A Kind of Accelerated AOS Difference Schemes For Dual Currency Option Pricing Model

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Abstract: Black-Scholes equation of Dual currency option pricing is a typical multi-asset option pricing model, and it is important to research it's numerical value. This paper uses the accelerated additive operator splitting (AOS) algorithm to transform the two-dimensional Black-Scholes equation into two equivalent one-dimensional equations, and then construct the 'explicit-implicit' and the 'implicit-explicit' scheme. These schemes proved to be stable and convergent unconditionally and they have second-order accuracy.The total computation of these schemes is only a quarter of the traditional AOS scheme. Finally, the numerical example shows the effectiveness of the accelerated AOS difference schemes.

Keywords: Dual currency option pricing model; accelerated AOS algorithm; 'explicit-implicit' scheme; 'implicit-explicit' scheme; second order accuracy

0 Introduction

In the financial market, the option is a kind of important financial derivatives. Along with the development of the financial market, it is difficult to meet the needs of financial traders by only using European, American and other single asset options. Therefore, the financial institution designs more complex multi-asset options.

The dual currency option is a typical multi-asset option, and it is an option contract of investing in foreign securities. Generally speaking, the dual-currency option pricing depends not only on changes of foreign securities price, but also on changes of foreign currency exchange rate, therefore the pricing of which is more complex. This paper mainly discusses the Black-Scholes equation of dual currency option pricing [1,2]:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\left[\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}+2\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2 V}{\partial S_1\partial S_2}+\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2}\right]+(r_1-\hat{q}_1)S_1\frac{\partial V}{\partial S_1}+(r_1-\hat{q}_2)S_2\frac{\partial V}{\partial S_2}-r_1V=0$$

$$\hat{q}_1=r_1-r_2+q+\sigma_1\sigma_2\rho,\quad \hat{q}_2=r_2. \tag{1}$$

Here, V is the price of the dual currency option, S_1 is foreign risk asset, S_2 is the exchange rate of foreign currency against domestic one, r_1 is the domestic interest rate without risk, r_2 is the foreign rate without risk, σ_1 is the volatility of S_1, σ_2 is the volatility of S_2, ρ is the correlation coefficient and q is the interest rate. The equation (1) has the analytical solution [1][2]:

$$V(S_1,S_2,t)=\frac{1}{2\pi(T-t)}e^{-r_1(T-t)}\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_0^\infty\frac{(\eta_1-K)^+}{\eta_1}\exp\left[\frac{\sigma_2^2\alpha_1^2-2\rho\sigma_1\sigma_2\alpha_1\alpha_2+\sigma_1^2\alpha_2^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)(T-t)}\right]d\eta_1d\eta_2$$

$$\alpha_1=\ln\frac{S_1}{\eta_1}+\left(r_2-q-\sigma_1\sigma_2\rho-\frac{\sigma_1^2}{2}\right)(T-t),$$

$$\alpha_2=\ln\frac{S_2}{\eta_2}+\left(r_1-r_2-\frac{\sigma_2^2}{2}\right)(T-t). \tag{2}$$

Although the Black-Scholes equation of dual currency option pricing has the analytical

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solution (2), it cannot meet the effective requirement in option pricing. In practice, the numerical method has been widely used, such as the Monte Carlo method and the Binary Tree method, the two methods have less accuracy than the finite difference method; Xiaozhong Yang, Yangguo Liu (2007) proposed the general difference scheme for solving the Black-Scholes equation [4], but it has lower accuracy; Lifei Wu, Xiaozhong Yang (2010) put forward the 'explicit-implicit' and 'implicit-explicit' difference schemes for the Black-Scholes equation of options payment [5], however it did not consider the multi-asset options; Weichert (1998) firstly used the additional operator splitting (AOS) method to solve the multi-dimensional partial differential equations [6]; Yi Zhang(2010) proposed the accelerated AOS schemes for nonlinear diffusion filtering [7], this method reduce the computation time and storage space, but it used the Hopscotch method to deal with the one dimensional equation, which is only conditionally compatible. This paper uses the accelerated AOS method to split the two dimensional Black-Scholes equation into two one dimensional equations, then constructs the 'explicit-implicit' and 'implicit-explicit' schemes for the every one dimensional equation. Meanwhile this paper analyzes the compatibility, stability, convergence and accuracy of the scheme. Finally, some numerical examples verify the effectiveness of this scheme.

