闵科夫斯基空间上的时向极值曲面的初边值问题

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摘要： 本文研究了闵科夫斯基时向极值曲面在第一象限内的混合初边值问题. 在初值有界且边值小的条件下，初边值问题存在唯一的整体光滑解. 基于整体存在性的结果，本文进一步证明了在合适的初边值条件下，当时间t 趋于无穷大时，解的一阶导数趋于C^1 行波解，从几何上看，这意味着极值曲面趋向于一个广义的圆柱面，即一个精确解.

关键词： 应用数学; 闵科夫斯基空间; 时向极值曲面; 初边值问题

中图分类号： 35F99, 35L60, 35Q80

Initial-boundary value problem for the equation of timelike extremal surfaces in Minkowski space

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Abstract: In this paper we investigate the mixed initial-boundary value problem for the equation of time-like extremal surfaces in Minkowski space $R^{1+(1+n)}$ in the first quadrant. Under the assumptions that the initial data is bounded and the boundary data is small, we prove the global existence and uniqueness of the C^2 solutions of the initial-boundary value problem for this kind of equation. Based on the existence results on global classical solutions, we also show that, as t tends to infinity, the first order derivatives of the solutions approach C^1 travelling wave, under the appropriate conditions on the initial and boundary datum. Geometrically, this means the extremal surface approaches a generalized cylinder.

Key words: Applied mathematics; Minkowski space; Timelike extremal surfaces; Initial-boundary value problem

0 Introduction and Main results

The extremal surfaces play an important role in the theoretical apparatus of elementary particle physics. A free string is a one-dimensional physical object whose motion is represented by a time-like extremal surfaces in the Minkowski space. Since the inhomogenous boundary

基金项目： Doctoral Foundation of Ministry of Education of China.

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conditions play an important role in string theory and particle physics, this paper concerns the global existence and asymptotic behavior of classical solutions of the mixed initial-boundary value problem for the equation of time-like extremal surfaces in Minkowski space $R^{1+(1+n)}$. By $(x_0, x_1, \cdots, x_{n+1})$ we denote a point in the $1+(1+n)$ dimensional Minkowski space endorsed with the metric

$$ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2 \tag{1.1}$$

Let

$$x_0 = t, x_1 = x, x_2 = \phi_1(t,x), \cdots, x_{n+1} = \phi_n(t,x) \tag{1.2}$$

be a two dimensional surface. Then the induced metric on the surface is

$$\begin{aligned} d_* s^2 &= -dt^2 + dx^2 + (d\phi_1)^2 + \cdots + (d\phi_n)^2 \\ &= -(1-|\phi_t|^2)dt^2 + (1+|\phi_x|^2)dx^2 + 2\phi_t \cdot \phi_x dxdt \end{aligned} \tag{1.3}$$

where $\phi = (\phi_1, \cdots, \phi_n)^T$, ϕ_t or ϕ_x denote partial differentiation with respect to t or x respectively and $\cdot$ denotes inner product in R^n. We assume that the surface is time-like, i.e. the induced metric is Lorentzian. Thus, it is easy to see that the area of the surface is

$$\int\int \sqrt{1-|\phi_t|^2+|\phi_x|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}dxdt \tag{1.4}$$

An extremal surface is defined to be the critical point of the area functional, hence it satisfies the Euler-Lagrange equations

$$\begin{aligned} &\left(\frac{(1+|\phi_x|^2)\phi_t - (\phi_t\cdot\phi_x)\phi_x}{\sqrt{1-|\phi_t|^2+|\phi_x|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}}\right)_t \\ &-\left(\frac{(1-|\phi_t|^2)\phi_x + (\phi_t\cdot\phi_x)\phi_t}{\sqrt{1-|\phi_t|^2+|\phi_x|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}}\right)_x = 0 \end{aligned} \tag{1.5}$$

This equation is called the generalized Born-Infeld equation. Recently the Born-Infeld theory has received much attention because of the development of the string theory and relativity theory. Gibbons gave a systematic study of the Born-Infeld equation theory in [1]. Brenier [2] carried out the theory in connection to the Vlasov-Maxwell system of classical hydrodynamics and electrodynamics in which he discussed an equation for generalized extremal surfaces in the five dimensional Minkowski space.

