Singular Nonlinear Boundary Value Problem in The Theory of Dilatant Non-Newtonian Fluids

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Abstract: A rigorous proof of existence and uniqueness of solutions to a singular nonlinear boundary value problem in the theory of Dilatant non-Newtonian fluids is given and a theoretical estimate formula for skin friction coefficient is presented, which is characterized by power law exponent. The reliability and efficiency of the analytical predictions are verified by the numerical results with good agree. The estimate formula can be successfully applied to give the value of skin friction coefficient.

Keywords: Dilatant non-Newtonian fluids, similarity solution, nonlinear boundary value problem.
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1. INTRODUCTION

The drag force due to "skin friction" is a fluid dynamic resistive force, which is a consequence of the fluid viscous and the pressure distribution on the surface of the object [1]. In view of the theoretical analysis of the boundary layer flow, Callegari and Friedman[2] developed an analytical solution to the classical Blasius boundary layer problem. Callegari and Nachman[3] established uniqueness and analyticity results for boundary conditions corresponding to flow behind weak expansion and shock waves and for the flow above a moving conveyor belt. Vajravelu et al([4]-[5]) studied the Blasius problem when the plate moves in the direction or opposite to the direction to that of the main stream.

Recently, a considerable attention has been devoted to the problem of how to predict the drag force behavior of non-Newtonian fluids. The main reason for this is probably that fluids (such as molten plastics, pulps, slurries, emulsions, etc.), which do not obey the Newtonian postulate that the tress tensor is directly proportional to the deformation tensor, are produced industrially in increasing quantities and are therefore in some cases just as likely to be pumped in a plant as the more common Newtonian fluids. Understanding the nature of this force by mathematical modeling with a view to predict the drag forces and the associated behavior of fluid flow has been the focus of research.

The theoretical analysis of an external boundary layer flow of a non-Newtonian fluid was first performed by Schowalter[6] and Acrivos et al[7]. The boundary layer was formulated, and the condition for the existence of similarity solution was established in [6]. A similarity solution to the boundary layer equations for a power fluid flowing along a flat plate at zero degree of angle of attack was obtained in [7]. Later, Nachman and Callegari [8] discussed the nonlinear singular boundary value problem in the theory of pseudoplastic fluids when the Crocco variable was introduced, existence, uniqueness and analyticity of solution to the problem are established. Howell [9] and Rao [10] examined momentum and heat transfer on a continuous moving surface in power law fluid. Recently, Akcay[11], Zheng et al [12-13], Lu and Zheng [14] studied a flat plate aligned with a uniform power law flow and continuous moving at constant speed in the direction or opposite to the direction to the mainstream in power fluid non-Newtonian, the existence, uniqueness or non-uniqueness of solutions to the problem are established by shooting technique.

All of the above-mentioned problems have had attention paid to Newtonian fluid or pseudoplastic non-Newtonian fluid, so that important question touching on well-posedness, i.e., Dilatant non-Newtonian fluid remain unanswered.

This paper makes a theoretical analysis for the boundary layer flow in Dilatant non-Newtonian fluid medium. A special emphasis is given to the formulation of boundary layer equations, which provide similarity solutions. A rigorous proof of existence and uniqueness of similarity solution to the problem is given a theoretical estimate for skin friction coefficient characterized by power law exponent is obtained.

2. FORMULATION OF THE PROBLEM

The problem considered here is the steady boundary layer flow in Dilatant non-Newtonian fluids with constant speed U_∞ over a semi-infinite flat plate at zero incidence. In the absence of body force, external pressure gradients and viscous dissipation, the laminar boundary layer equations expressing conservation of mass and momentum are governed by [6-7,12-13]:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0 \tag{2.1}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=\frac{1}{\rho}\frac{\partial \tau_{XY}}{\partial Y} \tag{2.2}$$

where the X and Y axes are taken along and perpendicular to the plate, U and V are the velocity components parallel and normal to the plate, and $\tau_{XY}=K\left|\frac{\partial U}{\partial Y}\right|^{N-1}\frac{\partial U}{\partial Y}$ is the shear stress, $\nu=\gamma\left|\frac{\partial U}{\partial Y}\right|^{N-1}$ is the kinematic viscosity. The case $N=1$ corresponds to a Newtonian fluid and the case $0<N<1$ is "power law" relation proposed as being descriptive of pseudo-plastic non-Newtonian fluids and $N>1$ describes the dilatant fluid. The appropriate boundary conditions are:

$$U\Big|_{Y=0}=0,V\Big|_{Y=0}=0,U\Big|_{Y=+\infty}=U_\infty \tag{2.3}$$

We are now to convert the boundary layer equations (2.1)-(2.3) into a singular nonlinear two-point boundary value problem. The derivation process is mathematically detail, rigorous.