1 Accelerated AOS difference scheme

1.1 Initial-boundary value problem

Assume that the underlying assets meet geometric Brown motion, and the market is arbitrage free. It does not consider the tax. By the $\Delta - hedging$ principle, we can get a portfolio Π:

$$\Pi = V - \Delta_1 S_2 S_1 - \Delta_2 S_2.$$

Select the proper Δ_1, Δ_2 to make Π no risk, namely:

$$d\Pi = r_2 \Pi dt,$$

$$d\Pi = dV - \Delta_1 d(S_1 S_2) - \Delta_2 dS_2 - \Delta_1 S_2 S_1 q dt - \Delta_2 S_2 r_2 dt$$

Then through the $It\hat{o}$ formula:

$$dV = \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\left[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right] \right\} dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2$$

We can get the Black-Scholes equation of dual currency pricing:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right] + (r_1 - \hat{q}_1) S_1 \frac{\partial V}{\partial S_1} + (r_1 - \hat{q}_2) S_2 \frac{\partial V}{\partial S_2} - r_1 V = 0$$

$$\hat{q}_1 = r_1 - r_2 + q + \sigma_1\sigma_2\rho, \quad \hat{q}_2 = r_2. \qquad (3)$$

In theory, the solving area of this equation is:

$$\{(S_1, S_2, t) | 0 < S_1 < \infty, 0 < S_2 < \infty, t \in [0, T]\}$$

But in the actual transaction, the price of the underlying asset will not always appear to be zero or infinity. Therefore, the financial institution provides a small enough value $S_{\min}$ $(S_{\min} > 0)$ as the lower bound and a large enough value $S_{\max}$ $(S_{\max} < \infty)$ as the upper bound for it. Then the pricing problem can be solved in a bounded area:

$$\Omega = \{(S_1, S_2, t) | S_{1\min} < S_1 < S_{1\max}, S_{2\min} < S_2 < S_{2\max}, t \in [0, T]\}$$

Assume that the foreign option is the call option, then construct the initial and boundary conditions for equation (3). For the reason that the option pricing is a backward problem, the initial condition is:

$$V(S_1,S_2,t)=S_2\max(S_1-K,0)$$

the boundary condition is:

$$V(S_{1\min},S_2,t)=0,\quad V(S_{1\max},S_2,t)=0.$$

$$V(S_1,S_{2\min},t)=0,\quad V(S_1,S_{2\max},t)=0.$$

In order to solve the equation (3), we can substitute its variable as follows:

$$x=\ln S_1, y=\ln S_2, \tau=T-t.$$

Then this pricing model will be transformed into the initial-boundary value problem of partial differential equation with constant coefficients:

$$\frac{\partial V}{\partial \tau}-\frac{1}{2}\left[\sigma_1^2\frac{\partial^2 V}{\partial x^2}+2\rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial x\partial y}+\sigma_2^2\frac{\partial^2 V}{\partial y^2}\right]-\left(r_1-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\frac{\partial V}{\partial x}-\left(r_1-\hat{q}_2-\frac{\sigma_2^2}{2}\right)\frac{\partial V}{\partial y}+r_1V=0$$

$$\hat{q}_1=r_1-r_2+q+\sigma_1\sigma_2\rho,$$

$$\hat{q}_2=r_2 \qquad (4)$$

Initial condition:

$$V(x,y,0)=e^y\max(e^x-K,0),$$

Boundary condition:

$$V(\ln S_{1\min},y,t)=0,\quad V(\ln S_{1\max},y,t)=0,$$

$$V(x,\ln S_{2\min},t)=0,\quad V(x,\ln S_{2\max},t)=0.$$

1.2 Construction of accelerated AOS difference scheme

Firstly, transform the equation (4) into equivalent equation set along with the x axis and the y axis:

$$\frac{\partial V}{\partial \tau}-\sigma_1^2\frac{\partial^2 V}{\partial x^2}-2\left(r_1-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\frac{\partial V}{\partial x}-\rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial x\partial y}+r_1V=0, \qquad (5)$$

$$\frac{\partial V}{\partial \tau}-\sigma_1^2\frac{\partial^2 V}{\partial y^2}-2\left(r_1-\hat{q}_2-\frac{\sigma_2^2}{2}\right)\frac{\partial V}{\partial y}-\rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial x\partial y}+r_1V=0. \qquad (6)$$