The extremal surfaces in the Minkowski space are C^2 surfaces with vanishing mean curvature. Then, the equation of the extremal surfaces can be reduced to

$$(1+|\phi_x|^2)\phi_{tt} - 2(\phi_t\cdot\phi_x)\phi_{tx} - (1-|\phi_t|^2)\phi_{xx} = 0$$

In mathematics, the extremal surfaces in the Minkowski space include the following four types: space-like, time-like, light-like and mixed type. The time-like case have been investigated by several authors (e.g.[3] and [4]). Milnor described all entire time-like minimal surfaces in the three-dimensional Minkowski space via a kind of Weierstrass representation [4]. Barbashov et al. studied the nonlinear differential equations describing in differential geometry the minimal surfaces in the pseudo-Euclidean space [5]. Recently Kong et al. studied the equation of the relativistic string moving and the equation for the time-like extremal surfaces in the Minkowski space R^{1+n} (see [6], [7]). For the case of space-like extremal surfaces, we can see the classical papers Calabi [8] and Cheng and Yau [9]. For the case of extremal surfaces of mixed type, Gu gave a series of papers(e.g.[10], [11]). In addition, for the multidimensional cases or more general framework we can refer to the papers by Hoppe et al. [12], Lindblad [13] and Chae and Huh [14].

In the following we consider the global existence and asymptotic behavior of classical solutions of the mixed initial-boundary value problem for the equation of time-like extremal surfaces in Minkowski space $R^{1+(1+n)}$ in the first quadrant.
On the domain

$$D = \{(t, x) | t \geq 0, x \geq 0\}$$

we consider the mixed initial-boundary value problem of system (1.5) with initial condition

$$t = 0: \ \phi(0, x) = f(x), \quad \phi_t(0, x) = g(x) \tag{1.6}$$

and Neumann boundary condition

$$x = 0: \ \phi_x(t, 0) = h(t) \tag{1.7}$$

or Dirichlet boundary condition

$$x = 0: \ \phi(t, 0) = H(t) \tag{1.8}$$

where f, H are vector-valued C^2 functions and g, h are vector-valued C^1 function.

In the following section we first consider system (1.5) with initial condition (1.6) and Neumann boundary condition (1.7). We suppose that at point $(0, 0)$ the conditions of C^2 compatibility are satisfied, i.e.

$$f'(0) = h(0), \quad g'(0) = h'(0) \tag{1.9}$$

Let

$$u = \phi_x, \quad v = \phi_t \tag{1.10}$$

Then, system (1.5) can be equivalently rewritten as a first order system of conservation laws for the unknown $U(t, x) = (u(t, x), v(t, x))$ as follows

$$\begin{cases} u_t - v_x = 0 \\ v_t - \frac{2(u \cdot v)}{1+|u|^2} v_x - \frac{1-|v|^2}{1+|u|^2} u_x = 0 \end{cases} \tag{1.12}$$

The initial condition (1.6) then becomes

$$t = 0: \ (u(0,x), v(0,x)) = U_0(x) = (f'(x), g(x)) \tag{1.13}$$

The Neumann boundary condition can be written as

$$x = 0: \ u = h(t) \tag{1.14}$$

The aim of this paper is to get the global existence and asymptotic behavior of classical solutions under some appropriate conditions on the initial and boundary datum.

For the general first order quasilinear hyperbolic systems, the global existence and asymptotic behavior of classical solutions of the Cauchy problem has been obtained by many authors (see [3], [15-21]). For the initial-boundary value problem in the first quadrant the global existence and asymptotic behavior of classical solutions is studied by Li and Wang [22] and Zhang [23].

Then, the initial-boundary value problem with Neumann boundary condition can be written as

$$\begin{cases} u_t - v_x = 0 \\ v_t - \frac{2(u\cdot v)}{1+|u|^2} v_x - \frac{1-|v|^2}{1+|u|^2} u_x = 0 \\ t = 0: \quad u = f'(x), v = g(x) \\ x = 0: \quad u = h(t) \end{cases} \tag{1.15}$$

The system have two n-constant multiple eigenvalues

$$\lambda_\pm(u,v) = \frac{1}{1+|u|^2}(-(u\cdot v) \pm \sqrt{\triangle(u,v)}) \tag{1.16}$$

where $\triangle(u,v) = 1 - |v|^2 + |u|^2 - |u|^2|v|^2 + (u\cdot v)^2 > 0$, i.e. the surface is time-like.
Let

$$R_i = v_i + \lambda_+ u_i \quad (i = 1, ..., n) \tag{1.17}$$

$$S_i = v_i + \lambda_- u_i \quad (i = 1, ..., n) \tag{1.18}$$

By direct computation (see [7]), The system of equations can be diagonalized as follows

