# 修正Sawada-Kotera 方程的推广及守恒律

耿献国 ，何国亮

郑州大学数学系，郑州 450001

**摘要**：从修正Sawada-Kotera 方程出发，引入一个具有两个位势的$3\times 3$ 的矩阵谱问题，并由此得到一族新的非线性演化方程，该族中第二个方程就是修正Sawada-Kotera 方程的推广形式，从而给出了修正Sawada-Kotera 方程的Lax 对。而且，借助于该族中前两个非线性演化方程和修正Sawada-Kotera 方程的Lax 对，得到了他们的无穷多个守恒量。

**关键词**：基础数学、谱问题、可积非线性演化方程、守恒律

**中图分类号**： O175.29

# An extension of the modified Sawada-Kotera equation and conservation laws

Xianguo Geng, Guoliang He

[1] Department of Mathematics, Zhengzhou University, Zhengzhou, Henan 450001

**Abstract:** Based on the modified Sawada-Kotera equation, we introduce a $3\times 3$ matrix spectral problem with two potentials and derive a hierarchy of new nonlinear evolution equations. The second member in the hierarchy is a generalization of the modified Sawada-Kotera equation, by which a Lax pair of the modified Sawada-Kotera equation is obtained. Moreover, infinite sequences of conserved quantities of the first two nonlinear evolution equations in the hierarchy and the modified Sawada-Kotera equation are constructed with the aid of their Lax pairs.

**Key words:** fundamental mathematics; spectral problem; integrable nonlinear evolution equations; conservation laws.

## 0 Introduction

It is well known that the two important fifth-order nonlinear evolution equations, the Sawada-Kotera (SK) equation [1-3]

$$q_t = q_{5x} + 5qq_{3x} + 5q_xq_{xx} + 5q^2q_x \tag{1}$$

and the Kaup-Kupershmidt (KK) equation [4, 5]

$$r_t = r_{5x} + 10rr_{3x} + 25r_xr_{xx} + 20r^2r_x, \tag{2}$$

基金项目： Research supported by SRFDP (200804590008) and NSFC (10871182)

作者简介： 耿献国（1957-），男，教授，主要研究方向：孤立子与可积系统。通信作者：何国亮（1983-），男，博士，主要研究方向：孤立子与可积系统。邮箱：glhemath@163.com

have been studied extensively; see Refs. 6-13, just to mention a few. For example, research has been conducted on their Hamiltonian structures, Lax pairs, Bäcklund transformations, bilinear transformations, soliton solutions and other properties. Equations (1) and (2) are fundamentally different despite their evident duality. Another fifth-order nonlinear evolution equation,

$$v_t = v_{5x} - (5v_x v_{xx} + 5vv_x^2 + 5v^2 v_{xx} - v^5)_x, \tag{3}$$

possesses the close connection with (1) and (2). If $v$ is a solution of (3), then functions $q$ and $r$ determined by the Miura transformations,

$$q = v_x - v^2, \quad r = -v_x - \tfrac{1}{2}v^2, \tag{4}$$

satisfy the SK equation (1) and the KK equation (2), respectively.[11] Equation (3) is called a modified SK equation.

In this paper, we introduce a $3 \times 3$ matrix spectral problem, from which a hierarchy of new nonlinear evolution equations is derived. The first typical member in the hierarchy is

$$\begin{aligned} u_t &= u_{xx} + 2uu_x - 2(uv)_x - 2v_{xx}, \\ v_t &= -v_{xx} + 2vv_x - 2(uv)_x + 2u_{xx}, \end{aligned} \tag{5}$$

and the second member is reduced to the modified SK equation (3) as $u = 2v$. Based on these results, a Lax pair of the modified SK equation and its explicit solutions are obtained. Moreover, infinite sequences of conserved quantities of the first two nonlinear evolution equations in the hierarchy and the modified SK equation are constructed with the aid of their Lax pairs.

