))) + e(t). \quad (3.1)$$

then (2.2) is equivalent to the following equation

$$(\varphi_p(x'(t)))' = h(t, x(t), x(t-\tau(t,|x|_\infty)), x'(t)) \quad (3.2)$$

By Lemma 2, there exists a constant $r > d$ such that, for any T-periodic solution $x(t)$ of (3.2)

$$\|x\| < r \quad (3.3)$$

Set

$$B = \{x : x \in C_T^1, \|x\| < r\} \quad (3.4)$$

By (3.1), we know that (3.2) has no solution on ∂B as $\lambda \in (0,1)$, so condition (i) of Lemma 1 is satisfied. By the definition of F in Lemma 1 we get

$$F(a) = \frac{1}{T}\int_0^T h(t, a, 0)dt = \frac{1}{T}\int_0^T [e(t) - g(t, a)]dt = \frac{1}{T}\int_0^T [\overline{e} - g(t, a)]dt$$

This together with (H1) yields that $F(r)F(-r) < 0$, condition (ii) of Lemma 1 is satisfied. Therefore, it follows from Lemma that there exists a T-periodic solution $x(t)$ of (1.1)

Theorem 2. Suppose (H1) and (H2) hold. Then

$$(\varphi_p(x'(t)))' + f(x(t))'x(t) + g(t, x(t-\tau)) = e(t) \quad (3.5)$$

has a unique T-periodic solution.

Proof. If (H1) and (H2) hold, it follows from Lemm3 and Theorem1 that (3.5) exists a unique T-periodic solution.

4 Example

Example1. Let $p = \sqrt{2}, g(t, x) = -1/(100 + \cos^2 t)|x|^{p-2} x$ for all $t \in R, x > 0$ and $g(t, x) = -x^{32}(x-1)$ for all $t \in R, x \le 0$ .Then, the following Lienard typep-Laplacian equation with a deviating argument

$$(\varphi_p(x'(t)))' + x^4(t)x'(t) + g(t, x(t - |\cos(t)|)) = \cos t$$

Has at least one 2π -periodic solution.

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The direction of ordinary differential equations