$$\begin{cases} \frac{\partial\lambda_+}{\partial t} + \lambda_- \frac{\partial\lambda_+}{\partial x} = 0 \\ \frac{\partial R_i}{\partial t} + \lambda_- \frac{\partial R_i}{\partial x} = 0 \quad (i = 1, ..., n) \\ \frac{\partial\lambda_-}{\partial t} + \lambda_+ \frac{\partial\lambda_-}{\partial x} = 0 \\ \frac{\partial S_i}{\partial t} + \lambda_+ \frac{\partial S_i}{\partial x} = 0 \quad (i = 1, ..., n) \end{cases} \tag{1.19}$$

$$t = 0: \lambda_+ = \Lambda_+(x); \ \lambda_- = \Lambda_-(x); \quad R_i = R_i^0(x); \quad S_i = S_i^0(x) \tag{1.20}$$

where

$$\Lambda_\pm(x) = (1 + |f'|^2)^{-1}[-(f'\cdot g) \pm \sqrt{1 - |g|^2 + |f'|^2 - |g|^2|f'|^2 + (f'\cdot g)^2}] \tag{1.21}$$

$$R_i^0(x) = g_i(x) + \Lambda_+(x) f_i'(x), \quad S_i^0(x) = g_i(x) + \Lambda_-(x) f_i'(x) \tag{1.22}$$

However, the boundary condition can not be decoupled, which is the main source of mathematical difficulty of our paper.

To prove the global existence of C^2 solutions of system (1.5)-(1.7) on the domain D, it suffices to prove that system (1.15) admits global classical solutions $U = (u, v)$ on the domain D. In this case, it is enough to get uniform a priori estimate on the C^1 norm of (u, v). Noting (1.17), (1.18), the global classical solutions of system (1.15) satisfy

$$u_i(t,x) = \frac{R_i(t,x) - S_i(t,x)}{\lambda_+(t,x) - \lambda_-(t,x)}, v_i(t,x) = \frac{\lambda_+ S_i(t,x) - \lambda_- R_i(t,x)}{\lambda_+(t,x) - \lambda_-(t,x)} \tag{1.23}$$

Then, it suffices to get uniform a priori estimate on the C^1 norm of $\lambda_\pm(t,x)$, $R_i(t,x)$, $S_i(t,x)$ and a positive lower bound for $\lambda_+(t,x) - \lambda_-(t,x)$.
Suppose that U_0, h are C^1 functions with bounded C^1 norm and the initial data satisfies

$$\sup_{x\in R^+} \Lambda_-(x) \le -a < 0 < b \le \inf_{x\in R^+} \Lambda_+(x) \tag{1.24}$$

Without loss of generality, we assume $a < b$. (Otherwise, we can always replace a by a smaller positive number.) If the Neumann boundary data is sufficiently small, for example

$$|h(t)| \le \frac{b-a}{2} \tag{1.25}$$

Then we have the following global existence result for the initial-boundary value problem (1.5)-(1.7):

Theorem 1.1 Suppose that the initial and Neumann boundary datum satisfy (1.24), (1.25) and the conditions of C^2 compatibility (1.9) are satisfied, then the initial-boundary value problem (1.5)-(1.7) admits a unique global C^2 solution $\phi = \phi(t,x)$ on $R^+ \times R^+$.

If we also suppose that the initial and boundary datum satisfy the following assumptions:

$$\sup_{x\in R^+} \{|f''(x)| + |g'(x)|\} \doteq N < +\infty, \quad \int_0^{+\infty} |f'(x)| + |g(x)| dx \doteq N_1 < +\infty \tag{1.26}$$

$$\sup_{x\in R^+} \{|f'(x)| + |g(x)|\} = N_0, \quad \int_0^{+\infty} |f''(x)| + |g'(x)| dx \doteq N_2 < +\infty \tag{1.27}$$

$$\int_0^{+\infty} |h(t)| dt \doteq M_1 < +\infty, \quad \sup_{t\in R^+} |h'(t)| \doteq M < +\infty \tag{1.28}$$

$$\int_0^{+\infty} |h'(t)| dt \doteq M_2 < +\infty \tag{1.29}$$

Based on the global existence of classical solutions, we also prove the following Theorem:

Theorem 1.2 Under the assumptions of Theorem 1.1 and above, there exists a unique C^1 vector-valued function $\Phi(x) = (\Phi_1(x), ..., \Phi_n(x))$ such that

$$((\phi_i)_x, (\phi_i)_t) \longrightarrow (\Phi_i(x-t), -\Phi_i(x-t)) \quad i = 1, ..., n \tag{1.30}$$

uniformly as t tends to infinity.

