

Optimal Filtration and a Pulsar Time Scale

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An algorithm is proposed for constructing a group (ensemble) pulsar time based on the application of optimal Wiener filters. This algorithm makes it possible to separate the contributions of variations of the atomic time scale and of the pulsar rotation to barycentric residual deviations of the pulse arrival times. The method is applied to observations of the pulsars PSR B1855+09 and PSR B1937+21, and is used to obtain corrections to UTC relative to the group pulsar time PT_{ens} . Direct comparison of the terrestrial time TT(BIPM06) and the group pulsar time PT_{ens} shows that they disagree by no more than $0.4 \pm 0.17 \mu\text{s}$. Based on the fractional instability of the time difference $TT(\text{BIPM06}) - PT_{\text{ens}}$, a new limit for the energy density of the gravitational-wave background is established at the level $\Omega_g h^2 \sim 10^{-9}$.

The discovery of pulsars in 1967 [1] and of millisecond pulsars in 1982 [2], as well as subsequent observations, have clearly shown that the rotational stability of pulsars is such that pulsars can be used as astronomical clocks. Currently, the accuracy with which pulse arrival times (PATs) can be measured is at the level of several microseconds for most pulsars, and reaches the submicrosecond level for some pulsars. If we compare this accuracy to the interval covered by observations, we obtain the relative accuracy 10^{-15} , which is essentially comparable to the fractional instability of the best atomic frequency standards.

A number of studies have been concerned with the stability of pulsar rotation and its application to time systems. Writing about the principles for establishing Terrestrial Time (TT), Guinot [3] concludes that Terrestrial Time as realized by the Bureau International des Poids et Mesures [TT(BIPMXX), where XX denotes the year] is the most expedient time system to use for pulsar chronometry. The principles of pulsar time and the definition of the

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pulsar second are given in [4]. It is shown in [5] that it is not possible to use the rotation of pulsars to improve the definition of the time unit. The studies [6–9] introduce a definition for a time based on the motion of a pulsar in a binary system – Binary Pulsar Time, or BPT – together with theoretical expressions for the variations in the rotational and orbital parameters as functions of the duration of the observing interval. The main conclusion of these studies is that BPT is less stable than the ordinary pulsar time PT over short intervals, but the fractional instability of BPT can reach 10^{-14} over long intervals ($10^2 \div 10^3$ yr). Petit and Tavella [10] present an algorithm for determining a group pulsar time based on weighted averaging, which is compared to an algorithm based on optimal filtration; they also consider some ideas connected with the stability of BPT. Foster and Backer [11] present a polynomial approach to describing clock and ephemerides variations and the influence of gravitational waves passing through the solar system on pulsar chronometry.

Here, we propose a method for forming a group pulsar time (PT) based on the application of optimal Wiener filters [12, 13]. Section 2 discusses the pulsar-chronometry (timing) algorithm from the point of view of time scales. Section 3 describes the principles for constructing optimal filters and the formation of a group pulsar time. Section 4 contains the results of computer simulations of detecting a signal against the background of correlated noise using optimal filtration. Section 5 discusses the results of applying Wiener filters to observations of the pulsars PSR B1855+09 and PSR B1937+21.

1. PULSAR CHRONOMETRY ALGORITHM

An observer located on the Earth, which is rotating about its axis and revolving around the Sun, receives a signal from a pulsar using a radio telescope over an accumulation time that is sufficient to obtain a specified signal-to-noise ratio. The pulse arrival time (PAT) is measured in a local time scale via crosscorrelation with a standard profile for the pulsar pulse. The resulting PATs τ_N can be translated into UTC, TAI, and TT using the relations [14]

$$\text{UTC} = \tau_N + \Delta\tau, \text{ TAI} = \text{UTC} + k, \text{ TT} = \text{TAI} + 32.184\text{ s}, \quad (1)$$

where $\Delta\tau$ is the local time correction, k is a whole number of seconds introduced to take into account the variation in the length of the day, and the quantity 32.184 s is added to remove a historical jump between atomic and ephemerides time. Since the duration of the

second in TT depends on the position and velocity of the Earth in its orbit, an additional transformation from TT to Barycentric Time TB is required, using the algorithm described in [15].