Introduce a stream function $\psi(x,y)$ and similarity variable η by the expressions

$$\psi = AX^{\alpha} f(\eta), \quad \eta = BX^{\beta} Y, \tag{2.4}$$

where A, B, α and β are constants to be determined, and $f(\eta)$ denotes the dimensionless stream function. Thus the velocity components become:

$$U = \frac{\partial \psi}{\partial Y} = ABX^{\alpha+\beta} f'(\eta) = U_{\infty} f'(\eta), \tag{2.5}$$

where we have chosen $\beta = -\alpha$ and $AB = U_{\infty}$, and

$$V = -\frac{\partial \psi}{\partial X} = -A\alpha X^{\alpha-1}[f(\eta) - \eta f'(\eta)] \ . \tag{2.6}$$

At this moment, the equation (2.1) is satisfied automatically. Inserting U and V defined by (2.5) and (2.6) into (2.2), we obtain

$$-U_{\infty}^{2} \alpha X^{-1} f(\eta) f''(\eta) = U_{\infty}^{N} B^{N+1} \gamma X^{-(N+1)\alpha} (|f''(\eta)|^{N-1} f''(\eta))' \ . \tag{2.7}$$

In terms of similarity law, by choosing

$$\alpha := \frac{1}{N+1}, \quad B = \left(\frac{U_{\infty}^{2-N}}{(N+1)\gamma} \right)^{\frac{1}{N+1}}, \tag{2.8}$$

we obtain

$$-f(\eta) f''(\eta) = (|f''(\eta)|^{N-1} f''(\eta))' \tag{2.9}$$

$$f(0) = 0, \ f'(0) = 0, \ f'(+\infty) = 1. \tag{2.10}$$

Assume that the solution of Eqs.(2.9)-(2.10) possesses a positive second derivative $f''(\eta)$ in $(0, +\infty)$ and $f''(+\infty) = 0$ (which is related the boundary conditions). Defining the general Crocco variable transformation as:

$$g(t) = [f''(\eta)]^{N}, \quad t = f'(\eta), \ t \in [0, 1), \tag{2.11}$$

where t is the dimensionless tangential velocity, $g(t)$ is the dimensionless shear force. Substituting (2.11) into (2.9)-(2.10) and applying the chain rule yield the following singular nonlinear two-point boundary value problems:

$$g''(t) = -tg^{-1/N}(t), \quad 0 < t < 1, \tag{2.12}$$

$$g'(0) = 0, \quad g(1) = 0. \tag{2.13}$$

Clearly, it may be seen from the derivation process, only the positive solutions of Eqs. (2.12)-(2.13) are physically significance.

3. UNIQUENESS OF SOLUTIONS AND SKIN FRICTION ESTIMATES

As the positive solutions of Eqs. (2.12)-(2.13) are concerned, Callegari and Friedman[2] has considered the case of $N=1$ and Nachman and Callegari [8] has studied the case of $0<N<1$. They demonstrated the existence, uniqueness and analyticity of the positive solutions to the problem. Recently, Zheng et al [12-14] discussed some general cases for $0<N\le 1$ and some general nonlinear boundary value problems corresponding to the surface moving in the direction or opposite to the direction of the stream. Sufficient conditions for existence, non-uniqueness, uniqueness and analytical positive solutions to the problem were obtained utilizing the perturbation and shooting techniques.

In what follows, we study the existence and uniqueness of solutions of Eqs. (2.12)-(2.13) for the case of $N>1$(Dilatant non-Newtonian fluids). We present the proofs of our main results in what we believe to be the clearest and most elegant form.

Since the end $t=0$ is singular, we consider the positive solution of Eq.(2.12) subject to the boundary conditions:

$$g'(0)=0,\quad \mathrm{g}(1)=v>0. \tag{3.1$_v$}$$

Clearly, when $v=0$, then the boundary conditions $(3.1)_v$ reduce to (2.13). Denote the solution of Eqs.(2.12)-$(3.1)_v$ by $g_v(t)$, we first show the following lemmas.