Remark 1.1 $u = \Phi(x-t), v = -\Phi(x-t)$ is obviously a solution of system (1.12). It can be proved that geometrically this extremal surface is a generalized cylinder. In fact, noting (1.3), the first fundamental form is

$$I = -(1 - |\Phi(x-t)|^2)dt^2 + (1 + |\Phi(x-t)|^2)dx^2 + 2\Phi(x-t)\cdot\Phi(x-t)dxdt$$

$$= d(x-t)(d(x+t) + |\Phi(x-t)|^2 d(x-t)) = d(x-t)d(x+t+\bar{\Phi}(x-t))$$

Then, the first fundamental form is flat through the transformation of the variable. where $\bar{\Phi}$ is the primitive function of $|\Phi|^2$.
Noting (1.2), we can calculate its second fundamental form

$$\mathbb{I}_i = -\Phi_i'(x-t)dt^2 - \Phi_i'(x-t)dx^2 + 2\Phi_i'(x-t)dtdx = -\Phi_i'(x-t)(d(x-t))^2$$

Remark 1.2

$$N_0 = \sup_{x\in R^+}\{|f'(x)| + |g(x)|\}, \text{we have } N_0 \le N_2.$$

In fact, noting $\int_0^{+\infty} |f''(x)| + |g'(x)|dx \doteq N_2 < \infty$, $\int_0^{+\infty} |f'(x)| + |g(x)|dx \doteq N_1 < \infty$, we conclude that

$$\lim_{x\longrightarrow+\infty} f'(x) = 0, \quad \lim_{x\longrightarrow+\infty} g(x) = 0 \tag{1.31}$$

This paper is organized as follows. In section 2, we get the global existence of classical solutions of the initial-boundary value problem with Neumann boundary condition in the first quadrant. In order to prove the asymptotic behavior of global classical solutions, in section 3 we obtain some uniform a priori estimate which play an important role in the proof of Theorem 1.2. In section 4, we study the asymptotic behavior of global classical solutions and obtain that , as t tends to infinity, the first order derivatives of global classical solution approach C^1 travelling wave. In section 5 we consider the initial-boundary value problem with Dirichlet boundary condition in the first quadrant and get the same conclusion as the initial-boundary value problem with Neumann boundary condition.

1 Global existence of classical solutions of the initial-boundary value problem with Neumann boundary condition in the first quadrant

In this section, we will prove Theorem 1.1 and obtain some uniform a priori estimate which also play an important role in the proof of Theorem 1.2. In order to prove Theorem 1.1, it suffices to prove that under the assumptions of Theorem 1.1, the initial-boundary value problem (1.15) have the uniform a priori estimate on the C^1 norm of global classical solutions, where $\|U\|_1 = \|U\|_0 + \|U_x\|_0$ and $\|\cdot\|_0$ denote the C^0 norm. Noting (1.23), it suffices to prove that $\|\lambda_\pm\|_1 + \|R_i\|_1 + \|S_i\|_1$ is bounded and $\lambda_+ - \lambda_-$ has a positive lower bound.
Firstly we consider the following system

$$\begin{cases} \frac{\partial \lambda_+}{\partial t} + \lambda_- \frac{\partial \lambda_+}{\partial x} = 0 \\ \frac{\partial \lambda_-}{\partial t} + \lambda_+ \frac{\partial \lambda_-}{\partial x} = 0 \end{cases} \tag{2.1}$$

$$t = 0 : \Lambda_\pm(x) = (1 + |f'|^2)^{-1}[-(f' \cdot g) \pm \sqrt{1 - |g|^2 + |f'|^2 - |g|^2|f'|^2 + (f' \cdot g)^2}]$$

By Kong et al. [7], we have

$$-1 < \lambda_+(t, x) \leq 1, \quad -1 \leq \lambda_-(t, x) < 1 \tag{2.2}$$

The system (2.1) enjoys the following property on the domain D.
Lemma 2.1 Under the assumptions of (1.9), (1.24) and (1.25), system (2.1) is strictly hyperbolicity. Furthermore, on the domain D we have

$$\lambda_-(t, x) \leq -a < 0 < b \leq \lambda_+(t, x) \tag{2.3}$$

Lemma 2.2 Let R_i, S_i be as system (1.19), then

$$\{|R_i(t, x)|, |S_i(t, x)|\} \leq C \tag{2.4}$$

where C is a positive constant only depending on a, b, N_0.