The observations that have been transformed into the uniform Barycentric Time system TB must then be reduced to the barycenter of the solar system, in order to exclude variations in the PATs due to the motion of the observer about the barycenter [14]

$$T = t - t_0 + \Delta_R(\alpha, \delta, \mu_\alpha, \mu_\delta, \pi) + \Delta_{\text{orb}} - DM/f^2 + \Delta_{\text{rel}}, \quad (2)$$

where t_0 is the initial epoch; t the topocentric PAT in TB; T the PAT at the barycenter of the solar system; $\Delta_R(\alpha, \delta, \mu_\alpha, \mu_\delta, \pi)$ the Remer correction along the Earth's orbit; $\alpha, \delta, \mu_\alpha, \mu_\delta$ and π the right ascension, declination, proper motion in right ascension and declination, and parallax of the pulsar, respectively; Δ_{orb} the Remer correction along the orbit of the pulsar, when it is located in a multiple system; DM/f^2 the delay for the signal propagation in the interstellar and interplanetary media at frequency f taking into account the Doppler shift; and Δ_{rel} the relativistic correction for the delay of the signal propagation in the gravitational field of the solar system.

The PATs at the solar-system barycenter are then used to calculate the rotational phase (number of rotations) of the pulsar:

$$N(T) = N_0 + \nu T + \frac{1}{2}\dot{\nu}T^2 + \varepsilon(T), \quad (3)$$

where N_0 is the initial phase at time $T = 0$, ν and $\dot{\nu}$ are the rotational frequency of the pulsar and its derivataive at $T = 0$, and $\varepsilon(T)$ represents variations in the rotational phase (timing noise). The procedure for determining the parameters includes refining $\alpha, \delta, \mu_\alpha, \mu_\delta$ and π , via a least-squares fit minimizing a weighted sum of the square differences between $N(T)$ and the nearest integer. The residual deviations in the rotational phase are usually expressed in units of the time $\delta t = \delta N/\nu$. Here, we consider variations in the intrinsic rotation of the pulsar and variations due to irregularity of the reference time scale $\Delta t_{\text{clock}}(T)$.

When intercomparing different realizations of atomic time [3], flicker noise in the frequency dominate on intervals of several months, while random walk noise in the frequency dominates on intervals of several years. Thus, clock variations have a power spectrum of the form $1/\omega^n$ in the frequency domain, and are expressed in the time domain by the polynomial

$$\Delta t_{\text{clock}}(\mathcal{T}) = c_0 + c\mathcal{T} + \frac{1}{2}\dot{c}\mathcal{T}^2 + \frac{1}{6}\ddot{c}\mathcal{T}^3 + \dots \quad (4)$$

It is clear that the appearance of the quantity Δt_{clock} in Eq. (3) leads to a redetermination of the rotational parameters of the pulsar:

$$N(\mathcal{T}) = N'_0 + (1 + c)f\mathcal{T} + \frac{1}{2}(f\dot{c} + (1 + c)^2\dot{f})\mathcal{T}^2, \quad (5)$$

where \mathcal{T} is ideal barycentric time and f and \dot{f} are the rotational frequency of the pulsar and its derivative undistorted by the clock parameters. For this reason, we must use the PAT values expressed in TT(BIPMXX), as best determined at the current time [5].

2. OPTIMAL FILTRATION

Let us consider n uniform measurements of a random quantity (the residual PAT deviations) $\mathbf{r} = (r_1, r_2, \dots, r_n)$. The quantity \mathbf{r} is the sum of two uncorrelated quantities, $\mathbf{r} = \mathbf{s} + \varepsilon$, where \mathbf{s} is a random signal to be estimated and associated with the clock contribution, and ε is the contribution of the rotational phase of the pulsar, which we treat here like noise. We are interested in estimating the signal \mathbf{s} against the background of the additive noise ε using the Wiener filtration method.

Wiener filtration consists of estimating the signal \mathbf{s} given the measurements \mathbf{r} and covariances (7) [16]. We must reconstruct the random signal with insufficient a priori information, since the covariance matrix of the signal is not known a priori, and is estimated from the observational data themselves via a crosscorrelation of all the data, assuming that the variations in the clicks (the estimated signal) and in the pulsar rotational phase (additive noise) are uncorrelated.

