An overview of f(R) theories

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Abstract. A brief introduction to theories of the gravitational field with a Lagrangian that is a function of the scalar curvature is given. The emphasis will be placed in formal developments, while comparison to observation will be discussed in the chapter by S. Jorás in this volume.

Keywords: modified theories of gravity, cosmology, dark energy

INTRODUCTION

The predictions of General Relativity (GR) are confirmed to an impressive degree by observations in number of situations [1]. In spite of this fact, theories that differ from GR either in the limit of low or high curvature have been intensively studied lately, and have a long tradition, starting with a paper by Weyl in 1918 [2]. Although Weyl's motivation was related to the unification of GR and Electrodynamics, the current revival of these theories is twofold. In the case of low curvature, the aim is to describe the accelerated expansion of the universe that follows from several observations [3] (when interpreted in the standard cosmological model [4]) [1]. Regarding the high-curvature regime, it is important to note that there is no observational evidence of the behaviour of the gravitational field for very large values of the curvature. This makes objects such as black holes and neutron stars the ideal places to look for deviations from General Relativity in the strong regime. In fact, the Kerr solution is not unique in $f(R)$ theories [33]. Consequently, any deviation from Kerr's spacetimee in compact objects will be unequivocally signaling the need of changes in our description of strong gravity. The task of understanding what kind of deviations can be expected, and their relation to observable quantities is of relevance in view of several developments that offer the prospect of observing properties of black holes in the vicinity of the horizon [11].

In this short review we will be concerned with gravitational theories described by the action

$$S = \int d^4x \sqrt{-g} f(R), \tag{1}$$

where g is the determinant of the metric $g_{\mu\nu}$ and f is an arbitrary function of the the curvature scalar R [2]. The function f must satisfy certain constraints, some of which

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[1] The possibility of describing the current accelerated expansion of the universe using $f(R)$ theories was first discussed in [5].

[2] This choice if favoured by a theorem by Ostrogradski [12] over Lagrangians built with invariants obtained from the the Ricci and Riemann tensors.

are necessary for the theory to be well-defined *ab initio*, and others to account for observational facts. Those in the first class will be discussed in this review, while those in the second class are presented in the chapter by S. Jorás in this volume [3]. We shall begin by reviewing in the next section some general features of this type of theories.

THE THREE VERSIONS OF $f(R)$ THEORIES

We shall see in this section that $f(R)$ theories can be classified in three different types, according to the role attibuted to the connection. In all of the versions, the equation for the energy-momentum conservation is valid, since the total (gravitational plus matter) action is diffeomorphism-invariant and gravity and matter are minimaly coupled by hypothesis (see for instance [10], [21]).

Metric version

In the metric version of $f(R)$ theories, the action

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_{\rm M}(g_{\mu\nu}, \psi), \tag{2}$$

is varied with respect to $g_{\mu\nu}$. Here, $S_{\rm M}$ is the matter action, which is independent of the connection. The resultant equations of motion are of fourth order in the derivatives of the metric tensor:

$$\frac{df(R)}{dR} R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right] \frac{df(R)}{dR} = \kappa T_{\mu\nu}, \tag{3}$$

where $T_{\mu\nu}$ is the energy-momentum of he matter fields, defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\rm M}}{\delta g^{\mu\nu}},$$

and the covariant derivative is defined using the usual Levi-Civita connection. Taking the trace, we obtain

$$\frac{df(R)}{dR} R - 2f(R) + 3\Box \frac{df(R)}{dR} = \kappa T,$$

which is to be compared to $R = -\kappa T$, the result in GR.

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[3] There are several reviews that deal with different aspects of $f(R)$ theories, see [6],[7],[8], and also the recent book [9].

