On critical cases of Sobolev's inequalities for Carnot groups

Yang Qiaohua
(School of Mathematics and Statistics, Wuhan University, WuHan 430072)

Abstract: In this paper we deal with the problem of Sobolev imbedding in thecritical cases on Carnot groups. We prove some Trudinger-type inequalities on the whole Carnot group, extending to this context the Euclidean results by T. Ozawa and the Heisenberg groups by the same author. The procedure depend also on optimal growth rate of Gagliardo-Nirenberg inequalities. We note the condition m>\max{Q/q,1} in [1], Theorem 1.4, can be replaced by m>Q/q though a new inequality on G. Using these inequalities, we also obtain the Brezis-Gallouet-Wainger inequality on Carnot group.

Keywords: Carnot group; Sobolev's inequality; Brezis-Gallouet-Wainger inequality

0 Introduction

This paper is a continuation of the paper [1]. In [1], it is proved that for every p with $1 < p < \infty$, it holds

$$\| u\|_{L^q(H_n)} \le C_{Q,p} q^{1-p} \| u\|^{p/q}_{L^p(H_n)} \|(-\Delta_H)^{\frac{Q}{2p}} u\|^{1-p/q}_{L^p(H_n)}, \qquad (1)$$

for all $u \in C_0^\infty(H_n)$ and for all q with $p \le q < \infty$, where Δ_H is the Kohn's sublaplace on H_n and $\mathrm{C}_{Q,p}$ is a constant depending only on Q and p, but not on q. Inequality (1) generalized the result of T. Ozawa [2]. Using inequality (1), the author obtain some other inequalities which characterize the critical imbedding in the Sobolev space and the Brezis-Gallouet-Wainger type inequality.

The aim of this note is to prove analogous inequality (1) on the Carnot group G. Recall that if G is a Carnot group, then the Lie algebra of of G can be written by $g = \bigoplus_{i=1}^{r} V_i$ which satisfy $[V_i, V_j] \subset V_{i+j}$. The homogeneous dimension of G is $Q = \sum_{i=1}^{r} i \dim V_i$. Let $X_1, \cdots, X_k$ be a basis of V_1. The second order differential operator

$$\Delta_G = -\sum_{j=1}^{k} X_j^* X_j = \sum_{j=1}^{k} X_j^2$$

is called a sub-Laplacian on G.

To this end we have:

Theorem 1. Let $1 < p < \infty$. Then there exists a constant $\mathrm{C}_{Q,p}$, depending only on Q and p such that for all $u \in C_0^\infty(G)$,

$$\| u\|_{L^q(G)} \le C_{Q,p} q^{1-p} \| u\|^{p/q}_{L^p(G)} \|(-\Delta_G)^{\frac{Q}{2p}} u\|^{1-p/q}_{L^p(G)}.$$

Theorem 1 implies the following result.

Theorem 2. Let $1 < p < \infty$. There exist positive constant α and C>0 such that for all

Foundations: Fundamental Research Funds for the Central Universities under Grant (No. 1082001)
Brief author introduction: Yang Qiaohua, (1977-),male,PhD, Heisenberg group. E-mail: qaohyang@163.com

$f \in C_0^\infty(G)$ with $\|(-\Delta_G)^{\frac{Q}{2p}} u\|_{L^p(G)} \le 1$,

$$\int_G \left(\exp(\alpha |f(\xi)|^{p'}) - \sum_{0 \le j < p-1;\, j \in \mathbb{N}} \frac{1}{j!} (\alpha |f(\xi)|^{p'})^j \right) d\xi \le C \| u\|^p_{L^p(G)}$$

where p'=(p-1)/p is the Holder conjugate of p.

Finally, we obtain the following Brezis-Gallouet-Wainger inequality on G.