The optimal Wiener estimate of the signal \mathbf{s} is given by [16]:

$$\hat{\mathbf{s}} = \mathbf{Q}_{sr} \mathbf{Q}_{rr}^{-1} \mathbf{r} = \mathbf{Q}_{ss} \mathbf{Q}_{rr}^{-1} \mathbf{r} = \mathbf{Q}_{ss} (\mathbf{Q}_{ss} + \mathbf{Q}_{\varepsilon\varepsilon})^{-1} \mathbf{r}, \quad (6)$$

where the covariance matrices \mathbf{Q}_{rr} , \mathbf{Q}_{sr} , and \mathbf{Q}_{ss} are formed as arrays of the corresponding covariance functions. The covariance functions for \mathbf{r} , \mathbf{s} and ε are written

$$\begin{aligned} \text{cov}(r, r) &= \langle r_i, r_j \rangle = \langle s_i, s_j \rangle + \langle \varepsilon_i, \varepsilon_j \rangle, \\ \text{cov}(s, s) &= \langle s_i, s_j \rangle, \\ \text{cov}(s, r) &= \langle s_i, r_j \rangle = \langle r_i, s_j \rangle = \langle s_i, s_j \rangle, \\ \text{cov}(\varepsilon, \varepsilon) &= \langle \varepsilon_i, \varepsilon_j \rangle. \end{aligned} \quad (i, j = 1, 2, \dots, n). \quad (7)$$

The angular brackets, $\langle \rangle$, denote ensemble averaging. We assume that the processes s and ε are stationary in a weak sense; i.e., only the first and second moments are stationary, since fitting a quadratic polynomial to the rotational phase excludes the nonstationary part of the random process [17]. Thus, the covariance functions depend only on the time difference $t_i - t_j$.

Since the pulsar-chronometry algorithm assumes that the PATs are defined relative to a reference time, distinguishing the covariances $\langle s_i, s_j \rangle$ and $\langle \varepsilon_i, \varepsilon_j \rangle$ requires observations of at least two pulsars using a single time system. In this case, combining the pulsar PATs and assuming that the cross-covariance $\langle {}^2\varepsilon_i, {}^1\varepsilon_j \rangle = \langle {}^1\varepsilon_i, {}^2\varepsilon_j \rangle = 0$, we can estimate

$$\langle s_i, s_j \rangle = (\langle {}^1r_i + {}^2r_i, {}^1r_j + {}^2r_j \rangle - \langle {}^1r_i - {}^2r_i, {}^1r_j - {}^2r_j \rangle) / 4 \quad (8)$$

or

$$\langle s_i, s_j \rangle = \langle {}^1r_i, {}^2r_j \rangle. \quad (9)$$

If M pulsars are observed to construct a group pulsar time, we will have $M(M - 1)/2$ estimates of the covariance matrices $\langle s_i, s_j \rangle = \langle {}^k r_i, {}^l r_j \rangle$, ($k, l = 1, 2, \dots, M$)

The matrix \mathbf{Q}_{rr}^{-1} in (6) is a whitening filter. The matrix \mathbf{Q}_{ss} forms a signal from the whitened data.

The averaged signal (group pulsar time) is written

$$\hat{\mathbf{s}}_{\text{ens}} = \frac{2}{M(M - 1)} \sum_{m=1}^{\frac{M(M-1)}{2}} {}^m \mathbf{Q}_{ss} \cdot \sum_{i=1}^M {}^i w {}^i \mathbf{Q}_{rr}^{-1} \cdot {}^i \mathbf{r}, \quad (10)$$

where ${}^i w$ is the relative weight of the i th pulsar, ${}^i w = \kappa / \sigma_i^2$, σ_i is the rms deviation of the whitened data ${}^i \mathbf{Q}_{rr}^{-1} \cdot {}^i \mathbf{r}$, and the constant κ serves to ensure that $\sum_i {}^i w = 1$. The first factor in (10) is the average cross-covariance function, and the second factor is the weighted sum of the whitened data.