## 1 Nonlinear evolution equations

In this section, we shall derive a hierarchy of new nonlinear evolution equations. We first introduce a $3 \times 3$ matrix spectral problem

$$\phi_x = U\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad U = \begin{pmatrix} u & 1 & 0 \\ 0 & v & \lambda \\ 1 & 0 & 0 \end{pmatrix}, \tag{6}$$

where $u$ and $v$ are two potentials, $\lambda$ a constant spectral parameter. In order to derive the hierarchy of nonlinear evolution equations associated with the spectral problem (6), we solve the stationary zero-curvature equation with the help of the approach in Refs. 15, 16:

$$V_x - [U, V] = 0, \quad V = (V_{ij})_{3\times 3}, \tag{7}$$

which is equivalent to

$$\begin{aligned}
V_{11,x}+V_{13}-V_{21}=0,\\
V_{12,x}+(v-u)V_{12}+V_{11}-V_{22}=0,\\
V_{13,x}+\lambda V_{12}-uV_{13}-V_{23}=0,\\
V_{22,x}+V_{21}-\lambda V_{32}=0,\\
V_{21,x}+(u-v)V_{21}+V_{23}-\lambda V_{31}=0,\\
V_{23,x}-vV_{23}+\lambda(V_{22}-V_{33})=0,\\
V_{31,x}+uV_{31}+V_{33}-V_{11}=0,\\
V_{32,x}+vV_{32}+V_{31}-V_{12}=0,\\
V_{33,x}+\lambda V_{32}-V_{13}=0,
\end{aligned} \tag{8}$$

where each entry $V_{ij}=V_{ij}(A,B,C,D)$ is a function of $A,B,C,D$ :

$$\begin{aligned}
V_{11}&=(\partial+u)B-(\partial^2+\partial v+u\partial+uv)A+\lambda C,\\
V_{12}&=B,\quad V_{13}=\lambda(A+\partial C),\quad V_{21}=\lambda D,\\
V_{22}&=(2\partial+v)B-(\partial^2+\partial v+u\partial+uv)A+\lambda C,\\
V_{23}&=\lambda(\partial-u)A+\lambda B+\lambda(\partial^2-u\partial)C,\\
V_{31}&=B-(\partial+v)A,\quad V_{32}=A,\quad V_{33}=\lambda C
\end{aligned} \tag{9}$$

with $\partial=\partial_x$. Substituting (9) into (8) yields

$$\begin{aligned}
(2\partial+v-u)A+(\partial^2-u\partial)C+(\partial+u-v)D=0,\\
(\partial u+u\partial+\partial v+v\partial)A-3\partial B-(\partial^3-\partial u\partial-v\partial^2+uv\partial)C=0,\\
(\partial^2+\partial u)B-(\partial^3+\partial^2 v+\partial u\partial+\partial uv)A+\lambda(2\partial C+A-D)=0,\\
(2\partial^2+\partial v)B-(\partial^3+\partial^2 v+\partial u\partial+\partial uv)A+\lambda(\partial C-A+D)=0.
\end{aligned} \tag{10}$$

The functions $A,B,C$ and $D$ are expanded into a Laurent series in $\lambda$:

$$A=\sum_{j\geq 0}A_j\lambda^{-j},\quad B=\sum_{j\geq 0}B_j\lambda^{-j},\quad C=\sum_{j\geq 0}C_j\lambda^{-j},\quad D=\sum_{j\geq 0}D_j\lambda^{-j}. \tag{11}$$

Substituting (11) into (10) and comparing coefficients for the same power of $\lambda$ yield the recursion equations $(j\geq 0)$

$$\begin{aligned}
&2\partial C_0+A_0-D_0=0,\quad \partial C_0-A_0+D_0=0,\\
&(2\partial+v-u)A_j+(\partial^2-u\partial)C_j+(\partial+u-v)D_j=0,\\
&(\partial u+u\partial+\partial v+v\partial)A_j-3\partial B_j-(\partial^3-\partial u\partial-v\partial^2+uv\partial)C_j=0,\\
&(\partial^2+\partial u)B_j-(\partial^3+\partial^2 v+\partial u\partial+\partial uv)A_j+2\partial C_{j+1}+A_{j+1}-D_{j+1}=0,\\
&(2\partial^2+\partial v)B_j-(\partial^3+\partial^2 v+\partial u\partial+\partial uv)A_j+\partial C_{j+1}-A_{j+1}+D_{j+1}=0.
\end{aligned} \tag{12}$$

