Periodic solutions for Duffing type p-Laplacian equations

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Abstract

In this paper, we study periodic solutions for a class of Duffing type p-Laplacian equations. By using the Manásevich-Mawhin continuation theorem, some new results on the existence of periodic solutions are obtained.

Keywords: Periodic solution; Duffing type p-Laplacian euqations; Manásevich-Mawhin continuation theorem

Mathematics Subject Classification: 34C25, 54H25

1 Introduction

In recent years, many works have focused on the investigation of existence and uniqueness of periodic solutions for Duffing equations, see, for instance, [3, 4, 5, 7, 8, 9, 10] and references therein.

In [1], Zhang and Li considered the one-dimensional Duffing type p-Laplacian equation

$$(\varphi_p(x'(t)))' + Cx'(t) + g(t, x(t)) = e(t), \tag{1}$$

where $t, x \in \mathbb{R}$, $p > 1$, $\varphi_p : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, C is a constant, $g(t, x)$ is continuous and $g(t, \cdot) = g(t + T, \cdot)$, $e(t)$ is a continuous function, $e(t) = e(t + T)$, $\int_0^T e(t)dt = 0$. They showed that if

(A1) $(g(t, u_1) - g(t, u_2))(u_1 - u_2) < 0$ for $u_1 \neq u_2$, $t \in \mathbb{R}$;

(A2) $xg(t, x) < 0$ for $|x| > 0$, $t \in \mathbb{R}$;

(A3) There exist constants $K > 0$ and $M > 0$, such that

$$2^{2-p}MT^p < 1, \quad g(t, x) \geqslant -M|X|^{p-1} - K, \quad \text{for} \quad x \geqslant 0, \quad t \in \mathbb{R},$$

*Supported by Program for New Century Excellent Talents in University, NSFC grant (10771083), SRFDP grant (20040183030).

hold, then equation (1) has a unique periodic solution.

Then Tang and Li[2] improved the above result. Under the assumptions (A1) and

(A2*) There exists a constant $d \geqslant 0$ such that $xg(t,x) < 0$ for $|x| > d$, $t \in \mathbb{R}$, they obtained the existence and uniqueness of periodic solution for equation (1).

However, these previously known results are just about one-dimensional case, and they exclude the cases $xg(t,x) \geqslant 0$ and $\int_0^T e(t)dt \neq 0$. The main purpose of this paper is to present a existence result of periodic solutions for Duffing type p-Laplacian equations.

We consider the following Duffing type p-Laplacian equations

$$(\varphi_p(x'(t)))' + \frac{d}{dt}\nabla F(x(t)) + g(t,x(t)) = e(t), \tag{2}$$

where $p > 1$ and $\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, $F : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function, $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with $g(t,\cdot) = g(t+T,\cdot)$, and $e : \mathbb{R} \to \mathbb{R}^n$ is continuous with $e(t) = e(t+T)$.

Let $C_T^1 := \{x \in C^1(\mathbb{R}^n) : x(0) = x(T), x'(0) = x'(T)\}$. For $x \in C_T^1$, define

$$\|x\| = |x|_\infty + |x'|_\infty,$$

where

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|.$$

Then C_T^1 is a Banach space.

Throughout this paper, we denote $B_r = \{x \in C_T^1 : \|x\| < r\}$ and we make the following assumptions:

(H1) There exist constants $d \geqslant 0$, $M \geqslant 0$ with $M(T/2)^p < 1$, such that for $|x| > d$,

$$\langle x, g(t,x(t))\rangle \leqslant M|x|^p \tag{3}$$

and

$$g(t,x(t)) - e(t) \neq 0; \tag{4}$$

(H2) There exists a consequence $\{r_i\}_{i=1}^{\infty}$, $r_i \in \mathbb{R}_+$, $r_i \to +\infty$, such that the Brouwer degree

$$deg(G, B_{r_i} \cap \mathbb{R}^n, 0) \neq 0,$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$G(a) = \frac{1}{T}\int_0^T (e(t) - g(t,a))dt.$$

We have the following result.

Theorem 1.1. Assume that (H1), (H2) hold. Then equation (2) has at least one T-periodic solution.

Remark 1.1. Theorem 1 can be regarded as the improvement of the results in [1] and [2]. In fact, when $n = 1$, $\nabla F(x) = Cx$, $\int_0^T e(t)dt = 0$, equations (2) is reduced to equation (1). Furthermore, under our assumptions, it is possible that $xg(t,x) \geqslant 0$. More precisely, $xg(t,x)$ can increase as $|x|^p$ increase.

Remark 1.2. It should be pointed out that the existence of T-periodic solution of (2) can not be ensured without (H2), by observing the equation

$$(\varphi_p(x'(t)))' + 1 = 0.$$

This paper is organized as follows. In section 2, we prove Theorem 1.1 by using Manásevich-Mawhin continuation theorem. Then in section 3, an example is given to illustrate our result.