The following algorithm was used to calculate the auto- and cross-covariances. The observational data ${}^k r_t$ were subjected to a rapid Fourier transform,

$${}^k x(\omega) = \frac{1}{\sqrt{n}} \sum_{t=1}^n {}^k r_t h_t e^{i\omega t}, \quad (k = 1, 2, \dots, M), \quad (11)$$

where the weights h_t , which are bell-like functions, were used to reduce leakage through the sidelobes [18]. They can be calculated to very good accuracy using the formula [18]

$$h_t = C'_0 \frac{I_0 \left(\pi W (n - 1) \sqrt{1 - \left(\frac{2t-1}{n} - 1 \right)^2} \right)}{I_0(\pi W (n - 1))}, \quad (12)$$

where C'_0 are scaling factors used to ensure that $\sum h_t^2 = 1$, I_0 , is a modified zeroth-order Bessel function of the first kind, and the parameter W serves to regularize the sidelobe level in the spectrum, and is usually taken to be in the range $W = 1 \div 4$. We used the value $W = 1$.

The power spectrum ($k = l$) and cross-spectrum ($k \neq l$) were calculated using the formula

$${}^{kl}X(\omega) = \frac{1}{2\pi} |{}^kx(\omega){}^lx^*(\omega)|, \quad (13)$$

where $(\cdot)^*$ denotes complex conjugation.

Finally, the auto- ($k = l$) and cross- ($k \neq l$) covariance were calculated using the formula

$$\text{cov}({}^k r, {}^l r) = \sum_{\omega=1}^n {}^{kl}X(\omega)e^{-i\omega t}, \quad (k, l = 1, 2, \dots, M). \quad (14)$$

3. COMPUTER MODELING

We carried out numerical simulations to estimate the quality of the signal reconstruction applying Wiener filtration and weighted averaging [10]. In contrast to [13], where a harmonic signal was taken as the initial signal, we estimated a time series of the difference UTC–TT(BIPM06) in the interval MJD = 46399–48949, from which we subtracted the quadratic polynomial from the least-squares fit. We added white noise with zero mean and various dispersions. For example, if we used 50 pulsars for the modeling, the dispersion of the white noise n_0 was $\sigma^2 = 1 \div 50$. The weights were taken to be inversely proportional to the dispersion of the obtained noise series. The quality of the reconstructed signal was calculated as the rms deviation of the difference between the reconstructed and input signals.

Figures 1a and 1b show the quality of the signal reconstruction using the weighted averaging (dashed curve) and Wiener filtration (solid curve) for white noise, as functions of the number of pulsars used and the length of the data series. As the length of the data series and the number of pulsars increase, the quality of the Wiener-filtration signal reconstruction becomes increasingly higher compared to the weighted-average reconstruction as the rotational phase variations increase.

4. RESULTS

The method described above was applied to observational data (barycentric residual deviations of the PATs) for the pulsars PSR B1855+09 and PSR B1937+21 [19]. Since these

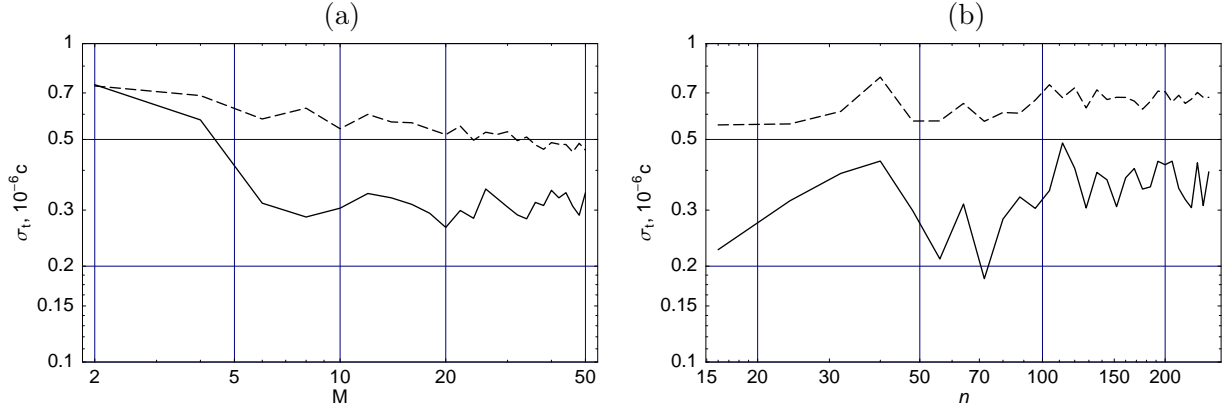


Figure 1. The rms difference between the reconstructed and initial signals as a function of (a) the number of pulsars M and (b) the length of the data series n , using weighted averaging (dashed) and Wiener filtration (solid).