Theorem 2. Let $1 < p < \infty$, $1 < q < \infty$ and m>Q/q. Then there exits a constant C such that for all $f \in H^{Q/p,p}(G) \cap H^{m,q}(G)$ with $\| f\|_{H^{Q/p,p}(G)} \le 1$,

$$\| f\|_{L^\infty} \le C(1 + \log(1 + \|(-\Delta_G)^{\frac{m}{2}} u\|_{L^q(G)}))^{\frac{1}{p'}}.$$

1 Notations and preliminaries

We begin by quoting some preliminary facts which will be needed inthe sequel and refer to [3] for more precise information on this subject. Let G be a Carnot group. Consider the Lie algebra $g = \bigoplus_{i=1}^{r} V_i$ of G. Let . By the assumption on the Lie algebra one immediately sees that the system $X_1, \cdots, X_k$ satisfies the well-know finite rank condition, therefore the operator $\Delta_G = -\sum_{j=1}^{k} X_j^* X_j = \sum_{j=1}^{k} X_j^2$ is hypoelliptic.

As a simply connected nilpotent group, G is differential with $\mathbb{R}^n$, $n = \sum_{i=1}^{r} \dim V_i$, via the exponential map $\exp : g \to G$. The Haar measure on G is induced by the exponential mapping from the Lebesgue measure on $\mathbb{R}^n$.

For $\lambda > 0$, we define $\delta_\lambda : g \to g$ by setting $\delta_\lambda(X) = \lambda^i X$ if $X \in V_i$ and extending by linearity. Via conjugation with exponential mapping, δ_λ induces an automorphism of G onto itself which we also denote by δ_λ. The Jacobian determinant of δ_λ (relative to Haar measure) is everywhere equal to λ^Q, where

$$Q = \sum_{i=1}^{r} i \dim V_i$$

is the so-called homogeneous dimension of G. We shall denote by the $\nabla_G = (X_1, \cdots, X_k)$ the related subelliptic gradient. Note that ∇_G is δ_λ-homogenous of degree one. In other words,

$$\nabla_G (u \circ \delta_\lambda) = \lambda (\nabla_G u) \circ \delta_\lambda.$$

We call a curve $\gamma : [a,b] \to G$ a horizontal curve connecting two points $\xi, \eta \in G$ if $\gamma(a) = \xi$, $\gamma(b) = \eta$ and $\dot\gamma(s) \in \mathrm{span}\{X_1, \cdots, X_n\}$ for all s. The Carnot-Caratheodory distance between ξ, η is defined as

$$d_{cc}(\xi, \eta) = \inf_\gamma \int_a^b \sqrt{\langle \dot\gamma(s), \dot\gamma(s) \rangle} ds$$

where the infimum is taken over all horizontal curves γ connecting ξ and η. It is known that any two points ξ and η on G can be joined by a horizontal curve of finite length and then d_{cc} is a metric on G. An important feature of this distance function is that the distance is left-invariant. For simplicity, we always write $d_{cc}(\xi)=d_{cc}(\xi,e)$, where e=(0,0) is the origin of G.

Given any $\xi\in G$, set $\xi^*=\delta_{d_{cc}(\xi)^{-1}}(\xi)$. The polar coordinates on G associated with d_{cc} is the following (cf. [4]):

$$\int_G f(\xi)d\xi=\int_0^\infty\int_\Sigma f(\lambda\xi^*)\lambda^{Q-1}d\sigma d\lambda,\ f\in L^1(G),$$

where $\Sigma=\{\xi\in G; d_{cc}(\xi)=1\}$ is the unit sphere related to d_{cc}.

Let P_h (h>0) denote the heat kernel (that is, the integral kernel of $e^{h\Delta_G}$ on G. For convenience, we set $\mathrm{P}_h(\mathrm{x},\mathrm{t})=\mathrm{P}_h((\mathrm{x},\mathrm{t});\mathrm{e})$. The following global estimates for P_h can be found in [5]: for every $\varepsilon>0$, there exist constants $C_{Q,\varepsilon}>0$ such that

$$P_h(x,t)\le C_{Q,\varepsilon}\frac{1}{h^{n+1}}e^{-\frac{d_{cc}^2(x,t)}{(4+\varepsilon)h}}. \tag{2}$$

2 The proofs

Consider the potential I_λ of order $\lambda\in(0,Q)$, defined by

$$(I_\lambda f)(\xi)=\int_G d_{cc}(\eta^{-1}\xi)^{\lambda-Q}f(\eta)d\eta.$$

Similar arguments to those in [1] show that

Lemma 1. There exists $\mathrm{C}_{Q,p}$ depends only on p and Q such that for any q with $1\le p\le q<\infty$ and $f\in C_0^\infty(G)$,

$$\|I_{Q(1/p-1/q)}f\|_{L^q(G)}\le C_{Q,p}q^{1-1/p}\|f\|_{L^p(G)}.$$

where p'=(p-1)/p is the Holder conjugate of p.