Remark 2.1 The positive constant C only depends on a, b, N_0 and is independent of M, M_1, M_2, N, N_1, N_2. In the following sections the meaning of C may change from line to line.
To estimate the first order derivatives of the solutions of system (1.19), we consider a linear system

$$\begin{cases} \frac{\partial S}{\partial t} + w(t, x)\frac{\partial S}{\partial x} = 0 \\ \frac{\partial Y}{\partial t} + z(t, x)\frac{\partial Y}{\partial t} = 0 \end{cases} \tag{2.5}$$

where z, w are regarded as given smooth functions. However, z, w are not arbitrary given. $S = z, \ Y = w$ itself is a solution of system (2.5). Assume that on the domain under consideration

$$w(t, x) - z(t, x) \geq \delta > 0 \tag{2.6}$$

where δ is positive constant. Then w and z are constant along characteristics respectively. Under these assumptions, system (2.5) enjoys the following remarkable properties:

Lemma 2.3 Let

$$T_1 = \frac{\partial}{\partial t} + w(t,x)\frac{\partial}{\partial x}, T_2 = \frac{\partial}{\partial t} + z(t,x)\frac{\partial}{\partial x} \tag{2.7}$$

Then

$$[T_1, T_2] = T_1T_2 - T_2T_1 = 0 \tag{2.8}$$

For any Lipschitz continuous functions F and G, system (2.5) implies the conservation laws:

$$\begin{cases} (\frac{F(S)}{w-z})_t + (\frac{wF(S)}{w-z})_x = 0 \\ (\frac{G(Y)}{w-z})_t + (\frac{zG(Y)}{w-z})_x = 0 \end{cases} \tag{2.9}$$

where $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$.

Remark 2.2 Lemma 2.3 generalizes corresponding result of Serre [24], see also Chen [25] and E and Kohn [26].

For any fixed $T \geq 0$, we introduce

$$W_\infty(T) = \max_{0\leq t\leq T} \sup_{x\in R^+} \{|\frac{\partial\lambda_+(t,x)}{\partial x}|, |\frac{\partial\lambda_-(t,x)}{\partial x}|, |\frac{\partial R_i(t,x)}{\partial x}|, |\frac{\partial S_i(t,x)}{\partial x}|\} \tag{2.10}$$

Lemma 2.4 Under the assumptions of Theorem 1.1, there exists a positive constant C only depending on a, b, N_0 such that

$$W_\infty(T) \leq C(M+N) \tag{2.11}$$

Remark 2.3 Obviously, from Lemma 2.4 we can get

$$W_\infty(\infty) \leq C(M+N) \tag{2.12}$$

Proof of Theorem 1.1 Under the assumptions of Theorem 1.1, by Lemma 2.1-2.4, on the domain D

$$\|\lambda_\pm\|_1, \|R_i\|_1, \|S_i\|_1 \leq C(M+N+1)$$

$$(\lambda_+(t,x) - \lambda_-(t,x)) \geq b + a$$

Noting (1.23), we can get uniform a priori estimate of C^1 norm of u and v, i.e. system (1.15) have the global C^1 solutions. Then, the system (1.5)-(1.7) have global C^2 solutions.

2 Uniform a priori estimate

In this section under the assumptions of Theorem 1.1 and (1.26)-(1.29), we will establish some uniform a priori estimate to prove Theorem 1.2.

For any fixed $T \geq 0$, we introduce

$$W_1(T) = \max_{i=1,\ldots,n} \sup_{0\leq t\leq T} \{\int_0^{+\infty} |\frac{\partial\lambda_+(t,x)}{\partial x}|dx, \int_0^{+\infty} |\frac{\partial\lambda_-(t,x)}{\partial x}|dx$$

$$\int_0^{+\infty}|\frac{\partial R_i(t,x)}{\partial x}|dx, \int_0^{+\infty}|\frac{\partial S_i(t,x)}{\partial x}|dx\} \tag{3.1}$$

$$\tilde{W_1}(T) = \max_{i=1,\dots,n}\{\sup_{\tilde{C}_1}\int_{\tilde{C}_1}|\frac{\partial\lambda_+(t,x)}{\partial x}|dt, \sup_{\tilde{C}_2}\int_{\tilde{C}_2}|\frac{\partial\lambda_-(t,x)}{\partial x}|dt$$

$$\sup_{\tilde{C}_1}\int_{\tilde{C}_1}|\frac{\partial R_i(t,x)}{\partial x}|dt, \sup_{\tilde{C}_2}\int_{\tilde{C}_2}|\frac{\partial S_i(t,x)}{\partial x}|dt\} \tag{3.2}$$

where $\tilde{C}_1$ stands for any given forward characteristic $\frac{dx}{dt} = \lambda_+$ on the domain $[0,T]\times R^+$; $\tilde{C}_2$ stands for any given backward characteristic $\frac{dx}{dt} = \lambda_-$ on the domain $[0,T]\times R^+$.