Lemma 1 If $v_1>v_2>0$, then $g_{v_1}(t)\ge g_{v_2}(t)$.

Proof If the inequality is not true, then there exists a point $t_0\in[0,\ 1)$ such that $g_{v_1}(t_0)<g_{v_2}(t_0)$.

Case (i) $g_{v_1}(0)<g_{v_2}(0)$.

Choose $t=0$ as $t=t_0$. Since $g_{v_1}(1)>g_{v_2}(1)>0$, there exists a maximal interval $[0,k]$ $(0<k<1)$, such that $g_{v_1}(t)<g_{v_2}(t)$ for $t\in[0,k)$ and $g_{v_1}(k)=g_{v_2}(k)=m>0$. Then, $g_{v_1}(t)$ and $g_{v_2}(t)$ are both the positive solutions of the integral equation

$$g(t)=m+\int_0^k G_1(t,s)\frac{s}{g^{\frac{1}{N}}(s)}ds, \tag{3.2}$$

where

$$G_1(t,s)=\begin{cases}k-t, & 0\le s\le t\le k<1\\ k-s, & 0\le t\le s\le k<1.\end{cases}$$

Eq.(3.2) implies

$$0<g_{v_2}(t)-g_{v_1}(t)=\int_{-1}^{k}G_1(t,s)\left(\frac{s}{g_{v_2}^{\frac{1}{N}}(s)}-\frac{s}{g_{v_1}^{\frac{1}{N}}(s)}\right)ds<0,$$

which is a contradiction.

Case (ii) If $g_{v_1}(0) \geq g_{v_2}(0)$

Since $g_{v_1}(0) > g_{v_2}(0) > 0$, then there exists a maximal interval $[a,b]$ $(0 \leq a < b < 1)$, containing the point t_0 such that $g_{v_1}(a) = g_{v_2}(a)$ and $g_{v_1}(b) = g_{v_2}(b)$, and $g_{v_1}(t) < g_{v_2}(t)$ for $t \in (a,b)$. Let $g_{v_1}(a) = g_{v_2}(a) = \alpha$ and $g_{v_1}(b) = g_{v_2}(b) = \beta$. Then, for $t \in [a,b]$, $g_{v_1}(t)$ and $g_{v_2}(t)$ are the positive solution of integral equation

$$g(t) = \frac{a\beta - b\alpha}{a-b} + \frac{\alpha - \beta}{a-b} t + \int_a^b G_2(t,s) \frac{s}{g^{\frac{1}{N}}(s)} ds, \tag{3.3}$$

where

$$G_2(t,s) = \begin{cases} \dfrac{(b-t)(s-a)}{a-b}, & 0 \leq a \leq s \leq t \leq b < 1 \\ \dfrac{(b-s)(t-a)}{a-b}, & 0 \leq a \leq s \leq t \leq b < 1. \end{cases}$$

Eq. (3.3) implies,

$$0 < g_{v_2}(t) - g_{v_1}(t) = \int_a^b G_2(t,s) \left(\frac{s}{g_{v_2}^{\frac{1}{N}}(s)} - \frac{s}{g_{v_1}^{\frac{1}{N}}(s)} \right) ds < 0,$$

which is impossible. This completes the proof.

Lemma 2 For each fixed $v > 0$, Eqs. (2.12)-$(3.1)_v$ have at most one solution.

Proof Suppose that Eqs. (2.12)-$(3.1)_v$ have two different solutions $g_1(t)$ and $g_2(t)$ for each fixed $v > 0$. We may assume that there exists a point $t_0 \in [0,1)$, $g_1(t_0) > g_2(t_0)$. Since $g_1(1) = g_2(1) = v$, there exists a maximal interval $[a_1, b_1] \subseteq [0, 1]$ such that $g_1(t) \geq g_2(t)$ for $t \in [a_1, b_1]$.

(i) If $a_1 = 0$, then $g_1(t) \geq g_2(t)$ for $t \in [0, b_1] \subseteq [0,1]$ and $w_1(b_1) = w_2(b_1)$.

(ii)If $a_1 \neq 0$, then $g_1(a_1) = g_2(a_1)$ for $t \in [a_1, b_1] \subset [0,1]$, $g_1(b_1) = g_2(b_1)$ and $g_1(t) > g_2(t)$ for $t \in (a_1, \mathrm{b}_1)$.

It follows along the same lines as the cases(i) and (ii) in lemma 1, we may show which is impossible.