Then $A_j, B_j, C_j$ and $D_j$ are recursively determined by (12). The recursion equation (12) can be solved successively to deduce the results

$$\begin{aligned}
&A_0 = D_0 = \alpha_0, \quad B_0 = \tfrac{1}{3}\alpha_0(u+v) + \alpha_1, \quad C_0 = \beta_0,\\
&C_1 = \tfrac{1}{9}\alpha_0(-3u_x + 3v_x - u^2 - v^2 + 4uv) - \tfrac{1}{3}\alpha_1(u+v) + \beta_1,\\
&A_1 = \tfrac{1}{9}\alpha_0[2u_{xx} - v_{xx} + 2vv_x - 3u_xv - uv_x - \tfrac{4}{9}(u^3+v^3) + \tfrac{2}{3}(uv^2 + u^2v)]\\
&\qquad -\tfrac{1}{9}\alpha_1(-3v_x + u^2 + v^2 - uv) + \alpha_2,\\
&D_1 = \tfrac{1}{9}\alpha_0[-u_{xx} - v_{xx} + 2uu_x - 2vv_x - u_xv + uv_x - \tfrac{4}{9}(u^3+v^3) + \tfrac{2}{3}(uv^2+u^2v)]\\
&\qquad -\tfrac{1}{9}\alpha_1(3v_x - 3u_x + u^2 + v^2 - uv) + \alpha_2,\\
&B_1 = \tfrac{1}{27}\alpha_0[3(u-v)_{xxx} + 3(u-v)(u-v)_{xx} - (2u^2+2v^2+uv)(u-v)_x + u_x^2 + v_x^2\\
&\qquad -7u_xv_x + \tfrac{1}{3}u^2v^2 + \tfrac{8}{9}uv(u^2+v^2) - \tfrac{7}{9}(u^4+v^4)] + \tfrac{1}{3}\alpha_2(u+v)\\
&\qquad -\tfrac{1}{27}\alpha_1[\tfrac{5}{3}(u^3+v^3) - 3(u+v)_{xx} + 3(u+v)(u-v)_x - uv^2 - u^2v] + \beta_2,
\end{aligned} \tag{13}$$

where $\alpha_j$ and $\beta_j$, $(j = 0, 1, 2)$, are arbitrary constants.

We now assume that $\phi$ satisfies the spectral problem (6) and an auxiliary problem

$$\phi_{t_m} = V^{(m)}\phi, \tag{14}$$

where each entry $V_{ij}^{(m)} = V_{ij}(A^{(m)}, B^{(m)}, C^{(m)}, D^{(m)})$ in the matrix $V^{(m)}$ is a polynomial of the spectral parameter $\lambda$ with

$$A^{(m)} = \sum_{j=0}^{m} A_j\lambda^{m-j}, \quad B^{(m)} = \sum_{j=0}^{m} B_j\lambda^{m-j}, \quad C^{(m)} = \sum_{j=0}^{m} C_j\lambda^{m-j}, \quad D^{(m)} = \sum_{j=0}^{m} D_j\lambda^{m-j}. \tag{15}$$

Then the compatibility condition of (6) and (14) yields the zero-curvature equation, $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to a hierarchy of new nonlinear evolution equations

$$\begin{aligned}
u_{t_m} &= (\partial^2 + \partial u)B_m - (\partial^3 + \partial^2 v + \partial u\partial + \partial uv)A_m,\\
v_{t_m} &= (2\partial^2 + \partial v)B_m - (\partial^3 + \partial^2 v + \partial u\partial + \partial uv)A_m, \quad m \geq 0.
\end{aligned} \tag{16}$$

The first nontrivial member in the hierarchy (16) is

$$\begin{aligned}
u_{t_0} &= \tfrac{\alpha_0}{3}[u_{xx} + 2uu_x - 2(uv)_x - 2v_{xx}] + \alpha_1u_x,\\
v_{t_0} &= \tfrac{\alpha_0}{3}[-v_{xx} + 2vv_x - 2(uv)_x + 2u_{xx}] + \alpha_1v_x,
\end{aligned} \tag{17}$$

which is just (5) as $\alpha_0 = 3$ and $\alpha_1 = 0$. And the second one is as follows