2 Proof of the Theorem 1.1

To prove the Theorem 1.1, we first introduce the following lemma.

Lemma 2.1. (Manásevich-Mawhin [6]). Consider the equations

$$(\varphi_p(x'(t)))' = f(t,x,x'), \tag{5}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(t,\cdot,\cdot) = f(t+T,\cdot,\cdot)$. Assume that

(1) For each $\lambda \in (0,1)$ the equations

$$(\varphi_p(x'(t)))' = \lambda f(t,x,x')$$

has no T-periodic solution on ∂B_r.

(2) $G(a) = 0$ has no solution on $\partial B_r \cap \mathbb{R}^n$, where

$$G(a) := \frac{1}{T}\int_0^T f(t,a,0)dt.$$

(3) The Brouwer degree

$$deg(G, B_r \cap \mathbb{R}^n, 0) \neq 0.$$

Then equation (5) has at least one T-periodic solution in $\overline{B}_r$.

Proof of Theorem 1.1. Consider the following homotopy equation

$$(\varphi_p(x'(t)))' + \lambda\frac{d}{dt}\nabla F(x(t)) + \lambda g(t,x(t)) = \lambda e(t), \quad \lambda \in [0,1]. \tag{6}$$

We first prove that the set of all possible T-periodic solutions of equation (6) is a bounded subset of C_T^1.

Let $x(t) \in C_T^1$ be an arbitrary T-periodic solution of the equation (6). Since $x(0) = x(T)$, $x'(0) = x'(T)$, integrating equation (6) from 0 to T, we get

$$\int_0^T (g(t, x(t)) - e(t))dt = 0. \tag{7}$$

So there exists $\xi \in [0, T]$ such that

$$g(\xi, x(\xi)) - e(\xi) = 0.$$

By (4) we know that

$$|x(\xi)| \leqslant d.$$

Therefore

$$|x(t)| = |x(\xi) + \int_\xi^t x'(s)ds| \leqslant d + \int_\xi^t |x'(s)|ds, \qquad t \in [\xi, \xi + T],$$

$$|x(t)| = |x(t-T)| = |x(\xi) - \int_{t-T}^\xi x'(s)ds| \leqslant d + \int_{t-T}^\xi |x'(s)|ds, \quad t \in [\xi, \xi + T].$$

Consequently, we have

$$\begin{aligned} |x|_\infty &= \max_{t\in[0,T]} |x(t)| = \max_{t\in[\xi,\xi+T]} |x(t)| \\ &\leqslant \max_{t\in[\xi,\xi+T]} \{d + \frac{1}{2}(\int_\xi^t |x'(s)|ds + \int_{t-T}^\xi |x'(s)|ds)\} \\ &\leqslant d + \frac{1}{2}\int_0^T |x'(s)|ds. \end{aligned} \tag{8}$$

Let

$$E_1 = \{t : t \in [0, T], |x(t)| > d\}, \quad E_2 = \{t : t \in [0, T], |x(t)| \leqslant d\}.$$

Multiplying equation (6) by $x(t)$ and integrating from 0 to T, by (3), we get

$$\begin{aligned} &\int_0^T |x'(t)|^p dt \\ =& -\int_0^T \langle (\varphi_p(x'(t)))', x(t)\rangle dt \\ =& \lambda \int_0^T \langle \frac{d}{dt}\nabla F(x(t)), x(t)\rangle dt + \lambda \int_0^T \langle g(t, x(t)), x(t)\rangle dt - \lambda \int_0^T \langle e(t), x(t)\rangle dt \\ =& \lambda \int_0^T \langle g(t, x(t)), x(t)\rangle dt - \lambda \int_0^T \langle e(t), x(t)\rangle dt \end{aligned}$$

$$\begin{aligned} &= \lambda\int_{E_1}\langle g(t,x(t)),x(t)\rangle dt+\lambda\int_{E_2}\langle g(t,x(t)),x(t)\rangle dt-\lambda\int_0^T\langle e(t),x(t)\rangle dt \\ &\leqslant \int_0^T M|x|^p dt+\int_0^T \max_{t\in[0,T],|x|\leqslant d}|g(t,x(t))||x(t)|dt+\int_0^T|e(t)||x(t)|dt \\ &\leqslant MT|x|_\infty^p+DT|x|_\infty, \end{aligned} \tag{9}$$

where $D=\max\{|g(t,x(t))|,t\in[0,T],|x|\leqslant d\}+|e|_\infty$.