Using Lemma 1, we have the following:

Lemma 2. There exists $\mathrm{C}_{Q,p}$ depends only on p and Q such that for any q with $1\le p\le q<\infty$ and $f\in C_0^\infty(G)$,

$$\|u\|_{L^q(G)}\le C_{p,Q}q^{p'}\|(-\Delta_G)^{\frac{Q}{2}(\frac{1}{p}-\frac{1}{q})}u\|_{L^p(G)},$$

where p'=(p-1)/p is the Holder conjugate of p.

Proof. By the following identity

$$(-\Delta_G)^{-\mu}=\frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}e^{t\Delta_G}dt,$$

we have, using the estimates (2),

$$|(-\Delta_G)^{Q/2(1/p-1/q)}f|(\eta)=\frac{1}{\Gamma(Q/2(1/p-1/q))}\left|\int_0^\infty h^{Q/2(1/p-1/q)-1}P_h*fdh\right|$$

$$\le \frac{C_{Q,\varepsilon}}{\Gamma(Q/2(1/p-1/q))}\left|\int_0^\infty h^{Q/2(1/p-1/q)-n-2}\int_G e^{-\frac{d_{cc}^2(\xi)}{(4+\varepsilon)h}}|f(\eta^{-1}\circ\xi)|d\xi dh\right|$$

$$=\frac{C_{Q,\varepsilon}}{\Gamma(Q/2(1/p-1/q))}\left|\int_G\int_0^\infty h^{Q/2(1/p-1/q)-n-2}e^{-\frac{d_{cc}^2(\xi)}{(4+\varepsilon)h}}|f(\eta^{-1}\circ\xi)|dhd\xi\right|$$

$$\le C'_{Q,\varepsilon}(I_{Q(1/p-1/q)}|f|)(\xi).$$

Therefore,

$$\|(-\Delta_G)^{Q(1/p-1/q)}f\|_{L^q(G)}\le C_{Q,\varepsilon}\|I_{Q(1/p-1/q)}|f|\|_{L^q(G)}\le C_{Q,\varepsilon,p}q^{1-1/p}\|f\|_{L^p(G)}.$$

The desired result follows by choosing $\varepsilon\in(0,1/4)$.

Proof of Theorem 1. It has been proved in [6] and [7], though the Littlewood-Paley analysis on G, that

$$\|(-\Delta_G)^{\frac{\sigma}{2}}f\|_{L^p(G)}\simeq\left\|\left(\sum_{j\in\mathbb{Z}}2^{2\sigma j}|\Delta_j f|^2\right)^{1/2}\right\|_{L^p(G)}.$$

By the Interpolation inequality,

$$\left\|\left(\sum_{j\in\mathbb{Z}}2^{2(1-\theta)\sigma j}|\Delta_j f|^2\right)^{1/2}\right\|_{L^p}\le C\left\|\left(\sum_{j\in\mathbb{Z}}2^{2\sigma j}|\Delta_j f|^2\right)^{1/2}\right\|_{L^p}^{1-\theta}\left\|\left(\sum_{j\in\mathbb{Z}}|\Delta_j f|^2\right)^{1/2}\right\|_{L^p}.$$

Therefore, the following Gagliardo-Niremberg inequality holds for all $f\in C_0^\infty(G)$ and $\theta\in(0,1)$:

$$\|(-\Delta_G)^{\frac{\sigma(1-\theta)}{2}}f\|_{L^p(G)}\le C_{Q,p}\|(-\Delta_G)^{\frac{\sigma}{2}}f\|_{L^p(G)}^{1-\theta}\|f\|_{L^p(G)}^{\theta}.$$

The desired result follows from Lemma 2.

Proof of Theorem 2. Recalling the results in Theorem 1, we can see that the proof is completely analogue to the context of $\mathbb{R}^n$ and Heisenberg group. These complete the proof of Theorem 2.