$$\begin{aligned}
u_{t_1} = \;& \tfrac{1}{243}\alpha_0[-7u^5 + 20u^4v - 15u^3v^2 - 10u^2v^3 + 5uv^4\\
&-5(2u^3 + 2v^3 - 3u^2v - 3uv^2 + 9vu_x - 9v_{xx})(u-2v)_x\\
&+45(u^2 + v^2 - uv + u_x + v_x)u_{xx} + 45u(u_x^2 - v_x^2) - 27u_{xxxx}]_x,\\
v_{t_1} = \;& \tfrac{1}{243}\alpha_0[-7v^5 + 20v^4u - 15v^3u^2 - 10v^2u^3 + 5vu^4\\
&-5(2u^3 + 2v^3 - 3u^2v - 3uv^2 - 9uv_x - 9u_{xx})(2u-v)_x\\
&+45(u^2 + v^2 - uv - u_x - v_x)v_{xx} - 45v(u_x^2 - v_x^2) - 27v_{xxxx}]_x
\end{aligned} \tag{18}$$

with taking $\alpha_1 = \alpha_2 = \beta_2 = 0$ for the sake of simplicity. It is easy to see that (18) is exactly reduced to the modified SK equation (3) if we choose $u = 2v$ and $\alpha_0 = -9$.

## 2 The Lax pair of the modified SK equation and conservation laws

In this section, we shall construct a Lax pair of the modified SK equation (3) and discuss infinite sequences of conserved quantities of equations (5), (18) and (3). Based on the above result, we deduce a Lax pair given by spectral problems

$$L\psi = \lambda\psi, \quad \psi_t = B\psi, \tag{19}$$

where $L$ and $B$ are two differential operators by

$$\begin{aligned} L =& \ \partial^3 - 3v\partial^2 + 2(v^2 - v_x)\partial, \\ B =& \ -9\partial^5 + 45v\partial^4 + 75(v_x - v^2)\partial^3 + (60v_{xx} - 165vv_x + 45v^3)\partial^2 \\ & +(20v_{xxx} - 45v_x^2 - 70vv_{xx} + 70v^2v_x - 5v^4)\partial. \end{aligned} \tag{20}$$

Then a direct calculation shows that the Lax equation, $L_t = [B, L]$, gives rise to the modified SK equation (3). Assume that $v$ is a solution of the modified SK equation (3). Then functions $q$ and $r$ determined by the Miura transformations (4) (see Ref. 14) satisfy respectively the SK equation (1) and the KK equation (2). In fact, a direct calculation gives that

$$\begin{aligned} & q_t - (q_{5x} + 5qq_{3x} + 5q_xq_{xx} + 5q^2q_x) \\ & \quad = (\partial - 2v)\{v_t - v_{5x} + 5(v_xv_{xx} + vv_x^2 + v^2v_{xx})_x - 5v^4v_x\}, \end{aligned} \tag{21}$$

$$\begin{aligned} & r_t - (r_{5x} + 10rr_{3x} + 25r_xr_{xx} + 20r^2r_x) \\ & \quad = -(\partial + v)\{v_t - v_{5x} + 5(v_xv_{xx} + vv_x^2 + v^2v_{xx})_x - 5v^4v_x\}. \end{aligned} \tag{22}$$

With the help of the transformation in Ref. 14, we obtain two explicit solutions of the modified SK equation (3)

$$v = -a\frac{c_1\sinh(ax+a^5t+\delta_0)+c_2\cosh(ax+a^5t+\delta_0)}{c_1\cosh(ax+a^5t+\delta_0)+c_2\sinh(ax+a^5t+\delta_0)}, \tag{23}$$

and

$$v = a\frac{c_1\sin(ax+a^5t+\delta_0)+c_2\cos(ax+a^5t+\delta_0)}{c_1\cos(ax+a^5t+\delta_0)-c_2\sin(ax+a^5t+\delta_0)}, \tag{24}$$

where $a$, $c_1$, $c_2$ and $\delta_0$ are arbitrary constants and $a > 0$. Under the Miura transformations (4), functions given by (23) and (24) correspond to trivial solutions $q = \pm a^2$ of the SK equation (1) and two nontrivial solutions of the KK equation (2):

$$r = \frac{2a^2(c_1^2-c_2^2)-a^2[c_1\sinh(ax+a^5t+\delta_0)+c_2\cosh(ax+a^5t+\delta_0)]^2}{2[c_1\cosh(ax+a^5t+\delta_0)+c_2\sinh(ax+a^5t+\delta_0)]^2}, \tag{25}$$

and

$$r = -\frac{2a^2(c_1^2+c_2^2)+a^2[c_1\sin(ax+a^5t+\delta_0)+c_2\cos(ax+a^5t+\delta_0)]^2}{2[c_1\cos(ax+a^5t+\delta_0)-c_2\sin(ax+a^5t+\delta_0)]^2}. \tag{26}$$