Then make a mesh partition on the area Ω, let h_1, h_2 as the space step and k as the time step:

$$\begin{cases} x_i=\ln S_{1\min}+ih_1, & i=0,1,2\ldots,M_1, \\ y_j=\ln S_{1\min}+jh_2, & j=0,1,2\ldots,M_2, \\ \tau_n=nk, & n=0,1,2,\ldots,N. \end{cases}$$

Here $h_i=\dfrac{S_{i\max}-S_{i\min}}{M_i}$, $k=\dfrac{T-t}{N}$. $V_{i,j}^n$ denotes $V(x_i,y_j,\tau_n)$, then the initial conditions will be transformed into:

$$V_{i,j}^0=e^{jh_2}\max(e^{ih_1}-K,0) \qquad (7)$$

and the boundary conditions will be transformed into:

$$V_{0,j}^n=0,\ V_{M_1,j}^n=0,\ V_{i,0}^n=0,\ V_{i,M_2}^n=0. \qquad (8)$$

In the x axis direction, the space derivative can be replaced by the central difference:

$$\left(\frac{\partial V}{\partial x}\right)_{i,j}^{n}=\frac{V_{i+1,j}^{n}-V_{i-1,j}^{n}}{2h_1},$$

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_{i,j}^{n}=\frac{V_{i+1,j}^{n}-2V_{i,j}^{n}+V_{i-1,j}^{n}}{h_1^2}.$$

the mixed partial derivative can be replaced by :

$$\left(V_{xy}\right)_{i,j}^{n}=\frac{V_{i+1,j+1}^{n}-V_{i+1,j-1}^{n}-V_{i-1,j+1}^{n}+V_{i,j}^{n}}{2\left(h_1+h_2\right)} \tag{9}$$

We still use $\left(V_{xy}\right)_{i,j}^{n}$ denote the equation (9) for convenience.

If the time derivative can be replaced by the one-order forward difference:

$$\left(\frac{\partial V}{\partial \tau}\right)_{i,j}^{n}=\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{k}$$

the equation (5) will be transformed into :

$$\frac{V_{i,,j}^{n+1}-V_{i,j}^{n}}{k}=\sigma_1^2\frac{V_{i+1,j}^{n}-2V_{i,j}^{n}+V_{i-1,j}^{n}}{h_1^2}+2\left(r_1-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\frac{V_{i+1,j}^{n}-V_{i-1,j}^{n}}{2h_1}+\rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{n}+rV_{i,j}^{n} \tag{10}$$

If the time derivative can be replaced by the one-order backward difference:

$$\left(\frac{\partial V}{\partial \tau}\right)_{i,j}^{n}=\frac{V_{i,j}^{n}-V_{i,j}^{n-1}}{k}$$

the equation (5) will be transformed into :

$$\frac{V_{i,,j}^{n}-V_{i,j}^{n-1}}{k}=\sigma_1^2\frac{V_{i+1,j}^{n}-2V_{i,j}^{n}+V_{i-1,j}^{n}}{h_1^2}+2\left(r_1-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\frac{V_{i+1,j}^{n}-V_{i-1,j}^{n}}{2h_1}+\rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{n}+rV_{i,j}^{n} \tag{11}$$

Then we construct the 'explicit-implicit' scheme [10]. We adopt the explicit scheme (10) at the odd number floor, and implicit scheme (11) at the even number floor.

$$\begin{cases}\dfrac{V_{i,j}^{2n+1}-V_{i,j}^{2n}}{k}=\sigma_1^2\dfrac{V_{i+1,j}^{2n}-2V_{i,j}^{2n}+V_{i-1,j}^{2n}}{h_1^2}+2\left(r_1-\hat{q}_1-\dfrac{\sigma_1^2}{2}\right)\dfrac{V_{i+1,j}^{2n}-V_{i-1,j}^{2n}}{2h_1}+\rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{2n}-rV_{i,j}^{2n}\\ \dfrac{V_{i,j}^{2n+2}-V_{i,j}^{2n+1}}{k}=\sigma_1^2\dfrac{V_{i+1,j}^{2n+2}-2V_{i,j}^{2n+2}+V_{i-1,j}^{2n+2}}{h_1^2}+2\left(r_1-\hat{q}_1-\dfrac{\sigma_1^2}{2}\right)\dfrac{V_{i+1,j}^{2n+2}-V_{i-1,j}^{2n+2}}{2h_1}+\rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{2n+2}-rV_{i,j}^{2n+2}\end{cases} \tag{12}$$