To prove the Theorem 1.3, we firstly need the following lemma:

Lemma 3. Let $1<p<\infty$. For every s with Qp'/(Q+p)<s<p', there holds

$$\int_G|I_{Q/p}(\xi\eta^{-1})-I_{Q/p}(\xi)|^s d\xi\le Cd_{cc}(\eta)^{Q(1-s/p')}.$$

Proof. Note

$$\int_G|I_{Q/p}(\xi\eta^{-1})-I_{Q/p}(\xi)|^s d\xi=\int_{d_{cc}(\xi)\le 2d_{cc}(\eta)}|I_{Q/p}(\xi\eta^{-1})-I_{Q/p}(\xi)|^s d\xi$$

$$+\int_{d_{cc}(\xi)>2d_{cc}(\eta)}|I_{Q/p}(\xi\eta^{-1})-I_{Q/p}(\xi)|^s d\xi$$

$$=:\ \mathrm{J}_1+\mathrm{J}_2.$$

Since $d_{cc}(\xi\eta^{-1})\le C(d_{cc}(\xi)+d_{cc}(\eta))$, we have

$$J_1\le\int_{d_{cc}(\xi)\le 2d_{cc}(\eta)}|I_{Q/p}(\xi\eta^{-1})|^s d\xi+\int_{d_{cc}(\xi)\le 2d_{cc}(\eta)}|I_{Q/p}(\xi)|^s d\xi$$

$$\le\int_{d_{cc}(\xi)\le(C+2)d_{cc}(\eta)}|I_{Q/p}(\xi)|^s d\xi+\int_{d_{cc}(\xi)\le 2d_{cc}(\eta)}|I_{Q/p}(\xi)|^s d\xi$$

$$\le Cd_{cc}(\eta)^{Q(1-s/p')}.$$

On the other hand, using the inequalities (see [2])

$$| I_{Q/p}(\xi\eta^{-1}) - I_{Q/p}(\xi) | \le C d_{cc}(\eta) d_{cc}(\xi)^{-Q/p-1}$$

When $d_{cc}(\xi) \ge 2 d_{cc}(\eta)$, we obtain

$$J_2 \le C d_{cc}(\eta)^s \int_{d_{cc}(\xi) > 2 d_{cc}(\eta)} d_{cc}(\xi)^{-Qs/p-s} d\xi \le C d_{cc}(\eta)^{Q(1-s/p')}$$

The desired results follow.

Lemma 3 implies the following embedding theorem:

$$| f(\xi) - f(\eta) | \le C \| (-\Delta_G)^{-m/2} f \| \, d_{cc}(\xi\eta^{-1})^{\sigma}$$

with $\sigma = m - Q/q \in (0,1)$.

Proof of Theorem 3. We assume 0<m-Q/q<1. Then

$$| f(\xi) - f(\eta) | \le C \| (-\Delta_G)^{-m/2} f \| \, d_{cc}(\xi\eta^{-1})^{\sigma}.$$

Now Let $0 < \epsilon < e^{-p}$ and let $\tau \in G$ satisfying $d_{cc}(\tau) \le 1$. Then,

$$| f(\xi) - f(\xi \bullet (\epsilon\tau)) | \le C \epsilon^{\sigma} \| (-\Delta_G)^{\frac{m}{2}} f \|_{L^q(G)},$$

where $\sigma = m - Q/q \in (0,1)$. By Holder inequality and Theorem 1, there exists a constant $C_{Q,p}$ such that for any $r \ge p$,

$$\int_{d_{cc}(\tau)\le 1} | f(\xi \circ (\epsilon\tau)) | d\tau \le \left(\int_{d_{cc}(\tau)\le 1} d\tau \right)^{1-1/r} \left(\int_{d_{cc}(\tau)\le 1} | f(\xi \circ (\epsilon\tau)) |^r d\tau \right)^{1/r}$$

$$\le \left(\int_{d_{cc}(\tau)\le 1} d\tau \right)^{1-1/r} \epsilon^{-Q/r} \| f \|_{L^r(G)} \le C_{Q,p} \epsilon^{-Q/r} r^{1-1/p} \| f \|_{H^{Q/p,p}(G)}$$

$$\le C_{Q,p} \epsilon^{-Q/r} r^{1-1/p}.$$

Let $\epsilon = e^{-r}$. Then for any ϵ with $0 < \epsilon < e^{-p}$ we have

$$\int_{d_{cc}(\tau)\le 1} | f(\xi \circ (\epsilon\tau)) | d\tau \le C_{Q,p} e^{Q} r^{1-1/p} \le C'_{Q,p} \left(-\log \epsilon \right)^{1-1/p}.$$