Lemma 3.1 Under the assumptions of Theorem 1.2, there exists a positive constant C only depending on a, b, N_0 such that, the following estimates hold:

$$\tilde{W_1}(T), W_1(T) \le C(N_2 + M_2 + M_1 N) \tag{3.3}$$

Remark 3.1 Obviously, from Lemma 3.1 we can get

$$\tilde{W_1}(\infty), W_1(\infty) \le C(N_2 + M_2 + M_1 N) \tag{3.4}$$

Lemma 3.2 Under the assumptions of Theorem 1.2, we have

$$\{\int_{L_1}(1-\lambda_+(t,x))dt, \int_{L_2}(1+\lambda_-(t,x))dt, \int_{L_1}|R_i(t,x)|dt, \int_{L_2}|S_i(t,x)|dt\} \le C(N_1 + M_1) \tag{3.5}$$

$$\{\int_{\tilde{C}_1}(1-\lambda_+(t,x))dt, \int_{\tilde{C}_2}(1+\lambda_-(t,x))dt, \int_{\tilde{C}_1}|R_i(t,x)|dt, \int_{\tilde{C}_2}|S_i(t,x)|dt\} \le C(N_1 + M_1) \tag{3.6}$$

where $\tilde{C}_1$ stands for any given forward characteristic $\frac{dx}{dt} = \lambda_+$ on the domain $[0,T]\times R^+$; $\tilde{C}_2$ stands for any given backward characteristic $\frac{dx}{dt} = \lambda_-$ on the domain $[0,T]\times R^+$; L_1 stands for any given radial that has the slope 1 on the domain $[0,T]\times R^+$; L_2 stands for any given radial that has the slope -1 on the domain $[0,T]\times R^+$.

3 Asymptotic behavior of global classical solutions of the initial-boundary value problem with Neumann boundary condition in the first quadrant

In this section we will study asymptotic behavior of the global classical solutions of system (1.5)-(1.7) and give the proof of Theorem 1.2.

Let

$$\frac{D}{D_1 t} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \tag{4.1}$$

Obviously, noting the system (1.19)

$$\frac{D\lambda_-}{D_1 t} = T_1\lambda_- + (1-\lambda_+)\frac{\partial\lambda_-}{\partial x} \tag{4.2}$$

In the following we consider Equation (4.2), i.e.

$$\frac{D\lambda_-}{D_1 t} = (1-\lambda_+)\frac{\partial\lambda_-}{\partial x} \tag{4.3}$$

For any fixed point $(t,x)\in D$, we define $\xi = x-t$

Case 1: $\xi \geq 0$, it follows Equation (4.3) that

$$\lambda_-(t,x) = \lambda_-(t,\xi+t) = \lambda_-(0,\xi) + \int_0^t (1-\lambda_+)\frac{\partial\lambda_-}{\partial x}(s,\xi+s)ds \tag{4.4}$$

By (2.12) and Lemma 3.2, we have

$$\begin{aligned} |\int_0^t (1-\lambda_+)\frac{\partial\lambda_-}{\partial x}(s,\xi+s)ds| &\leq W_\infty(\infty)\int_0^{+\infty} |1-\lambda_+(s,\xi+s)|ds \\ &\leq C(M+N)(M_1+N_1) \end{aligned} \tag{4.5}$$

This implies that the integral $\int_0^t (1-\lambda_+)\frac{\partial\lambda_-}{\partial x}(s,\xi+s)ds$ converges uniformly for $\xi\in R^+$, On the other hand, noting that all functions in the right-hand side in Equation (4.4) are continuous with respect to ξ, then, we observe that there exists a unique function $\tilde{\psi}(\xi)\in C^0(R^+)$ such that

$$\lambda_-(t,x) \longrightarrow \tilde{\psi}(x-t) \quad t\longrightarrow +\infty \tag{4.6}$$

Case 2: $\xi \leq 0$, it follows Equation (4.3) that

$$\lambda_-(t,x) = \lambda_-(t,\xi+t) = \lambda_-(-\xi,0) + \int_{-\xi}^t (1-\lambda_+)\frac{\partial\lambda_-}{\partial x}(s,\xi+s)ds \tag{4.7}$$

