Existence of Periodic Solution on a Class of Discrete Difference System *

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Abstract: Using trigonometrical series theory and contraction mapping principle, This paper study difference systems $X(n+1)=\sum_{j=-\infty}^{+\infty}A(j)X(n-j)+f(n,x(n+\cdot))$ and $X(n+1)=\sum_{j=-\infty}^{+\infty}A(j)X(n-j)+G(n,X(n+\cdot))$, sufficient and necessary conditions on the existence of periodic solutions for the first equation and sufficient conditions on the uniqueness of the second equation are obtained.

Keywords Discrete systems; periodic solution; trigonometrical series; contraction mapping principle

CLC number: O175.14 Document code: A

1. Introduction

Paper[1] studied the existence of Periodic Solution of linear inhomogeneous differential equations

$$\dot{x}(t)=\int_{-\infty}^{+\infty}[dE(s)]x(t+s)+f(t), \tag{A}$$

where $E:R\to C^{n\times n}$is continue to the left and of bounded total variation on R, i.e., $\gamma=\int_{-\infty}^{\infty}|dE(s)|<\infty, f\in C_T$, and $x\in BC(R;C^n):=\{\psi\in C(R;C^n):\psi$is bounded on R$\}$. Paper[1] obtained some sufficient and necessary conditions on the existence of periodic solutions for Eq. (A).

Paper[2] studied the uniqueness of Periodic Solution of quasilinear functional differential equations

$$\dot{x}(t)=\int_{R}[dE(s)]x(t+s)+G(t,x(t+\cdot)), \tag{B}$$

where $x(t)\in R^n, E:R\to R^{n^2}$is continue to the left and of bounded total variation on R, i.e., $\gamma=\int_{-\infty}^{\infty}|dE(s)|<\infty$, $G:R\times BC(R;R^n)\to R^n$ is continue, G is $T>0$ periodic with respect to its first variable t, and G maps bounded set to bounded set .Paper[2] obtained some sufficient and necessary conditions on the uniqueness of periodic solutions for Eq.(B).

Using trigonometrical series theory and contraction mapping principle, This paper study discrete linear inhomogeneous difference systems and qunasilinear delay difference systems,and obtain sufficient

*Supported by 973 Program of the Ministry of Science and Technology of the People's Republic of China No.:(1999064911)

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and necessary conditions on the existence of periodic solutions for the discrete linear inhomogeneous difference systems with sufficient conditions on the existence of periodic solutions for the qunasilinear delay difference systems,morever,The main results in [1],[2]are extended and improved to difference systems.

2. Main Results

In this paper ,we investigate the following linear inhomogeneous difference periodic systems

$$X(n+1)=\sum_{j=-\infty}^{+\infty}A(j)X(n-j)+f(n). \tag{1}$$

where $A(j)\in C^{n\times n},\quad X(n)\in C^n,\ f\in l_N=\{\{\phi(n)\}|\ \ \phi(n+N)=\phi(n),\},N\geq 1$ is positive integer.

Lemma 1 Suppose that $f(n)\in l_N$,then $f(n)$ can be uniquely expressed as $f(n)=\sum\limits_{k=0}^{N-1}\hat{f}(k)e^{\mu_k n}$, where $\hat{f}(k)=\frac{1}{N}\sum\limits_{n=0}^{N-1}f(n)e^{-\mu_k n},\mu_k=\frac{2k\pi i}{N},k\in\omega:=\{0,1,2,...,N-1\}$.

Proof: Assume that

$$f(n)=\sum_{k=0}^{N-1}a(k)e^{\mu_k n} \tag{2}$$

hold,multiplying Eq.(2) by $e^{-\mu_j n}$ and adding form 0 to N-1 ,we obtain

$$\begin{aligned}\sum_{n=0}^{N-1}f(n)e^{-\mu_j n}&=\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}a(k)e^{u_k n}e^{-u_m n}\\&=\sum_{k=0}^{N-1}a(k)\sum_{n=0}^{N-1}e^{(\mu_k-\mu_j)n}.\end{aligned}$$

Since

$$e^{(\mu_k-\mu_j)n}=\begin{cases}N & k=j\\ 0 & k\neq j,\end{cases}$$

We have

$$\sum_{n=0}^{N-1}f(n)e^{-\mu_m n}=Na(k),$$

Hence

$$a(k)=\frac{1}{N}\sum_{n=0}^{N-1}f(n)e^{-\mu_m n}=\hat{f}(k),$$

This implies $f(n)$ can be uniquely expressed as $f(n)=\sum\limits_{k=0}^{N-1}\hat{f}(k)e^{\mu_k n}$.