In the following, we shall derive infinitely many conservation laws of (5), (18) and (3). We first eliminate $\phi_1$ and $\phi_2$ from (6) by writing the column vector $\phi$ in components as $\phi = (\phi_1, \phi_2, \phi_3)^T$ to get a single equation for $\psi = \phi_3$, namely

$$\psi_{xxx} - (u+v)\psi_{xx} + (uv - u_x)\psi_x - \lambda\psi = 0. \tag{27}$$

If we set $\rho = (\ln\psi)_x$ in (27), it is clear that $\rho$ satisfies

$$\rho_{xx} + 3\rho\rho_x + \rho^3 - (u+v)(\rho_x + \rho^2) + (uv - u_x)\rho - \lambda = 0. \tag{28}$$

Then the corresponding $t$ evolution of $\psi$ implies a conservation law for $\rho$, which is

$$\rho_t = \theta_x, \quad \theta = V_{31}^{(m)}\rho + V_{32}^{(m)}(\rho_x + \rho^2 - u\rho) + V_{33}^{(m)}. \tag{29}$$

**Case 1**. In order to get conservation laws of (5), we choose $V_{31}^{(0)} = u - 2v, V_{32}^{(0)} = 3, V_{33}^{(0)} = 0$. In this case, $\theta$ can be written as

$$\theta = -2(u+v)\rho + 3(\rho_x + \rho^2). \tag{30}$$

In (28) and (30), functions $\rho$ and $\theta$ can be expanded as a series in powers of $\lambda$ with the coefficients which are called conserved densities ($\lambda = \eta^3$):

$$\begin{aligned} &\rho = \eta + \sum_{j=0}^{\infty} \rho_j \eta^{-j}, \\ &\theta = 3\eta^2 + \sum_{j=0}^{\infty} \theta_j \eta^{-j}. \end{aligned} \tag{31}$$

Substituting the expansions (31) into (28), (30) and comparing the same power of $\lambda$ yield two infinite sequences of conserved quantities for (5). The first some densities and currents are given by

$$\begin{aligned}
\rho_0 =\ & \tfrac{1}{3}(u+v), \\
\rho_1 =\ & \tfrac{1}{9}(u^2 + v^2 - uv - 3v_x), \\
\rho_2 =\ & \tfrac{1}{81}[9(u_x - v_x)(2v - u) + 9(2v-u)_{xx} - 3uv(u+v) + 2(u^3+v^3)], \\
\rho_3 =\ & \tfrac{1}{27}[u_x^2 + 4v_x^2 - 7u_xv_x + (v^2 - 2u^2 + 2uv)u_x + (u^2 - 2v^2 + 2uv)v_x \\
& +(u-5v)u_{xx} + (4v-2u)v_{xx} + 3(u-v)_{xxx})], \\
\rho_4 =\ & \tfrac{1}{729}[-2(u^5+v^5) + (u^3+v^3)(6v_x - 9u_x + 5uv) - (u+v)(2u^2v^2 + 9uvv_x + 45u_xv_x) \\
& -9v(4u_x^2 - 8v_x^2 - 9u_{xxx} + 6v_{xxx}) + 9u(8u_x^2 - 4v_x^2 + 3v_{xxx}) + 27(v-2u)_{xxxx} \\
& +9(5u^2 - v^2 - 5uv + 18v_x)u_{xx} + 9(5v^2 - u^2 - 5uv - 21v_x + 15u_x)v_{xx}], \\
\theta_0 =\ & \tfrac{1}{3}(u^2 + v^2 - 4uv + 3u_x - 3v_x), \\
\theta_1 =\ & \tfrac{1}{27}[4(u^3+v^3) - 6uv(u+v) + 27vu_x + 9(u-2v)v_x + 9(v-2u)_{xx}].
\end{aligned} \tag{32}$$