By (2.12) and Lemma 3.2, we can get

$$\begin{aligned} \int_{-\xi}^t (1-\lambda_+)\frac{\partial\lambda_-}{\partial x}(s,\xi+s)ds &\leq W_\infty(\infty)\int_{-\xi}^t (1-\lambda_+)(s,\xi+s)ds \\ &\leq C(M+N)(M_1+N_1) \end{aligned} \tag{4.8}$$

Then,we obtain that there exists a unique function $\bar{\psi}(\xi)\in C^0(R^-)$ such that

$$\lambda_-(t,x) \longrightarrow \bar{\psi}(x-t) \quad t\longrightarrow +\infty \tag{4.9}$$

Case 3: When $\xi\longrightarrow 0$, noting the above cases we can get

$$\tilde{\psi}(\xi)\longrightarrow\tilde{\psi}(0) \ \text{ and } \ \bar{\psi}(\xi)\longrightarrow\bar{\psi}(0) \tag{4.10}$$

Moreover,

$$\tilde{\psi}(0) = \bar{\psi}(0) \tag{4.11}$$

We define

$$\psi(\xi) = \begin{cases} \tilde{\psi}(\xi), & \xi \in R^+; \\ \bar{\psi}(\xi), & \xi \in R^-; \end{cases}$$

Hence from above we have proved the following lemma

Lemma 4.1 There exists a unique function $\psi(x-t) \in C^0(R)$, such that

$$\lambda_-(t,x) \longrightarrow \psi(x-t) \quad t \longrightarrow +\infty \tag{4.12}$$

Remark 4.1 In the same way, we can obtain that there exists a unique function $\psi_i(x-t) \in C^0(R)$ such that

$$S_i(t,x) \longrightarrow \psi_i(x-t) \quad t \longrightarrow +\infty \quad i = 1, ..., n \tag{4.13}$$

Lemma 4.2 When $t \longrightarrow +\infty$, we have

$$\lambda_+(t,x) \longrightarrow 1 \tag{4.14}$$

$$R_i(t,x) \longrightarrow 0 \qquad i = 1, ..., n \tag{4.15}$$

uniformly for all $x \geq 0$.

Noting (1.23), when $x \geq 0$, we can get

$$\lim_{t \longrightarrow +\infty} u_i(t,x) = \lim_{t \longrightarrow +\infty} \frac{R_i(t,x) - S_i(t,x)}{\lambda_+(t,x) - \lambda_-(t,x)} = \frac{-\psi_i(x-t)}{1 - \psi(x-t)} \doteq \Phi_i(x-t) \tag{4.16}$$

$$\lim_{t \longrightarrow +\infty} v_i(t,x) = \lim_{t \longrightarrow +\infty} \frac{\lambda_+ S_i(t,x) - \lambda_- R_i(t,x)}{\lambda_+(t,x) - \lambda_-(t,x)} = \frac{\psi_i(x-t)}{1 - \psi(x-t)} \doteq -\Phi_i(x-t) \tag{4.17}$$

We next prove that $\Phi_i(\xi) \in C^1(R)$. Noting $\psi_i(\xi), \psi(\xi) \in C^0(R)$, we need to show that $d(\psi_i(\xi))/d\xi, d(\psi(\xi))/d\xi \in C^0(R)$. It suffices to show that $\psi(\xi), \psi_i(\xi) \in C^1(R)$.

In the following we only prove $\psi(\xi) \in C^1(R)$.

Lemma 4.3 Under the assumptions of Theorem 1.2, the limit

$$\lim_{t \longrightarrow +\infty} \frac{\partial \lambda_-}{\partial x}(t, x_1(t,\beta)) \doteq \psi^*(\beta) \tag{4.18}$$

exists and is continuous with respect to $\beta \in R$. Moreover

$$|\psi^*(\beta)| \leq C(M+N)(N_2 + M_2 + M_1 N) \tag{4.19}$$

Remark 4.2 In the similar way, for any $i \in \{1, ..., n\}$ we can obtain

$$\lim_{t \longrightarrow +\infty} \frac{\partial S_i}{\partial x}(t, x_1(t,\beta)) \doteq \psi_i^*(\beta) \tag{4.20}$$

$\psi_i^*(\beta)$ are continuous with respect to $\beta \in R$ and

$$|\psi_i^*(\beta)| \leq C(M+N)(M_2+N_2+M_1N) \tag{4.21}$$

Lemma 4.4 The limit

$$\lim_{t\longrightarrow+\infty}\frac{\partial\lambda_-}{\partial x}(t,\xi+t)$$

exists and is continuous with respect to $\xi \in R$.