Similarly, in the y axis direction we can get:

$$\begin{cases}\dfrac{V_{i,j}^{2n+1}-V_{i,j}^{2n}}{k}=\sigma_2^2\dfrac{V_{i,j+1}^{2n}-2V_{i,j}^{2n}+V_{i,j-1}^{2n}}{h_2^2}+2\left(r_1-\hat{q}_2-\dfrac{\sigma_2^2}{2}\right)\dfrac{V_{i,j+1}^{2n}-V_{i,j-1}^{2n}}{2h_2}+\rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{2n}-rV_{i,j}^{2n}\\ \dfrac{V_{i,j}^{2n+2}-V_{i,j}^{2n+1}}{k}=\sigma_1^2\dfrac{V_{i,j+1}^{2n+2}-2V_{i,j}^{2n+2}+V_{i,j-1}^{2n+2}}{h_2^2}+2\left(r_1-\hat{q}_2-\dfrac{\sigma_2^2}{2}\right)\dfrac{V_{i,j+1}^{2n+2}-V_{i,j-1}^{2n+2}}{2h_2}+\rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{2n+2}-rV_{i,j}^{2n+2}\end{cases} \tag{13}$$

If the computation is done in the x axis direction, the result $V_{i,j}$ of equation (12) is denoted as $\left(V_{i,j}\right)_x$ .In the y axis direction the result is denoted as $\left(V_{i,j}\right)_y$. Then the arithmetic mean value of $\left(V_{i,j}\right)_x$ and $\left(V_{i,j}\right)_y$ is the final value:

$$V_{i,j} = \frac{\left(V_{i,j}\right)_x + \left(V_{i,j}\right)_y}{2} \tag{14}$$

Similarly, if we adopt the implicit scheme (11) at the odd number floor, and explicit scheme (10) at the even number floor, we can construct the 'implicit-explicit' scheme as follow:

In the x axis direction:

$$\begin{cases} \dfrac{V_{i,j}^{2n+1} - V_{i,j}^{2n}}{k} = \sigma_1^2 \dfrac{V_{i+1,j}^{2n+1} - 2V_{i,j}^{2n+1} + V_{i-1,j}^{2n+1}}{h_1^2} + 2\left(r_1 - \hat{q}_1 - \dfrac{\sigma_1^2}{2}\right)\dfrac{V_{i+1,j}^{2n+1} - V_{i-1,j}^{2n+1}}{2h_1} + \rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{2n+1} - rV_{i,j}^{2n+1} \\ \dfrac{V_{i,j}^{2n+2} - V_{i,j}^{2n+1}}{k} = \sigma_1^2 \dfrac{V_{i+1,j}^{2n+1} - 2V_{i,j}^{2n+1} + V_{i-1,j}^{2n+1}}{h_1^2} + 2\left(r_2 - \hat{q}_1 - \dfrac{\sigma_1^2}{2}\right)\dfrac{V_{i+1,j}^{2n+1} - V_{i-1,j}^{2n+1}}{2h_1} + \rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{2n+1} - rV_{i,j}^{2n+1} \end{cases}$$