Equivalence with Brans-Dicke theory

As shown for instance in [13], the gravitational part of the action given in Eqn.(2) is equivalent to the following action:

$$S = \int d^4x \sqrt{-g} \left[\frac{\phi R}{2\kappa} - U(\phi) \right], \tag{4}$$

where

$$U(\phi) = \frac{\phi \chi(\phi) - f(\chi(\phi))}{2\kappa}, \tag{5}$$

$\phi = f_{,\chi}(\chi)$, and $\chi = R$, corresponding to a Brans-Dicke theory with $\omega = 0$ [15]. Note that the absence of a kinetic term for the scalar field does not mean that it is non-dynamical: its evolution, due to the non-minimal coupling with the gravitational field, is given by the variation of the action wrt ϕ:

$$3 \Box \phi + 2U(\phi) - \phi \frac{dU}{d\phi} = \kappa T. \tag{6}$$

Through a conformal transformation of the metric and a redefinition of the scalar field, the action given in Eqn.(4) can be taken to that of a scalar field minimally coupled with gravity, and with nonzero kinetic term and potential. These representations of $f(R)$ theories show that there is a massive scalar degree of freedom, which manifests as a longitudinal mode in gravitational radiation (see for instance [23] for the cosmological case).

We close this section by stating that the equivalent representations are convenient since the associated equations of motion are of order two, but a word of caution is needed because sometimes the potential in Eqn.(5) is typically multivalued (see for instance [16][17]). It may be better to work directly in the original representation, as for instance in Ref.[14] in the case of compact stars.

Palatini version

In this second type of $f(R)$ theories, the metric and the connection are taken as independent fields, and the matter action S_{M} is independent of the connection. So the starting point is the action

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, f(\mathscr{R}) + S_{\mathrm{M}}(g_{\mu\nu}, \psi), \tag{7}$$

where $\mathscr{R} = g^{\mu\nu} \mathscr{R}_{\mu\nu}$, and the corresponding Riemann tensor is constructed with a connection Γ a priori independent of the metric.

From the variation of the action wrt the metric and Γ we get [4] (see for instance [8])

$$f'(\mathscr{R})\mathscr{R}_{(\mu\nu)} - \frac{1}{2}f(\mathscr{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{8}$$

$$\bar{\nabla}_\lambda(\sqrt{-g}\,f'(\mathscr{R})g^{\mu\nu}) = 0, \tag{9}$$

where the prime denotes derivative wrt $\mathscr{R}$, and the barred covariant derivative is built with the connection Γ. GR is recovered by setting $f(\mathscr{R}) = \mathscr{R}$ in these equations. Taking the trace of Eqn.(8) we obtain

$$f'(\mathscr{R})\mathscr{R} - 2f(\mathscr{R}) = \kappa T, \tag{10}$$

which shows that in this case the relation between $\mathscr{R}$ and T is algebraic, hence no scalar mode is present.

From Eqn.(9), it follows that [8]

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{f'(\mathscr{R})}g^{\lambda\sigma}\left[\partial_\mu(f'(\mathscr{R})g_{\nu\sigma}) + \partial_\nu(f'(\mathscr{R})g_{\mu\sigma}) - \partial_\sigma(f'(\mathscr{R})g_{\mu\nu})\right]. \tag{11}$$

Since this expression relates Γ to $\mathscr{R}$ and the metric, and $\mathscr{R}$ and T are in principle interchangeable through Eqn.(10), the connection can be in principle expressed in terms of the matter fields and the metric. In other words, it is an auxiliary field. In fact, Eqn.(8) can be rewritten as

$$\begin{aligned}
G_{\mu\nu} = & \frac{\kappa}{f'}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(\mathscr{R} - \frac{f}{f'}\right) + \frac{1}{f'}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f' \\
& -\frac{3}{2}\frac{1}{f'^2}\left[(\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2}g_{\mu\nu}(\nabla f')^2\right]
\end{aligned} \tag{12}$$

where the Einstein tensor and the covariant derivatives are built with the Levi-Civita connection, and $\mathscr{R}$ is expressed in terms of T using Eqn.(10). It follows that this version of $f(R)$ theories can be interpreted as GR with a modified source. Perhaps the most important modification is that third order derivatives of the matter fields appear on the rhs of Eqn.(13). As reported in [24], this feature may cause serious problems in static spherically symmetric solutions with a polytropic fluid with index $3/2 < \gamma < 2$ as a source. Note however that this result result was challenged in the review [25] [5].