data are not uniform, we applied a spline interpolation to regularize the data, with the aim of simplifying the subsequent reduction using matrix-algebra methods. We chose a step of 10 days, which preserved the original quantity of data. This transformation of the data distorts the high-frequency component, but leaves the low-frequency component that is of interest to us unchanged [20].

We took the part of the data that was common to both pulsars (251 points for each), in the interval $\text{MJD} = 46450\text{--}48950$, for this reduction. Since, generally speaking, the series of residual deviations have different means and slopes after their lengths are shortened, they were reduced by fitting a quadratic polynomial (Fig. 2).

The data for PSR B1855+09 and PSR B1937+21 were obtained in UTC based on the information from [19]. Consequently, the signals that were distinguished from the observational data for PSR B1855+09 and PSR B1937+21 were the corrections UTC-PT_{1855} and UTC-PT_{1937} . The combined signal (group time) $\text{UTC-PT}_{\text{ens}}$ is shown in Fig. 3 by the thin curve. It shows behavior similar to that for $\text{UTC-TT}(\text{BIPM06})$, with the correlation coefficient $\rho = 0.75 \pm 0.04$. A direct comparison of terrestrial time $\text{TT}(\text{BIPM06})$ and the group Pulsar Time PT_{ens} shows that they disagree by no more than $0.4 \pm 0.17 \mu\text{s}$.

We calculated the fractional instability σ_z of the time difference $\text{TT-PT}_{\text{ens}}$, which was $\sigma_z = (0.5 \pm 2) \cdot 10^{-15}$ on a time interval of seven years.

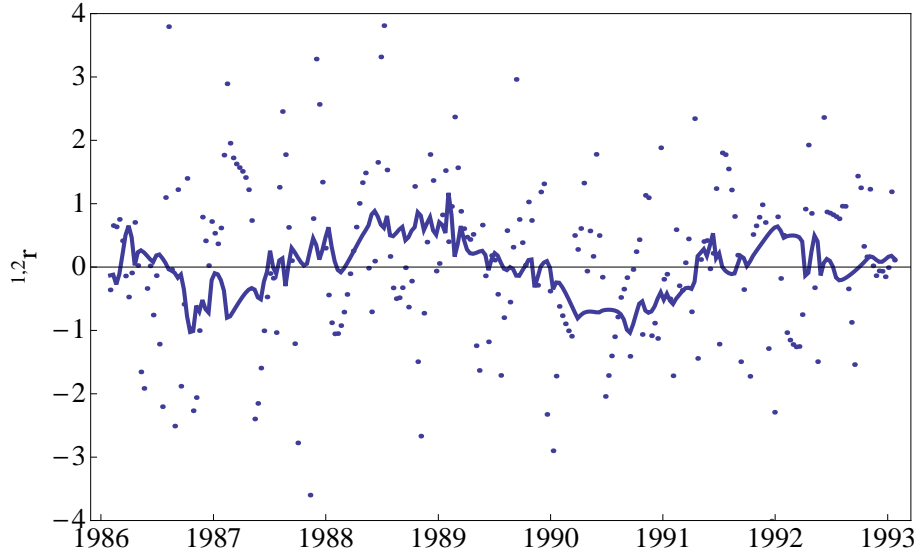


Figure 2. Barycentric residual deviations ${}^{1,2}r$ (in μs) for PSR B1855+09 (points) and PSR B1937+21 (curve) after fitting a quadratic polynomial.