Lemma 4.5

$$\frac{d\psi(\xi)}{d\xi}=\lim_{t\longrightarrow+\infty}\frac{\partial\lambda_-}{\partial x}(t,\xi+t). \tag{4.22}$$

Remark 4.3 In the similar way, we can also prove

$$\lim_{t\longrightarrow+\infty}\frac{\partial S_i}{\partial x}(t,\xi+t)=\frac{d\psi_i(\xi)}{d\xi} \tag{4.23}$$

Lemma 4.6

$$\lim_{t\longrightarrow+\infty}\frac{\partial\lambda_-}{\partial x}(t,\xi+t)=\psi^*(\vartheta(\xi)) \tag{4.24}$$

is continuous with respect to $\xi \in R$. Moreover

$$\frac{d\psi(\xi)}{d\xi}=\psi^*(\vartheta(\xi)) \tag{4.25}$$

Remark 4.4 By the same method, we obtain that $\frac{\partial S_i}{\partial x}(t,\xi+t)$ have the similar conclusion. Moreover,

$$\frac{d\psi_i(\xi)}{d\xi}=\psi_i^*(\vartheta(\xi)) \tag{4.26}$$

The proof of Theorem 1.2 The conclusion of Theorem 1.2 follows from Lemmas and Remarks above.

4 Initial-boundary value problem with Dirichlet boundary condition in the first quadrant

Since the inhomogenous Dirichlet boundary conditions play an important role in the string theory and particle physics (see [2]), in this section we consider the mixed initial-boundary value problem with Dirichlet boundary condition of system (1.5) with the initial data (1.6) and the boundary data (1.8). Then system (1.5), (1.6), (1.8) can be rewritten as

$$\begin{cases} u_t - v_x = 0 \\ v_t - \frac{2(u\cdot v)}{1+u^2}v_x - \frac{1-v^2}{1+u^2}u_x = 0 \\ t=0: \quad u=f'(x), v=g(x) \\ x=0: \quad v=H'(t) \end{cases}$$

Similarly, we suppose U_0, H' are C^1 functions with bounded C^1 norm and the initial data satisfies $\sup_{x\in R^+}\Lambda_-(x) \leq -a < 0 < b \leq \inf_{x\in R^+}\Lambda_+(x)$. Without loss of generality, we assume $a < b$. If the first order derivative of Dirichlet boundary data is sufficiently small, for example

$$|H'(t)| \leq b - a \tag{5.1}$$

and the conditions of C^2 compatibility are satisfied, i.e.

$$f(0) = H(0), \quad g(0) = H'(0) \tag{5.2}$$

and

$$H''(0) - \frac{2f'(0)\cdot g(0)}{1+|f'(0)|^2}g'(0) - \frac{1-|g(0)|^2}{1+|f'(0)|^2}f''(0) = 0 \tag{5.3}$$

We have the similar global existence result:

Theorem 5.1 Suppose that the above assumptions (1.24) and (5.1)-(5.3) are satisfied, then the initial-boundary value problem (1.5), (1.6) and (1.8) admits a unique global C^2 solution $\phi = \phi(t,x)$ on $R^+ \times R^+$.

Similarly, under the assumptions (1.26), (1.27) and

$$\sup_{t\in R^+}|H''(t)| \doteq M < +\infty, \quad \int_0^{+\infty}|H'(t)|dt \doteq M_1 < +\infty \tag{5.4}$$

$$\int_0^{+\infty}|H''(t)|dt \doteq M_2 < +\infty \tag{5.5}$$

Using the same method as Theorem 1.2, we can prove the following result:

Theorem 5.2 Under the assumptions of Theorem 1.2 and above, there exists a unique C^1 vector-valued function $\Psi(x) = (\Psi_1(x), ..., \Psi_n(x))$ such that

$$((\phi_i)_x, (\phi_i)_t) \longrightarrow (\Psi_i(x-t), -\Psi_i(x-t)) \quad i = 1, ..., n \tag{5.6}$$

uniformly as t tends to infinity.

Remark 5.1 Obviously $u = \Psi(x-t), v = -\Psi(x-t)$ is also a solution of system (1.12).

Remark 5.2 The similar conclusion can be obtained for the initial-boundary value problem with Neumann boundary condition or Dirichlet boundary condition for the motion of relativistic closed strings in the Minkowski space R^{1+n} (for the definition of the relativistic closed strings see Kong et al. [27]).

Acknowledgements The authors would like to thank Prof. De-Xing Kong for helpful discussions.

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