Lemma 3 For each fixed $v > 0$, the Eqs. (2.12)-$(3.1)_v$ have a positive solution $g_v(t)$.

Proof For each fixed $v > 0$, if $g(t)$ is the positive solution of Eqs. (2.12)-$(3.1)_v$ then $g(t)$ is convex on $[0,1]$ and must be a positive solution of the integral equation

$$g(t)=v+\int_0^1 G_3(t,s)\frac{s}{g^{\frac{1}{N}}(s)}ds\,, \tag{3.4}$$

where

$$G_3(t,s)=\begin{cases}1-t, & 0\le s\le t\le 1\\ 1-s, & 0\le t\le s\le 1.\end{cases}$$

Defining the Mapping M :

$$(Mg)(t)=v+\int_0^1 G_3(t,s)\frac{s}{g^{\frac{1}{N}}(s)}ds\,.$$

Where $g(s)\in\Omega$, $\Omega=\left\{g(s)\in C[0,1];v\le g(s)\le(Mv)(s)\right\}$, and $C[0,1]$ is the set of all real –valued continuous functions defined on $[0,1]$. It is easy to verify that M is a compactly continuous mapping from $\Omega\to\Omega$ [15-17]. The Schauder Fixed Point Theorem asserts that the mapping M has at least one fixed point, denoted by $g_v(t)$, in set Ω. Clearly, $g(t)\ge v$ is a positive solution to the problem Eqs.(2.12)-$(3.1)_v$.

Lemma 4 For any fixed $v>0$ and $N>0$, the positive solutions $g_v(t)$ of Eqs. (2.12)-$(3.1)_v$ satisfy $g_v(0;N)>\left(\frac{4}{25}\right)^{\frac{N}{N+1}}>\frac{4}{25}$.

Proof We denote $g(0)=\sigma$ and consider the initial value problem

$$\begin{cases}g''(x)=-\dfrac{t}{g^{\frac{1}{N}}(t)}, & 0<t<1\\ g(0)=\sigma>0, & g'(0)=0.\end{cases} \tag{3.5}$$

The problem now is to find $\sigma>0$ such that the solution $g(t)$ of Eq. (3.5) is defined and positive on $[0,1)$ and $\lim_{t\to1^-}g(t)=0$. It follows from Eq (3.5) that for $t>\frac{24}{25}$,

$$g(t)\le\sigma-\frac{4}{25\sigma^{1/N}}t^2\,.$$

Choose function $f(t)=\sigma-\frac{4}{25\sigma^{1/N}}t^2$ as the governing function of $g(t)$. Then, the solutions of Eq.(3.5) satisfy $g(t)\le f(t)$ for $t\in(0,1)$. Let $f(t)$ intersects the t-axis at point t_0^*, then $t_0^{*2}=\frac{25}{4}\sigma^{\frac{N}{N+1}}$. Especially, for $t_0^*=1$, we obtain $\sigma=g(0;N)=\left(\frac{4}{25}\right)^{\frac{N}{N+1}}>\frac{4}{25}$.

For $\sigma \le \frac{4}{25}$, the solution $g(t)$ can't intersect the t-axis at the point $t \ge 1$. This shows that for $\sigma \le \frac{4}{25}$, the solution of initial value problem (3.5) only holds for $g(1) < 0$.

Theorem 1 For any $N \in (1,+\infty)$, Eqs.(2.12)-(3.1)$_0$ have a unique positive solution $g(t)$.

Proof Lemmas 2 and 3 imply that for any $v > 0$, Eqs.(2.12)-$(3.1)_v$ have a unique positive solution. Then, for any $v_2 > v_1 > 0$, it follows from Eq. (3.5) and Lemma1 that

$$0 < g_{v_2}(t) - g_{v_1}(t) = v_2 - v_1 + \int_{-1}^{0} G_3(t,s)\left(\frac{s}{g_{v_2}^{\frac{1}{N}}(s)} - \frac{s}{g_1^{\frac{1}{N}}(s)}\right) ds \le v_2 - v_1 .$$

It indicates the series of solutions $\{g_v(t)\}$ converges to a limit uniformly with v on $[0,1]$, denoted by $g_0(t)$. Then, $\lim_{v\to 0} g_v(t) \overset{uniformly}{=} g_0(t)$ on $[0,1]$.