In the y axis direction:

$$\begin{cases} \dfrac{V_{i,j}^{2n+1} - V_{i,j}^{2n}}{k} = \sigma_2^2 \dfrac{V_{i,j+1}^{2n+1} - 2V_{i,j}^{2n+1} + V_{i,j-1}^{2n+1}}{h_2^2} + 2\left(r_1 - \hat{q}_2 - \dfrac{\sigma_2^2}{2}\right)\dfrac{V_{i,j+1}^{2n+1} - V_{i,j-1}^{2n+1}}{2h_2} + \rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{2n+1} - rV_{i,j}^{2n+1} \\ \dfrac{V_{i,j}^{2n+2} - V_{i,j}^{2n+1}}{k} = \sigma_2^2 \dfrac{V_{i,j+1}^{2n+1} - 2V_{i,j}^{2n+1} + V_{i,j-1}^{2n+1}}{h_2^2} + 2\left(r_1 - \hat{q}_2 - \dfrac{\sigma_2^2}{2}\right)\dfrac{V_{i,j+1}^{2n+1} - V_{i,j-1}^{2n+1}}{2h_2} + \rho\sigma_1\sigma_2\left(V_{xy}\right)_{i,j}^{2n+1} - rV_{i,j}^{2n+1} \end{cases}$$

When the traditional AOS scheme is adopted to calculate, it needs to solve an equation set that contains a triple diagonal matrix every step. Generally, we use the thomas method to solve it, and the computation is $O\left(M_1 \times M_2 \times N\right)$. However, if we use the accelerated AOS algorithm to construct the 'explicit-implicit' and 'implicit-explicit' scheme, it only needs to solve the triple diagonal matrix every two step in the x and y axis direction. Therefore, the total computation of the accelerated AOS scheme is a quarter of the traditional one.

2 Analysis of the compatibility and accuracy

Firstly, we consider the 'explicit-implicit' scheme. Add up the two equations of (12), we can eliminate $V_{i,j}^{2n+1}$:

$$\begin{aligned} \frac{V_{i,j}^{2n+1} - V_{i,j}^{2n}}{k} &= \sigma_1^2\left(\frac{V_{i+1,j}^{2n} - 2V_{i,j}^{2n} + V_{i-1,j}^{2n}}{h_1^2} + \frac{V_{i+1,j}^{2n+2} - 2V_{i,j}^{2n+2} + V_{i-1,j}^{2n+2}}{h_1^2}\right) \\ &+ 2\left(r - \hat{q}_1 - \frac{\sigma_1^2}{2}\right)\left(\frac{V_{i+1,j}^{2n} - V_{i-1,j}^{2n}}{2h_1} + \frac{V_{i+1,j}^{2n+2} - V_{i-1,j}^{2n+2}}{2h_1}\right) + \rho\sigma_1\sigma_2\left(\left(V_{xy}\right)_{i,j}^{2n} + \left(V_{xy}\right)_{i,j}^{2n+2}\right) - r\left(V_{i,j}^{2n} + V_{i,j}^{2n+2}\right) \end{aligned} \tag{15}$$

The above equation is the well known Crank-Nicloson scheme. Suppose $V\left(x, y, t\right)$ is the analytical solution of (12), and substitute $V\left(x_i, y_j, z_n\right)$ by $V_{i,j}^{2n}$ in the above equation. Then make difference between the two side of the equation, and we will get the truncation error:

$$\left(R_{i,j}^{2n}\right)_x=\frac{V_{i,j}^{2n+1}-V_{i,j}^{2n}}{k}-\sigma_1^2\left(\frac{V_{i+1,j}^{2n}-2V_{i,j}^{2n}+V_{i-1,j}^{2n}}{h_1^2}+\frac{V_{i+1,j}^{2n+2}-2V_{i,j}^{2n+2}+V_{i-1,j}^{2n+2}}{h_1^2}\right)$$

$$-2\left(r-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\left(\frac{V_{i+1,j}^{2n}-V_{i-1,j}^{2n}}{2h_1}+\frac{V_{i+1,j}^{2n+2}-V_{i-1,j}^{2n+2}}{2h_1}\right)-\rho\sigma_1\sigma_2\left(\left(V_{xy}\right)_{i,j}^{2n}+\left(V_{xy}\right)_{i,j}^{2n+2}\right)-r\left(V_{i,j}^{2n}+V_{i,j}^{2n+2}\right)$$