The recursion relations for $\rho_j$ and $\theta_j$, $(j \geq 0)$, are

$$\begin{aligned}3\rho_{j+2} = & -3\rho_{j+1,x} + 2(u+v)\rho_{j+1} - \rho_{j,xx} + (u+v)\rho_{j,x} - (uv - u_x)\rho_j \\ & -3\sum_{\substack{i+k=j+1,\\ i,k\geq 0}} \rho_i\rho_k - \sum_{\substack{i+k=j,\\ i,k\geq 0}} (3\rho_i\rho_{k,x} - (u+v)\rho_i\rho_k) - \sum_{\substack{i+k+l=j,\\ i,k,l\geq 0}} \rho_i\rho_k\rho_l,\end{aligned} \tag{33}$$

and

$$\theta_j = 6\rho_{j+1} + 3\rho_{j,x} - 2(u+v)\rho_j + 3\sum_{\substack{i+k=j,\\ i,k\geq 0}} \rho_i\rho_k. \tag{34}$$

Then the first two conservation laws of (5) are just

$$\begin{aligned}(u+v)_t = & (u^2+v^2-4uv+3u_x-3v_x)_x, \\ (u^2+v^2-uv-3v_x)_t = & \tfrac{1}{3}[4(u^3+v^3) - 6uv(u+v) \\ & +27vu_x + 9(u-2v)v_x + 9(v-2u)_{xx}]_x.\end{aligned} \tag{35}$$

**Case 2**. Next we will give the conservation laws of (18), by choosing

$$\theta = [-\frac{2}{3}\alpha_0(u+v)\rho + \alpha_0(\rho_x+\rho^2) + C_1]\lambda + [B_1 - (\partial + v + u)A_1]\rho + A_1(\rho_x+\rho^2), \tag{36}$$

where $A_1, B_1, C_1, \alpha_0$ appear in (13), and functions $\rho$ and $\theta$ are expanded as$(\lambda = \eta^3)$:

$$\begin{aligned}\rho &= \eta + \sum_{j=0}^{\infty} \rho_j \eta^{-j}, \\ \theta &= \alpha_0\eta^5 + \sum_{j=0}^{\infty} \theta_j \eta^{-j}.\end{aligned} \tag{37}$$

Similar to Case 1, we can arrive at the first some densities and currents expressed by

$$\begin{aligned}\rho_0 = & \tfrac{1}{3}(u+v), \\ \rho_1 = & \tfrac{1}{9}(u^2+v^2-uv-3v_x), \\ \rho_2 = & \tfrac{1}{81}[9(u_x-v_x)(2v-u) + 9(2v-u)_{xx} - 3uv(u+v) + 2(u^3+v^3)], \\ \rho_3 = & \tfrac{1}{27}[u_x^2 + 4v_x^2 - 7u_xv_x + (v^2-2u^2+2uv)u_x + (u^2-2v^2+2uv)v_x \\ & +(u-5v)u_{xx} + (4v-2u)v_{xx} + 3(u-v)_{xxx}], \\ \rho_4 = & \tfrac{1}{729}[-2(u^5+v^5) + (u^3+v^3)(6v_x - 9u_x + 5uv) - (u+v)(2u^2v^2 + 9uvv_x + 45u_xv_x) \\ & -9v(4u_x^2 - 8v_x^2 - 9u_{xxx} + 6v_{xxx}) + 9u(8u_x^2 - 4v_x^2 + 3v_{xxx}) + 27(v-2u)_{xxxx} \\ & +9(5u^2 - v^2 - 5uv + 18v_x)u_{xx} + 9(5v^2 - u^2 - 5uv - 21v_x + 15u_x)v_{xx}], \\ \theta_0 = & \tfrac{1}{729}\alpha_0[-7(u^5+v^5) + 25uv(u^3+v^3-u^2v-uv^2) + 45(u^2+v^2-uv)(u+v)_{xx} \\ & +15(3uv^2+3u^2v-2u^3-2v^3)(u-v)_x + 45(u-2v)u_x^2 + 90(u+v)u_xv_x \\ & +45(v-2u)v_x^2 + 27(5u_xu_{xx} - 5v_xv_{xx} - u_{xxxx} - v_{xxxx})].\end{aligned} \tag{38}$$