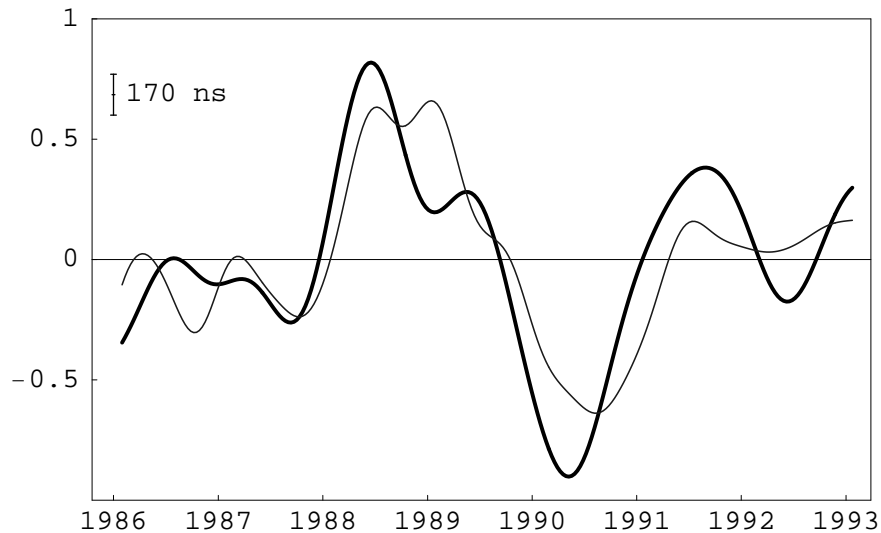


Figure 3. UTC-TT(BIPM06) (in μs , bold curve) and UTC-PT_{ens} (thin curve).

5. DISCUSSION

The fractional instability of time systems is characterized by the so-called Allan variance, which is numerically equal to the second-order finite difference of the clock phase variations. A chronometric analysis of observational data contains a determination of the rotational parameters of the pulsar, including at least the first frequency derivative; this corresponds

to fitting a quadratic polynomial in time to the rotational phase of the pulsar, which, in turn, is equivalent to excluding the second-order derivative from the PAT data. For this reason, the use of the classical Allan variance is not expedient. Instead, another quantity was proposed as a means of characterizing the fractional instability of the rotation of pulsars – σ_z [21]. A detailed algorithm for calculating σ_z is given in [22].

Figure 4 plots the fractional instability of the intrinsic rotation of the pulsars PSR B1855+09 and PSR B1937+21 taking into account the contribution of the reference time and the variations TT–PT_{ens}. Theoretical curves for the behavior of σ_z when noise due to the presence of a stochastic gravitational-wave background with relative energy densities $\Omega_g h^2 = 10^{-9}$ and 10^{-10} [19] are also shown in the lower right corner of the figure. The σ_z curve crosses the line $\Omega_g h^2 = 10^{-10}$; however, the upper limit for the gravitational-wave background should be an order of magnitude higher, taking into account the uncertainty in this quantity. Thus, as a conservative estimate of this upper limit, we adopt $\Omega_g h^2 \lesssim 10^{-9}$.

The fractional instability of TT(BIPM06) relative to PT_{ens} is at the level $< 10^{-15}$ for a seven-year interval, and is an order of magnitude better than the fractional instability of the rotation of PSR B1855+09 and PSR B1937+21, taking into account the contribution of the reference time system. As the numerical simulations show, when Wiener filtration is used, the accuracy with which TT–PT_{ens} is estimated grows with the number of pulsars used. We estimate the accuracy of the optimal filtration method to be 170 ns. This accuracy was calculated as the rms deviation between the obtained and smoothed signals. The smoothing was carried out in the frequency domain using a low-frequency Kaiser filter with a bandwidth of $f_{\max}/32$, where $f_{\max} = 2/\Delta t$ and $\Delta t = 10$ days is the sample interval. Our choice of this bandwidth was based on the desire to obtain smoothness similar to the UTC–TT(BIPM06) curve. Thus, the uncertainty in our estimation of PT_{ens} can, in principle, reach several tens of nanoseconds if the most stable millisecond pulsars are used.

The proposed method does not distinguish quadratic trends in the reference time and in the rotational phase of the pulsar, because the pulsar itself displays secular deceleration of the rotational frequency. In our opinion, this does not hinder the use of pulsars as independent clocks, since longer-period variations in the reference clocks will be revealed as data are accumulated over increasingly longer time intervals.