Lemma 2 Assume that $f(n)\in l_N$ and $f(n)$ can be expressed as $f(n)=\sum\limits_{k=0}^{N-1}\hat{f}(k)e^{\mu_k n}$,then

$$\sum_{k=0}^{N-1}|\hat{f}(k)|^2=\frac{1}{N}\sum_{n=0}^{N-1}|f(n)|^2.$$

Proof: Since $f(n)=\sum\limits_{k=0}^{N-1}\hat{f}(k)e^{\mu_k n}$,we have

$$\begin{aligned}|f(n)|^2 &= <f(n),f(n)>\\ &= <\sum_{k=0}^{N-1}\hat{f}(k)e^{\mu_k n},\sum_{l=0}^{N-1}\overline{\hat{f}(l)}e^{-\mu_l n}>\\ &= \sum_{k,l=0}^{N-1}<\hat{f}(k),\overline{\hat{f}(k)}>e^{(\mu_k-\mu_l)n},\end{aligned}$$

Then

$$\begin{aligned}\sum_{n=0}^{N-1}|f(n)|^2 &= \sum_{k,l=0}^{N-1}\hat{f}(k)\overline{\hat{f}(l)}\sum_{n=0}^{N-1}e^{(\mu_k-\mu_l)n}\\ &= \sum_{k=l=0}^{N-1}|\hat{f}(k)|^2 N,\end{aligned}$$

So

$$\sum_{k=0}^{N-1}|\hat{f}(k)|^2=\frac{1}{N}\sum_{n=0}^{N-1}|f(n)|^2.$$

and the proof is complete.

Theorem 1 Eq.(1) has a unique N-periodic solution if and only if e^{μ_k} are not roots of the characteristic equation

$$\det\Delta(\mu)=0.\quad \mu_k=\frac{2k\pi}{N}i,\quad k\in\omega,\quad \Delta(\mu)=\mu I-\sum_{j=-\infty}^{+\infty}A(j)\mu^{-j}.$$

Proof Assume that Eq.(1) has a unique N-periodic solution $x(n)$. Since $x(n+N)=x(n), f(n+N)=f(n)$,by Lemma 1,$x(n), f(n)$ all can be uniquely expressed as $x(n)=\frac{1}{N}\sum\limits_{n=0}^{N-1}\hat{x}(k)e^{\mu_k n}$,and $f(n)=\frac{1}{N}\sum\limits_{n=0}^{N-1}\hat{f}(k)e^{\mu_k n}$,

Multiplying Eq.(1) by $e^{-\mu_k n}$ and summing from 0 to N-1 ,we obtain

$$\sum_{n=0}^{N-1}x(n+1)e^{-\mu_k n}=\sum_{n=0}^{N-1}\sum_{j=-\infty}^{+\infty}A(j)x(n-j)e^{-\mu_k n}+\sum_{n=0}^{N-1}f(n)e^{-\mu_k n},$$

Then

$$\sum_{n=0}^{N-1}x(n+1)e^{-\mu_k(n+1)}e^{\mu_k}=\sum_{j=-\infty}^{+\infty}A(j)\sum_{n=0}^{N-1}x(n-j)e^{-\mu_k(n-j)}e^{-\mu_k j}+N\hat{f}(k),$$

Hence

$$(e^{\mu_k}I-\sum_{j=-\infty}^{+\infty}A(j)e^{-\mu_k j})N\hat{x}(k)=N\hat{f}(k),$$

That is

$$\Delta(e^{\mu_k})\hat{x}(k)=\hat{f}(k). \tag{3}$$

Since linear equation

$$\Delta(e^{\mu_k})y = \hat{f}(k) \tag{4}$$

has solutions,assume that Eq.(4) has a unique solution ,then

$$\det \Delta(e^{\mu_k}) \neq 0. \tag{5}$$

Assume that Eq.(4) has many solutions, let $\hat{y}(k)$ is other solution of Eq.(4), It is immediate that $y(n) = \sum_{k=0}^{N-1} \hat{y}(k)e^{\mu_k n}$ satisfy the following equation

$$y(n+1) = \sum_{j=-\infty}^{+\infty} A(j)y(n-j) + \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}. \tag{6}$$

In addition, $x(n) = \sum_{k=0}^{N-1} \hat{x}(k)e^{\mu_k n}$ also satisfy the following equation

$$x(n+1) = \sum_{j=-\infty}^{+\infty} A(j)x(n-j) + \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}. \tag{7}$$

Let Eq.(6) subtract Eq.(7),we obtain

$$g(n+1) = \sum_{j=-\infty}^{+\infty} A(j)g(n-j), \tag{8}$$

where $g(n) = x(n) - y(n)$.Since Eq.(1)has a unique solution ,then the corresponding homogeneous linear Eq.(8) has a unique null solution, therefore Eq.(5) hold,that is , e^{u_k} are not roots of the characteristic equation $\det \Delta(\mu) = 0$.