Then expand $\left(R_{i,j}^{2n}\right)_1$ as the Taylor Series at the point $\left(x_i,y_j,\tau_{2n}\right)$, and simplify it to get: [5][8]

$$\left(R_{i,j}^{2n}\right)_x=O\left(h_1^2+k^2\right)$$

Similarly, in the y axis direction, we will get:

$$\left(R_{i,j}^{2n}\right)_y=O\left(h_2^2+k^2\right)$$

Finally, take the arithmetic mean value of $\left(R_{i,j}^{2n}\right)_x$ and $\left(R_{i,j}^{2n}\right)_y$:

$$R_{i,j}^{2n}=\frac{1}{2}\left[\left(R_{i,j}^{2n}\right)_x+\left(R_{i,j}^{2n}\right)_y\right]=O\left(h_1^2+h_2^2+k^2\right) \tag{16}$$

Therefore we will get the following theorem:

Theorem 1: The 'explicit-implicit' scheme of accelerated AOS difference scheme of dual currency option pricing has two order accuracy, and it is compatible with the equation set (5),(6) unconditionally.

If we apply the same method on the 'implicit-explicit' scheme, we will get similar theorem.

Theorem 2: The 'implicit-explicit' scheme of accelerated AOS difference scheme of dual currency option pricing has two order accuracy, and it is compatible with the equation set (5),(6) unconditionally.

3 Analysis of the stability and convergence

Firstly, the stability of the scheme will be considered. As to the 'explicit-implicit' scheme, take the Fourier transformation on the two sides of the equation (15), and simplify it to get: [5][9]

$$\left[1+rk-\sigma_1^2\frac{k}{h_1^2}\left(e^{i\xi}-2+e^{-i\xi}\right)-\left(r-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\frac{k}{h_1}\left(e^{i\xi}-e^{-i\xi}\right)\right]\tilde{V}^{2n+2}(\xi)=$$

$$\left[1-rk+\sigma_1^2\frac{k}{h_1^2}\left(e^{i\xi}-2+e^{-i\xi}\right)+\left(r-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\frac{k}{h_1}\left(e^{i\xi}-e^{-i\xi}\right)\right]\tilde{V}^{2n}(\xi)$$

Therefore, the growth factor is:

$$G(\xi)=\frac{1-rk+4\sigma_1^2\frac{k}{h_1^2}\sin^2\left(\frac{\xi}{2}\right)+2\frac{k}{h_1}\left(r-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\sin(\xi)i}{1+rk-4\sigma_1^2\frac{k}{h_1^2}\sin^2\left(\frac{\xi}{2}\right)-2\frac{k}{h_1}\left(r-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\sin(\xi)i} \tag{17}$$

Denote that:

$$P=rk-4\sigma_1^2\frac{k}{h_1^2}\sin^2\left(\frac{\xi}{2}\right),$$

$$Q=2\frac{k}{h_1}\left(r-\hat{q}_1-\frac{\sigma_1^2}{2}\right)\sin\left(\frac{\xi}{2}\right).$$

Then the equation (17) can be transformed into:

$$G(\xi)=\frac{1-P+Qi}{1+P-Qi}$$

and we can get that:

$$|G(\xi)|^2=\frac{(1-P)^2+Q^2}{(1+P)^2+Q^2}\le 1.$$

In practice, we can get that $P\ge 0$. Therefore $|G(\xi)|\le 1$ is always true.

By the Von Neumann Theorem, we can get that the AOS difference scheme in the x axis direction is stable unconditionally. Similarly, we can get that the scheme in the y axis direction is also unconditionally stable. Therefore we can get the following theorems:

Theorem 3: The 'explicit-implicit' scheme of accelerated AOS difference scheme of dual currency option pricing is unconditionally stable.

In addition, due to the Lax Theorem [9], we can get:

Corollary1: The 'explicit-implicit' scheme of accelerated AOS difference scheme of dual currency option pricing is convergent.

If we apply the same method on the 'implicit-explicit' scheme, we will get the similar theorem.

Theorem 4: The 'implicit-explicit' scheme of accelerated AOS difference scheme of dual currency option pricing is unconditionally stable and convergent.

4 Numerical example

Here, we consider an American investor buy the Nikkei index call option. Assuming the current price of the Nikkei is 20,000 yen, the dividend rate of the Nikkei is 0.03, the volatility is 0.2, the Japanese yen against the dollar as 0.01, the volatility of exchange rate is 0.1, the correlation coefficient between the Nikkei and the yen is 0.2, the risk-free interest rate of American is 0.08, the risk-free interest rate of Japan is 0.04, the Strike price of option is 19,000 yen. Consider the deadline of the option is 3,6,9 and 12 months, and the final exchange rate is the spot exchange rate [2].