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[4] In the case of GR, this method furnishes the same result as the metric case, but this is not the case in more general theories as discussed for example in [19].

[5] It has also been claimed that the Cauchy problem is not well-posed for the Palatin version of $f(R)$ theories [8]. For an updated discussion see [26].

Metric-affine theories

In this case, the matter action depends of the connection, which is a priori independent of the metric. The action is given by

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\mathcal{R}) + S_{\mathrm{M}}(g_{\mu\nu}, \Gamma^\lambda_{\mu\nu}, \psi). \tag{13}$$

Depending on the matter fields, the theory may display non-propagating torsion and non-metricity (see [8] and [35] for details).

NONMINIMAL COUPLING

Metric $f(R)$ theories have been generalized by allowing a nonminimal coupling between the curvature and the matter Lagrangian, with action given by

$$S = \int \left\{ \frac{1}{2} f_1(R) + [1 + \lambda f_2(R)] \mathcal{L}_{\mathrm{m}} \right\} \sqrt{-g} d^4x, \tag{14}$$

where f_1 and f_2 are arbitrary functions of R, and λ is a constant. A particular case of this action was considered in [27] in the context of the accelerated expansion of the universe. Later, it was shown in [28] that this type of theory leads to a modification of the conservation law of the matter energy-momentum tensor, namely

$$\nabla^\mu T^{(\mathrm{m})}_{\mu\nu} = \frac{\lambda}{1 + \lambda f_2} f_2' \left[g_{\mu\nu} \mathcal{L}_{\mathrm{m}} - T^{(\mathrm{m})}_{\mu\nu} \right] \nabla^\mu R. \tag{15}$$

The presence of a nonzero rhs leads to non-geodesic motion, and it was suggested in [28] that this may be related to MOND.

A more general type of theories was propposed in [29], with action given by

$$S = \int f(R, \mathcal{L}_{\mathrm{m}}) \sqrt{-g} d^4x, \tag{16}$$

where f is an arbitrary function of R and of the matter Lagrangian. As in the previous case, an extra force, perpendicular to the 4-velocity, accelerates the particles.

ASSORTED APPLICATIONS

$f(R)$ theories have been used to describe different aspects of relativistic astrophysics and cosmology. Since the low curvature limit, which has been studied primarily to explain the accelerated expansion of the universe, is discussed in the chapter by S. Joras in this volume, only one example will be given here in this regime. Afterwards, some applications in the strong-curvature regime will be discussed.

Low curvature

In the case of the $k = 0$ Friedmann-Lemâitre-Robertson-Walker metric, the EOM (3) can be written as

$$\rho = -f' R_{tt} - \frac{f}{2} + 3f'' \frac{\dot{a}\dot{R}}{a}, \tag{17}$$

$$p = -\frac{f'}{3}(R_{tt} + R) + \frac{f}{2} - f''\left(\ddot{R} - \frac{2\dot{a}\dot{R}}{a}\right) - f''' \dot{R}^2. \tag{18}$$

Let us remark that it is safe to asssume that most of the current matter content of the universe (assumed here to be normal matter, as opposed to dark energy) is pressureless. This matter must satisfy the conditions $\rho_0 \geq 0$ and $p_0 = 0$, where the subindex 0 means that the quantity is evaluated today. Using Eqns.(17) and (18), we shall rewrite these conditions in terms of following kinematical parameters: the Hubble and deceleration parameters, the jerk, and the snap, respectively given by [41]

$$H = \frac{\dot{a}}{a}, \qquad q = -\frac{1}{H^2}\frac{\ddot{a}}{a}, \qquad j = \frac{1}{H^3}\frac{\dddot{a}}{a}, \qquad s = \frac{1}{H^4}\frac{\ddddot{a}}{a}.$$