On the other hand ,assume that $\det \Delta(e^{\mu_k}) \neq 0$ hold to each $k \in \omega$, then for every certain k ,linear Eq.(4) has a unique solution.That is ,to each $k \in \omega$,Eq.(4) uniquely define a $c(k)$ such that

$$\Delta(e^{\mu_k})c(k) = \hat{f}(k). \tag{9}$$

It is obvious that $z(n) = \sum_{k=0}^{N-1} c(k)e^{\mu_k n}$ satisfy Eq.(7). Let $\phi(n) = \sum_{k=0}^{N-1} \beta(k)e^{\mu_k n}$ is other solution of Eq.(7),then $\phi(n) - z(n)$ is a solution of the corresponding homogeneous linear Eq.(8) ,Since $\det \Delta(e^{\mu_k}) \neq 0$, hence Eq.(8) has a unique null solution,then $\phi(n) = z(n)$, that is ,$z(n)$ is the unique N-periodic solution of Eq.(8).

By Lemma 1, it is easy to know that Eq.(7) is the equivalent equation of Eq.(1) , hence $z(n) = \sum_{k=0}^{N-1} c(k)e^{\mu_k n}$ is the unique N-periodic solution of Eq.(1). and the proof is complete.

Next,We prove a more general result giving a necessary and sufficient condition for the existence of N-petiodic solutions also in the general case when $det\Delta(e^{\mu_k}) = 0$ for some integers.That is,

Theorem 2 Eq.(1) has a N-periodic solution if and only if $f(n) \in l_N(E)$. where $A(K) = \{a \in C^{n^*} | a\Delta(e^{\mu_k}) = 0\}, (k \in \omega), C^{n^*}$ is the space of n-dimension row vector , $l_N(E) = \{f \in l_N | a\hat{f}(k) = 0$ for all $a \in A(K), k \in \omega\}$.

Proof First assume that $x(n)$ is a N-periodic solution of Eq.(1). Multiplying Eq.(1) by $e^{-\mu_k n}$ and summing from 0 to N-1 ,we obtain

$$\Delta(e^{\mu_k})\hat{x}(k) = \hat{f}(k),$$

that is ,linear equation Eq.(4) has solutions.

From elementary linear algebra ,Eq.(4) has solutions if and only if $a\hat{f}(k) = 0$ for all $a \in A(K)$ such that $a\Delta(\mu_k) = 0$. Thus ,the existence of a N-periodic solution of Eq.(1) implies $f_n \in l_N(E)$.

On the other hand,Now assume that $f(n) \in l_N(E)$, then Eq.(4) has solutions. Choose $c(k)$ such that

$$\Delta(e^{\mu_k})c(k) = \hat{f}(k).$$

It is obvious that $z(n) = \sum_{k=0}^{N-1} c(k)e^{\mu_k n}$ is the N-periodic solution of

$$x(n+1) = \sum_{j=-\infty}^{+\infty} A(j)x(n-j) + \sum_{k=0}^{N-1} \hat{f}(k)e^{\mu_k n}.$$

By Lemma 1,Eq.(1) and Eq.(7) have same solutions,hence $\sum_{k=0}^{N-1} c(k)e^{\mu_k n}$ is N-periodic solution of Eq.(1).Therefore if $f_n \in l_N(E)$,then Eq.(1) has at least one N-periodic solution. and the proof is complete.

Finally,we consider the quasilinear delay difference equations

$$X(n+1) = \sum_{j=-\infty}^{+\infty} A(j)X(n-j) + G(n, X(n+\cdot)), \tag{10}$$

where $A(j) \in C^{n\times n}, X(n) \in C^n$, G is N-periodic with respect to its first variable n,and G maps bounded set to bounded set. Let $|\cdot|$ denote any norm of C^n, for any matrix $D \in C^{n\times n}$,$|D|$ denotes operator norm induced by the norm in C^n. From theorem 1,we know that Equation(1) has a unique N-periodic solution if and only if $det\Delta(e^{\mu_k}) \neq 0$ for all $k \in \omega$, at the same time , $x(n)$ can be expressed as following

$$x(n) = \sum_{k=0}^{N-1} \Delta^{-1}(e^{\mu_k})\hat{f}(k)e^{\mu_k n}. \tag{11}$$

On Eq.(10),we have

Theorem 3 Assume that $det\Delta(e^{\mu_k}) \neq 0$, for each $k \in \omega$, $G(n, \varphi)$ satisflies Lipschitz condition for $\varphi \in l_N$,and Lipschitz constant L such that

$$L^2 \sum_{k=0}^{N-1} |\Delta^{-1}(e^{\mu_k})|^2 < 1, \tag{12}$$

Then Eq.(10) has a unique N-periodic solution .