The numerical experiment is done in Matlab 7.6 environment. The comparison among analytical solution and numerical solution, such as the result of the accelerated AOS difference scheme and the Crank-Nicloson scheme is shown as follows:

Tab.1 The compasion of analytical and numerical solution table

Time/(month)	3	6	9	12	relative error
Analytical solution	21.6886	23.7346	26.5386	27.5893	0
AOS scheme	21.6386	23.5391	25.1436	26.4882	0.0198
Crank-Nicloson	21.7279	24.5015	27.7231	31.2501	0.0668

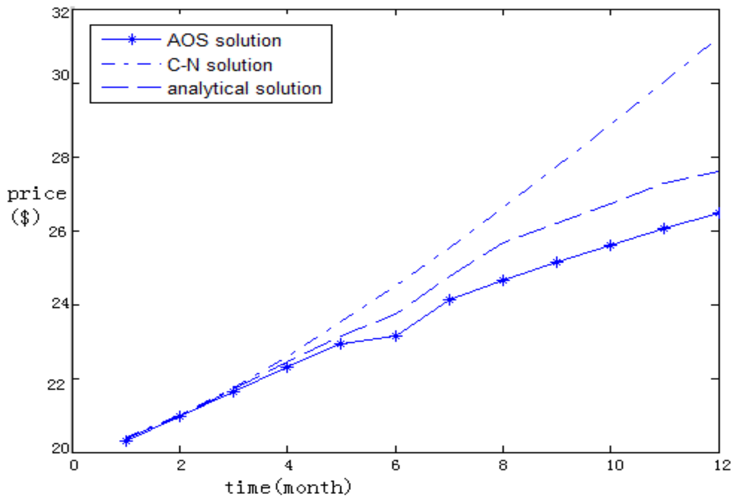

Fig 1:The compasion of analytical and numerical solution

From table 1 and figure 1, we can see that the accelerated AOS difference scheme has higher calculation accuracy than the Crank-Nicloson scheme. With a longer deadline of the option, the advantage of the scheme is more obvious. The numerical result demonstrates the theoretic analysis that the accelerated AOS difference scheme is effective.

5 Conclusion

In this paper we construct the accelerated AOS scheme. The main idea of the scheme is to split the two-dimensional Black-Scholes equation into two equivalent one-dimensional equations, and then construct the 'explicit-implicit' and the 'implicit-explicit' scheme. This scheme is second-order accuracy, stable and convergent unconditionally, and the total computation of these schemes is only a quarter of the traditional AOS scheme.

The main advantages of accelerated AOS difference scheme are as follows. Firstly, the AOS algorithm splits the high dimensional equation into low ones. This method can avoid the complexity of using difference method directly on high dimensional equation. AOS difference scheme is very applicable to deal with the high dimensional equations. Secondly, the 'explicit-implicit' and 'implicit-explicit ' scheme also has prominent advantage. Classical implicit scheme hides the potential stability, which is no use in the calculation, but when it is applied in the alternate scheme, this potential stability just cover the stability shortage of explicit scheme. Finally, the implicit scheme calculates the approximate value of the analytical solution from above, and the explicit scheme calculates it from below. Every two steps produce errors with the opposite symbol, which can counteract with each other, and then obtain the more accurate result.

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双币种期权定价模型的一种快速 AOS 差分解法

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摘要：双币种期权定价的 Black-Scholes 方程是典型的多资产期权定价模型，研究其数值解法在金融衍生品定价中有十分重要的意义。本文使快速加性算子分裂算法(AOS)将二维 Black-Scholes 方程转化为等价的一维方程组,再分别对两个一维方程构造‘显-隐’，‘隐-显’格式。从理伦上分析了对该格式的相容性、稳定性和收敛性，并证明了其具有二阶精度。最后通过数值算例证明了该格式的有效性，进一步说明通过本文快速 AOS 降维的方法能有效地避免高维方程的计算复杂性，还能较大幅度地提高计算速度，此方法能较好的适用于实时性要求较高的多资产期权定价。

关键词：双币种期权定价模型；快速 AOS 算法；‘显-隐’格式；‘隐-显’格式；二阶精度
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