Similar arguments to those in [5] and [7], we have

$$| f(\xi) | = \frac{1}{\int_{d_{cc}(\tau)\le 1} d\tau} \int_{d_{cc}(\tau)\le 1} | f(\xi) | d\tau \le C_{Q,p} \epsilon^{\sigma} \| (-\Delta_G)^{\frac{m}{2}} f \|_{L^q(G)} + C'_{Q,p} \left(-\log \epsilon \right)^{1-1/p}$$

The desired result follows by setting

$$\epsilon = \frac{1}{e^p + \| (-\Delta_H)^{\frac{m}{2}} f \|_{L^q(H_n)}^{1/\sigma}}.$$

In the case $m - Q/q \ge 1$, we choose s satisfying $s - Q/q \in (0,1)$ and $s \ge 1$. Similar arguments to those in [5], we have

$$\| f \|_{L^\infty} \le C (1 + \log(1 + \| (-\Delta_H)^{\frac{s}{2}} u \|_{L^q(G)}))^{\frac{1}{p'}}.$$

On the other hand, by Gagliardo-Niremberg inequalities in the proof of Theorem 1,

$$\| (-\Delta_G)^{\frac{s}{2}} f \|_{L^q(G)} \le C \| (-\Delta_G)^{\frac{m}{2}} f \|_{L^q(G)}^{s/m} \| f \|_{L^q(G)}^{1-s/m} \le C \| (-\Delta_G)^{\frac{m}{2}} f \|_{L^q(G)} + C \| f \|_{H^{Q/p,p}(G)}$$

$$\le C\|(-\Delta_G)^{\frac{m}{2}} f\|_{L^q(G)} + C$$

The result now follows.

References

[1] Yang Qiaohua, On critical cases of Sobolev's inequalities for Heisenberg groups，中国科技论文在线，论文编号：201009-466.
[2] T. Ozawa, On critical cases of Sobolev inequalities, J. Funct. Anal., 1995, 127: 259-269.
[3] G. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups [J], Ark. Math., 1975, 13: 161-207.
[4] R. Monti, Serra Cassano, Surface measures in Carnot-Caratheodory spaces [J], Calc. Var. Partial Differential Equations, 2001, 13(3): 339-376.
[5] N.T.Varopoulos, Saloff-Coste, Coulhon, Analysis and Geometry on Groups [M], Cambridge Tracts in Math., vol. 100, Cambridge, Cambridge Univ. Press, 1992.
[6] D. Chamorro, Improved Sobolev inequalities and Muckenhoupt weights on stratified Lie groups [J], J. Math. Anal. Appl.,2011, 377: 695-709.
[7] Giulia Furioli, Camilo Melzi, Alessandro Veneruso, Littlewood-Paley decomposition and Besov spaces on Lie groups of polynomial growth [J], Math. Nachr., 2006, 279 (9-10): 1028-1040.

Carnot 群上临界情形的 Sobolev 不等式

杨乔华
（武汉大学数学与统计学院，武汉 430072）

摘要：本文主要考虑在一般的 Carnot 幂零李群上临界情形的 Sobolev 嵌入问题。我们证明了在整个 Carnot 群上存在一类 Trudinger 型不等式，这一点推广了由 T. Ozawa 所证明的欧式空间和本文作者所证明的 Heisenberg 群情形。该证明过程依赖于一类 Gagliardo-Nirenberg 不等式的估计。在本文还得到了一个新的嵌入不等式，利用该不等式，文献[1]中定理 1.4 的条件 m>\max{Q/q,1} 可以改进为 m>Q/q ．进一步利用上述不等式，我们还得到了 Carnot 群上的 Brezis-Gallouet-Wainger 不等式.

关键词：Carnot 群；Sobolev 不等式；Brezis-Gallouet-Wainger 不等式
中图分类号：O152.5；O178