In this case, the recursion relations for $\rho_j$ and $\theta_j$, $(j \geq 0)$, are

$$\begin{aligned}3\rho_{j+2} = & -3\rho_{j+1,x} + 2(u+v)\rho_{j+1} - \rho_{j,xx} + (u+v)\rho_{j,x} - (uv - u_x)\rho_j \\ & -3\sum_{\substack{i+k=j+1,\\ i,k\geq 0}} \rho_i\rho_k - \sum_{\substack{i+k=j,\\ i,k\geq 0}} (3\rho_i\rho_{k,x} - (u+v)\rho_i\rho_k) - \sum_{\substack{i+k+l=j,\\ i,k,l\geq 0}} \rho_i\rho_k\rho_l,\end{aligned} \tag{39}$$

and

$$\begin{aligned}\theta_j = \;& 2\alpha_0\rho_{j+4} + \alpha_0\rho_{j+3,x} - \frac{2}{3}\alpha_0(u+v)\rho_{j+3} + 2A_1\rho_{j+1} + (B_1 - (\partial + u + v)A_1)\rho_j \\ & + A_1\rho_{j,x} + A_1 \sum_{\substack{i+k=j,\\ i,k\geq 0}} \rho_i\rho_k + \alpha_0 \sum_{\substack{i+k=j+3,\\ i,k\geq 0}} \rho_i\rho_k.\end{aligned} \tag{40}$$

Then the first conservation law of (18) is

$$\begin{aligned}(u+v)_t = \;& \frac{1}{81}[-7(u^5+v^5) + 25uv(u^3+v^3-u^2v-uv^2) + 45(u^2+v^2-uv)(u+v)_{xx} \\ & +15(3uv^2+3u^2v-2u^3-2v^3)(u-v)_x + 45(u-2v)u_x^2 + 90(u+v)u_xv_x \\ & +45(v-2u)v_x^2 + 27(5u_xu_{xx} - 5v_xv_{xx} - u_{xxxx} - v_{xxxx})]_x.\end{aligned} \tag{41}$$

Specially, if we choose $u = 2v, \alpha_0 = -9$, we can obtain infinite sequence of conserved quantities of the modified SK equation (3)

$$\begin{aligned}\rho_0 = \;& v, \\ \rho_1 = \;& \frac{1}{3}(v^2 - v_x), \\ \rho_2 = \;& 0, \\ \rho_3 = \;& \frac{1}{9}(-2v_x^2 - 2vv_{xx} + v_{xxx}), \\ \rho_4 = \;& \frac{1}{9}(-2v^3v_x + v^2v_{xx} + 5v_xv_{xx} + 2vv_x^2 + 2vv_{xxx} - v_{xxxx}), \\ \rho_5 = \;& \frac{1}{81}[-v^6 + 3v^4v_x + (57v^2 - 23v_x)v_x^2 + (24v^3 - 60vv_x - 27v_{xx})v_{xx} \\ & -(12v^2 + 36v_x)v_{xxx} - 12vv_{4x} + 6v_{5x}], \\ \theta_0 = \;& v^5 - 5vv_x^2 - 5v^2v_{xx} - 5v_xv_{xx} + v_{xxxx}, \\ \theta_1 = \;& \frac{1}{9}[5v^6 - 15v^4v_x + (25v_x - 15v^2)v_x^2 + (30vv_x + 18v_{xx} - 30v^3)v_{xx} \\ & +(15v^2 + 9v_x)v_{xxx} + 6vv_{4x} - 3v_{5x}].\end{aligned} \tag{42}$$

The recursion relations for $\rho_j$ and $\theta_j$, $(j \geq 0)$, are as follows

$$\begin{aligned}3\rho_{j+2} = \;& -3\rho_{j+1,x} + 6v\rho_{j+1} - \rho_{j,xx} + 3v\rho_{j,x} - 2(v^2 - v_x)\rho_j \\ & -3\sum_{\substack{i+k=j+1,\\ i,k\geq 0}} \rho_i\rho_k - \sum_{\substack{i+k=j,\\ i,k\geq 0}} 3(\rho_i\rho_{k,x} - v\rho_i\rho_k) - \sum_{\substack{i+k+l=j,\\ i,k,l\geq 0}} \rho_i\rho_k\rho_l,\end{aligned} \tag{43}$$