Considering the operator $E : l_N \to l_N$ defined by

$$Ef(n) = \sum_{k=0}^{N-1} \Delta^{-1}(e^{\mu_k})\hat{f}(k)e^{\mu_k n}. \tag{13}$$

Then it is easy to know $Ef(n)$ is the unique N-periodic solution od Eq.(1).

Lemma 3 Assume that $E : l_N \to l_N$ is the operator defined by Eq.(13),then E is a linear operator ,and

$$\|E\| \leq (\sum_{k=0}^{N-1} |\Delta^{-1}(e^{\mu_k})|^2)^{\frac{1}{2}},$$

where $\|E\|$ is the norm of the operator E.

Proof It is obvious that the operator E is linear operator.

By lemma 2 and Cauchy inequality, we have

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}|Ef(n)|^2 &= \sum_{k=0}^{N-1}\left|\Delta^{-1}(e^{\mu_k})\hat{f}(k)\right|^2 \\ &\le \left(\sum_{k=0}^{N-1}|\Delta^{-1}(e^{\mu_k})|^2\right)\left(\sum_{k=0}^{N-1}|\hat{f}(k)|^2\right) \\ &= \left(\frac{1}{N}\sum_{n=0}^{N-1}|f(n)|^2\right)\left(\sum_{k=0}^{N-1}|\Delta^{-1}(e^{\mu_k})|^2\right).\end{aligned}$$

Hence

$$\sum_{n=0}^{N-1}|Ef(n)|^2 \le \left(\sum_{n=0}^{N-1}|f(n)|^2\right)\left(\sum_{k=0}^{N-1}|\Delta^{-1}(e^{\mu_k})|^2\right).$$

Since

$$\|E\| = \sup_{f\neq 0}\frac{|Ef|}{\|f\|} = \sup_{\|f\|\neq 0}\frac{\left(\sum\limits_{n=0}^{N-1}|Ef(n)|^2\right)^{\frac{1}{2}}}{\left(\sum\limits_{n=0}^{N-1}|f(n)|^2\right)^{\frac{1}{2}}},$$

Then $\|E\| \le \left(\sum\limits_{k=0}^{N-1}|\Delta^{-1}(e^{\mu_k})|^2\right)^{\frac{1}{2}}$.

here $|Ef|$ denote the norm of the function $Ef(n)$.

The proof of theorem 3:

Considering the operator $T : l_N \to l_N$ defined by

$$(TF)(n) = G(n, Ef(n+\cdot)). \tag{14}$$

By thoorem 1,$f \in l_N$ is the fixed point of T if and only if $Ef(n)$ is a N-periodic solution of Eq.(1). So ,it suffices to prove that T has a unique fixed point in l_N.

For $f_1, f_2 \in l_N$, since

$$\begin{aligned}|Tf_1(n) - Tf_2(n)| &= |G(n, Ef_1(n+\cdot)) - G(n, Ef_2(n+\cdot))| \\ &\le L\|Ef_1 - Ef_2\| \\ &\le L\left(\sum_{k=0}^{N-1}|\Delta^{-1}(e^{u_k})|^2\right)^{\frac{1}{2}}\|f_1 - f_2\|,\end{aligned}$$

Then ,by (12),$T : l_N \to l_N$ is a contraction mapping. By contraction mapping principle,T has a unique fixed point f^* in l_N. Hence ,Ef^* is the unique N-periodic solution of Eq.(10). and the proof is complete.

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一类离散系统周期解的存在性

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摘要 利用三角级数和压缩映射原理，研究了离散系统 $X_{n+1} = \sum_{j=-\infty}^{+\infty} A_j X_{n-j} + f(n, x(n+\cdot))$ 和 $X(n+1) = \sum_{j=-\infty}^{+\infty} A_j X_{n-j} + G(n, X(n+\cdot))$, 得到了前者存在周期解的充要条件及后者存在唯一周期解的充分条件.

Keywords 离散系统； 周期解； 三角级数理论； 压缩映射原理

Subject Classification O175.14