Lemma 4 implies $g_0(0;N) > \frac{4}{25}$. For any $v \ge 0$, by the convexity of $g_v(t)$,

$$g_v(t) \ge (g_v(0;N) - v)(1-t) + v \ge (\frac{4}{25} - v)(1-t) + v \ge \frac{4}{25}(1-t) . \qquad (3.6)$$

Thus,

$$G_3(t,s)\frac{s}{g_v^{\frac{1}{N}}(s)} \le G_3(t,s)\frac{s}{[\frac{4}{25}(1-s)]^{\frac{1}{N}}} = \left(\frac{25}{4}\right)^{1/N} G_3(t,s)\frac{s}{(1-s)^{1/N}} . \qquad (3.7)$$

Letting $v \to 0^+$ in integral equation (3.5), by the Monotone Convergence Theorem[14-16], we obtain that for $N > 1$,

$$g_0(t) = \lim_{v\to 0}\int_0^1 G_3(t,s)\frac{s}{g^{\frac{1}{N}}(s)} ds = \int_0^1 \lim_{v\to 0} G_3(t,s)\frac{s}{g_v^{\frac{1}{N}}(s)} ds$$

i.e.,

$$g_0(t) = \int_0^1 G_3(t,s)\frac{s}{g_0^{\frac{1}{N}}(s)} ds . \qquad (3.8)$$

This completes the proof Theorem 1.

Nachman, A. and Callegari[8], have shown that the positive solution of Eqs. (2.12)-(3.1)$_0$ has a power series expansion:

$$g(t) = \sum_{i=0}^{\infty} \frac{g^{(i)}(0;N)}{i!} t^i \qquad (3.9)$$

which converges at $t=1$ to $g(1)=0=\sum_{i=0}^{\infty}\frac{g^{(i)}(0;N)}{i!}$.

Here $g^{(i)}(0;N)$ can be established by the induction:

$$g^{(m+3)}(0;N)=-g^{-1}[m!g^{1-\frac{1}{n}}\sum_{\lambda=p(m)}g^{-|\lambda|}\binom{1-\frac{1}{n}}{\lambda}\prod_{i=1}^{m}\left(\frac{g^{(i)}}{i!}\right)^{\lambda(i)}$$
$$+\frac{1}{n}\sum_{i=0}^{m}\binom{m}{i}g^{(i+1)}g^{(m+2-i)}+\sum_{i=0}^{m}\binom{m}{i}g^{(i)}g^{(m+3-i)}] \tag{3.10}$$

where $g^{(i)}=\frac{d^i g}{dt^i}$, $\binom{m}{i}=\frac{m!}{i!\cdot(m-i)!}$, $p(m)$ is a partition of the integer m, and λ is a vector of the partition, whose first component $\lambda(1)$ is the number 1's in the partition and second component $\lambda(2)$ is the number 2's in the partition, etc.,

$$|\lambda|=\sum_{i=1}^{m}\lambda(i),\ \binom{p}{\lambda}=p(p-1)\cdots(p-|\lambda|+1)/(1)!(2)!\cdots\lambda(m)!$$

and the first sum is a sum over all partition m.

Equation (3.10) indicates that each derivative of $g(t)$, of fourth or higher order, can be expressed in terms of those lower order, thus all derivatives of $g(t)$ depend only on the first three. Let $g(0;N)=\sigma$ (skin friction), then $g'(0;N)=0$, $g''(0;N)=0$, $g'''(0;N)=-\sigma^{1/N}$; i.e., all derivative of $g(t)$ depend only on the value of $g(0;N)=\sigma$ (skin friction coefficient). The following theorem provides a possibility for us to make an estimate of the value of $g(0;N)$.

Theorem 2 For any $N\in(1,+\infty)$, the positive solutions of Eqs.(2.12)-(3.1)$_0$ satisfy the estimate formula

$$\left(\frac{4}{25}\right)^{\frac{N}{N+1}}<g(0;N)<\left(\frac{25}{4}\right)^{\frac{1}{N+1}}\frac{N^2}{(2N-1)(3N-1)}.$$

where $g(0;N)=\sigma>0$ is the skin friction coefficient, which is very important in physical significance.