We claim that there exists a constant $M_1>0$, such that $|x|_\infty\leqslant M_1$. In fact, by (9), there exists a constant $M_*>M$ with $M_*(T/2)^p<1$ such that for large $|x|_\infty$,

$$\int_0^T|x'(t)|^p dt\leqslant M_*T|x|_\infty^p. \tag{10}$$

Hölder inequality follows

$$\int_0^T|x'(t)|dt\leqslant(\int_0^T|x'(t)|^p dt)^{\frac{1}{p}}(\int_0^T 1dt)^{\frac{p-1}{p}}=T^{\frac{p-1}{p}}(\int_0^T|x'(t)|^p dt)^{\frac{1}{p}}. \tag{11}$$

Therefore, by (8), (10), (11), we obtain

$$|x|_\infty\leqslant d+\frac{1}{2}T^{\frac{p-1}{p}}(\int_0^T|x'(t)|^p dt)^{\frac{1}{p}}\leqslant d+\frac{1}{2}TM_*^{\frac{1}{p}}|x|_\infty. \tag{12}$$

Since $M_*(T/2)^p<1$, (12) implies that

$$|x|_\infty\leqslant d(1-\frac{T}{2}M_*^{\frac{1}{p}})^{-1}.$$

Hence, there exists a constant M_1, such that

$$|x|_\infty\leqslant M_1. \tag{13}$$

Now we show there exists a constant $M_2>0$ such that $|x'|_\infty\leqslant M_2$. Since $x(0)=x(T)$, there exists $t_0\in[0,T]$ such that $x'(t_0)=0$. By $\varphi_p(0)=0$ we have

$$\begin{aligned} |x'|_\infty^{p-1} &= \max_{t\in[0,T]}|\varphi_p(x'(t))|=\max_{t\in[t_0,t_0+T]}|\int_{t_0}^t(\varphi_p(x'(s)))'ds| \\ &\leqslant |\int_{t_0}^t\frac{d}{ds}\nabla F(x(s))ds|+\int_{t_0}^t|g(s,x(s))|ds+\int_{t_0}^t|e(s)|ds \\ &\leqslant |\nabla F(x(t))-\nabla F(x(t_0))|+\int_0^T|g(t,x(t))|dt+\int_0^T|e(t)|dt \\ &\leqslant 2\max_{|x|\leqslant M_1}|\nabla F(x)|+T\max_{t\in[0,T],|x|\leqslant M_1}|g(t,x)|+T|e|_\infty. \end{aligned}$$

Thus, there exists M_2, such that

$$|x'|_\infty\leqslant M_2. \tag{14}$$

Combining (13) and (14), we get

$$\|x\| = |x|_\infty + |x'|_\infty \leqslant M_1 + M_2. \tag{15}$$

This means that the set of all possible T-periodic solutions of equation (6) is a bounded subset of C^1_T.

Define

$$G(a) = \frac{1}{T}\int_0^T (e(t) - g(t,a))dt.$$

Then from assumption (H2), there exists a constant $r > M_1 + M_2 + d + 1$, such that the Brouwer degree

$$deg(G, B_r \cap \mathbb{R}^n, 0) \neq 0.$$

By (15), the homotopy equation (6) has no T-periodic solution on ∂B_r. Furthermore, by (4), we know that $G(a) = 0$ has no solution on $\partial B_r \cap \mathbb{R}^n$. Hence, by the Manásevich-Mawhin theorem, equation (2) has at least one solution in $\overline{B_r}$. This completes the proof.□

3 An Example

Example 3.1. To illustrate our result, we consider the one-dimensional Duffing type p-Laplacian equation

$$(\varphi_4(x'(t)))' + 2x(t)x'(t) + C_0(2+\cos t)x^3 = \sin^2(t), \tag{16}$$

where constant C_0 satisfies $|C_0| < \dfrac{1}{3\pi^4}$.

Note that $\int_0^{2\pi}\sin^2(t) \neq 0$, and when $C_0 \geqslant 0$ we have $C_0(2+\cos t)x^4 \geqslant 0$. Furthermore, the second term of the left side is $2x(t)x'(t)$ but not $Cx'(t)$. Therefore the results in [1] or [2] are not applicable to (16).

Let $d = 1$. Then we can easily check that (H1) holds. Furthermore, for any $r_i > |C_0|^{-\frac{1}{3}}$, we have $G(r_i) < 0$, $G(-r_i) > 0$, so there exists a consequence $\{r_i\}_{i=1}^{\infty}$, $r_i \in \mathbb{R}_+$, $r_i \to +\infty$, such that the Brouwer degree

$$deg(G, B_{r_i} \cap \mathbb{R}^n, 0) \neq 0.$$

Thus (H2) holds. By Theorem 1.1, the equation has at least one 2π-periodic solution.

Acknowledgements

The authors are grateful to Professor Yong Li for useful discussions and suggestions.

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