and

$$\begin{aligned}\theta_j = \;& -18\rho_{j+4} - 9\rho_{j+3,x} + 18v\rho_{j+3} + (v^4 - 3v_x^2 - 14v^2v_x + 2vv_{xx} + 2v_{xxx})\rho_j \\ & +3(2vv_x - v_{xx})(2\rho_{j+1} + \rho_{j,x} + \sum_{\substack{i+k=j,\\ i,k\geq 0}} \rho_i\rho_k) - 9\sum_{\substack{i+k=j+3,\\ i,k\geq 0}} \rho_i\rho_k.\end{aligned} \tag{44}$$

The first two conservation laws of the modified SK equation (3) are

$$\begin{aligned}v_t = \;& (v^5 - 5vv_x^2 - 5v^2v_{xx} - 5v_xv_{xx} + v_{xxxx})_x, \\ (v^2 - v_x)_t = \;& \frac{1}{3}[5v^6 - 15v^4v_x + (25v_x - 15v^2)v_x^2 + (30vv_x + 18v_{xx} - 30v^3)v_{xx} \\ & +(15v^2 + 9v_x)v_{xxx} + 6vv_{4x} - 3v_{5x}]_x.\end{aligned} \tag{45}$$

## 参考文献（References）


[1] K. Sawada and T. Kotera, A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-Like Equation[J]. Prog. Theor. Phys., 1974, 51(5): 1355-1367.

[2] P.J. Caudrey, R.K. Dodd, and J.D. Gibbon, A new hierarchy of KdV equations[J]. Proc. Roy. Soc. London Ser. A, 1976, 351(1666): 407-422.

[3] R.K. Dodd and J.D. Gibbon, The prolongation structure of a higher order Korteweg-de Vries equation[J]. Proc. Roy. Soc. London Ser. A, 1978, 358(1694): 287-296.

[4] D.J. Kaup, On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q(x)\psi_x + 6R(x)\psi = \lambda\psi$[J]. Stud. Appl. Math., 1980, 62(3): 189-216.

[5] B.A. Kupershmidt, A super Korteweg-De Vries equation: an integrable system[J]. Phys. Lett. A, 1984, 102(5-6): 213-215.

[6] A. Parker, A reformulation of the "dressing method" for the Sawada-Kotera equation[J]. Inverse Problems, 2001, 17(4): 885-895.

[7] K. Konno, Conservation laws of modified Sawada-Kotera equation in complex plane[J]. J. Phys. Soc. Japan, 1992, 61(1): 51-54.

[8] M. Musette, Bäcklund transformation of partial differential equations from the Painlevé-Gambier classification. I. Kaup-Kupershmidt equation[J]. J. Math. Phys., 1998, 39(10): 5617-5630.

[9] M. Musette and C. Verhoeven, Nonlinear superposition formula for the Kaup-Kupershmidt partial differential equation[J]. Physica D, 2000, 144(1-2): 211-220.

[10] J.M. Dye and A. Parker, On bidirectional fifth-order nonlinear evolution equations, Lax pairs, and directionally dependent solitary waves[J]. J. Math. Phys., 2001, 42(6): 2567-2589.

[11] J.M. Dye and A. Parker, A bidirectional Kaup-Kupershmidt equation and directionally dependent solitons[J]. J. Math. Phys., 2002, 43(10): 4921-4949.

[12] Z. Popowicz, The two-component Kaup-Kupershmidt equation[J]. J. Phys. A, 2004, 37(42): L511-L516.

[13] P.F. Allan and J. Gibbons, Some remarkable nonlinear transformations[J]. Phys. Lett. A, 1980, 75(5): 325.

[14] X.G. Geng, The Darboux transformation for a two dimensions Sawada-Kotera equation. Appl. Math. J. Chinese Univ., 1989, 4, 40-43.

[15] X.G. Geng and B. Xue, An extension of integrable peakon equations with cubic nonlinearity[J]. Nonlinearity, 2009, 22(8): 1847-1856.

[16] X.G. Geng and B. Xue, A three-component generalization of Camassa-Holm equation with $N$-peakon solutions[J]. Adv. Math., 2011, 226(1): 827-839.