While the current value of the first two parameters is relatively well-established today, the value of j_0 is not known with high precision, and no acceptable value of s_0 has been reported yet [42]. By writing $\rho_0 \geq 0$ in terms of the kinematical parameters we get

$$3q_0 H_0^2 f_0' - \frac{f_0}{2} - 18H_0^4 f_0''(j_0 - q_0 - 2) \geq 0. \tag{19}$$

This inequality gives a relation that the parameters and the derivatives of a given $f(R)$ must satisfy today and, as shown in [43], it limits the possible values of the parameters of a given theory. Notice that Eqn.(18) involves the snap (through $\ddot{R}$). If we had a measurement of s_0, we could use the equation $p_0 = 0$ to obtain another constraint on $f(R)$. Since this is not the case, we shall express $p_0 = 0$ in such a way that it gives a forecast fo the possible current values of the snap for a given $f(R)$:

$$s_0 = \frac{f_0'}{6H_0^2 f_0''}(q_0 - 2) + 6H_0^2 \frac{f_0'''}{f_0''}(-q_0 + j_0 - 2)^2 - [q_0(q_0 + 6) + 2(1 + j_0)] - \frac{f_0}{12H^4 f_0''}. \tag{20}$$

Strong curvature

- The possibility of nonsingular cosmological solutions in $f(R)$ theories has been considered in [30] and [31]. In the latter article, a necessary condition for a bounce to occur in a Friedmann-Lemâtre-Robertson-Walker setting was obtained, and it is given by

$$\frac{\ddot{a}_0}{a_0} = -\frac{\rho_0}{f_0'} + \frac{f_b}{2f_0'}, \tag{21}$$

with

$$R_0 = 6 \left(\frac{\ddot{a}_0}{a_0} + \frac{K}{a_0^2} \right),$$ (22)

and the subindex b means that the quantity is evaluated at the bounce. Contrary to the case of GR, a bounce may occur for any value of K.

- It was shown in [32] that the theory given by

$$f(R) = R + \frac{R^2}{(6M)^2}$$ (23)

has an inflationary solution given by

$$H \approx H_i - \frac{M^2}{6}(t - t_i),$$

$$a \approx a_i \exp \left[H_i(t - t_i) - \frac{M^2}{12}(t - t_i)^2 \right],$$

where t_i marks the beginning of the inflationary epoch. Several features of this model have been studied in detail (see references in [7]). The results of WMAP constraint $M \approx 10^{13}$ GeV, and the spectral index for this model is $n_{\mathscr{R}} \approx 0.964$, which is in the range allowed by WMAP 5-year constraint. The tensor to scalar ratio r also satisfies the current observational bound, but is different from that of the chaotic inflation model. Hence, future observations such as the Planck satellite may be able to discriminate between these two models.

- Compact stars have been repeatedly studied for a number of $f(R)$ theories, either in the conformal representation [36], or directly in the fourth-order version (see for instance [14]).

- Regarding black holes in $f(R)$ theories, it was shown in [33] that an observational verification of the Kerr solution for an astrophysical object cannot be used in distinguishing between GR and $f(R)$ theories. Hence, the observation of deviations from the Kerr spacetime may point to changes in our understanding of gravitation. Other features of black holes in $f(R)$ theories have been analyzed in [34].

CONCLUSIONS

In this short review, I intended to show that several aspects of $f(R)$ theories (in its various representations) have been extensively studied in the literature. There are many other aspects that I had to left aside such as "good propagation" (*i.e.* absence of shocks) [37], the loop representation [38], and the Hamiltonian representation [39]. Although the low-curvature regime and its consequences has attracted a lot of attention due to its possible relevance in Cosmology, the high-curvature regime is also of interest independently of the low-curvature regime, and the consequences of a modification in such regime may be testable in the near future (as in the inflationary model in [32], and through the observation of electromagnetic [11] and gravitational waves [40] in the case of compact objects).

ACKNOWLEDGMENTS

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