The low relative accuracy of \dot{P} noted in [5] likewise does not pose problems, since the rotational phase of the pulsar is not predicted. However, if it is required to predict the

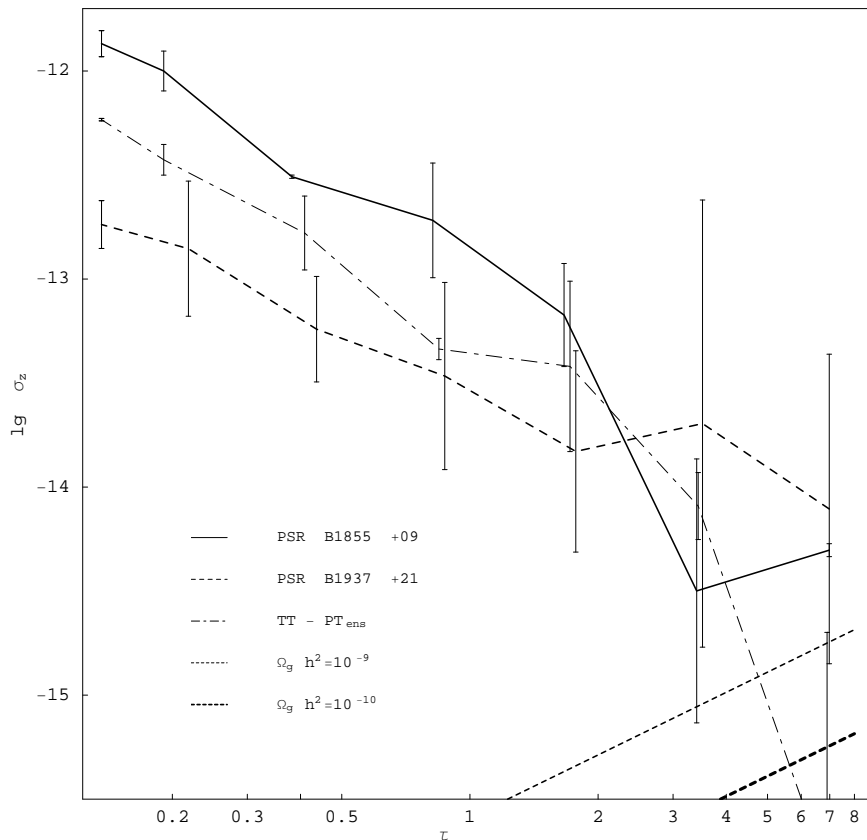


Figure 4. Fractional instability σ_z , based on the chronometry data for PSR B1855+09 (solid) and PSR B1937+21 (dashed) and corresponding to TT-PT_{ens} (dot-dashed) as a function of the observing interval τ (in years). The theoretical curves of σ_z for $\Omega_g h^2 = 10^{-9}$ and 10^{-10} are shown in the lower-right corner of the figure.

behavior of some reference atomic time, such as UTC, relative to the group pulsar time, this can be done based on variations of UTC-PT_{ens} using standard methods for the prognosis of time series, provided that this prognosis is obtained relative to short-period fluctuations, without linear and quadratic trends. In this approach, the low relative accuracy of the derivative of the rotational period of the pulsar will not play a role, since the absolute phase is not predicted.

6. CONCLUSION

We have presented an algorithm for forming a group pulsar time based on optimal Wiener filtration. The algorithm makes it possible to distinguish the contributions to barycentric residual deviations of PATs due to irregularities in the intrinsic rotation of the pulsar and variations in the reference clocks. Both irregularity in the pulsar rotation and variations in the reference time scale are obtained relative to an *ideal* time system. Realization of the algorithm requires observations of at least two pulsars relative to a common reference time.

The proposed approach has better accuracy than the use of weighted averaging to form the group time scale, since it uses additional information about the signal via its correlation function or power spectrum.

The presence of a pulsar time that is independent of terrestrial conditions makes it possible to carry out independent checks of terrestrial time scales, which immediately provides possibilities for revealing the presence of variations that are common to all terrestrial standards that would otherwise be undetectable.

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