Proof The convexity of $g_v(t)$ and Lemma 4 imply that for any $v\geq 0$ and N ($N>1$),

$$g_v(t)\geq(g_v(0;N)-v)(1-t)+v$$
$$\geq[(\frac{4}{25})^{\frac{N}{N+1}}-v](1-t)+v\geq(\frac{4}{25})^{\frac{N}{N+1}}(1-t)>0$$

Thus,

$$G_3(t,s)\frac{s}{g_v^{\frac{1}{N}}(s)}\leq G_3(t,s)\frac{s}{[(\frac{4}{25})^{\frac{N}{N+1}}(1-s)]^{\frac{1}{N}}}=(\frac{25}{4})^{\frac{1}{N+1}}G_3(t,s)\frac{s}{(1-s)^{1/N}}. \tag{3.11}$$

It follows from (3.8) and (3.11) that,

$$\left(\frac{4}{25}\right)^{\frac{N}{N+1}} < g(0;N) < \left(\frac{25}{4}\right)^{\frac{1}{N+1}} \frac{N^2}{(2N-1)(3N-1)}. \tag{3.12}$$

This completes the proof Theorem 2.

Remark 1 If we denote the lower-bound of $g(0;N)$ by $\sigma_{lower-bound} = \left(\frac{4}{25}\right)^{\frac{N}{N+1}}$ and upper-bound by $\sigma_{upper-bound} = \left(\frac{25}{4}\right)^{\frac{1}{N+1}} \frac{N^2}{(2N-1)(3N-1)}$ in formula (3.12), it is seen that both $\sigma_{lower-bound}$ and $\sigma_{upper-bound}$ decrease with N increasing. Let $N \to +\infty$ in (3.12), we have immediately that

$$0.16 = \frac{4}{25} < g(0;+\infty) < \frac{1}{6} < 0.16667.$$

Remark 2 The reliability and efficiency of estimate formula (3.12) are significantly determined by the governing function of $g(t)$, we choose the governing function $f(t) = \sigma - \frac{4}{25\sigma^{1/N}} t^2$. The main purpose in this paper is to provide a method to obtain an estimate formula which can be successfully applied to give the value of skin friction coefficient. One can obtain a better estimate than (3.12) by choosing a suitable governing function.

4. NUMERICAL SOLUTIONS AND DISCUSSIONS

Eqs.(2.12)-$(3.1)_0$ are solved for values of power law exponent of N by using the shooting technique. The skin friction coefficient $\sigma(= g(0;N))$ calculated by the shooting method denoted by $\sigma_{com.}$. A comparison is presented in Table 1.

Table1 Skin friction coefficient obtained by estimates and numerical results

Exponent	$\sigma_{\text{lower-bound}}$	$\sigma_{\text{com.}}$	$\sigma_{\text{upper-bound}}$
N=1.5	0.3330	0.3727	0.6690
N=2.0	0.2947	0.3244	0.4912
N=3.0	0.2530	0.2728	0.3558
N=5.0	0.2172	0.2294	0.2693
N=8.0	0.1961	0.2043	0.2274
N=10.0	0.1890	0.1959	0.2144
N=30.0	0.1697	0.1734	0.1818
N=50.0	0.1659	0.1689	0.1757
N=80.0	0.1637	0.1664	0.1723
N=100.0	0.1629	0.1655	0.1711

N=200.0	0.1615	0.1638	0.1689
N=300.0	0.1610	0.1633	0.1682
N=500.0	0.1606	0.1628	0.1676
N=1000.0	0.1603	0.1625	0.1671
N=5000.0	0.1601	0.1622	0.1668
N=10000.0	0.1600	0.1622	0.1667

The reliability and efficiency of the proposed estimate formula are verified by numerical results with the absolute errors obtained by lower-bound no more than 0.0069 and obtained by upper-bound no more than 0.0185 for power law exponent $n \geq 10$. For power law exponent $n \geq 100$, the absolute errors obtained by lower-bound and upper-bound are respectively 0.0026 and 0.0056. It may be seen that for power law exponent $n \geq 500$, the values of skin friction coefficient approach a constant. Numerical results indicated that the estimate formula may be successfully applied to give the value of skin friction coefficient.

5. CONCLUSIONS

This paper presents a similarity analysis for laminar boundary layer flow in power law non-Newtonian fluids. The boundary layer equations are changed into a singular nonlinear two-point boundary value of ordinary differential equation when Crocco variables are introduced. Sufficient conditions for existence and uniqueness of positive solutions are established and a theoretical estimate formula for skin friction coefficient is given, which is characterized by power law exponent. The proposed estimate formula can be successfully applied to give the value of skin friction coefficient.

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