Symmetry group analysis and an optimal system

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Abstract: In this paper，Smmetry group properties of Benjamin-Bona-Mahoney equation are analyzed.The symmetries and adjoint representations for this equation are given.Second,an optimal system of one-dimensional subalgabras is derived.At last,the similarity reductions are obtained on the optimal system method.

Keywords: Benjamin-Bona-Mahoney equation; symmetry group; optimal systerm;reduction equation.

0 Introduction

Symmetry analysis has played an important role in the analysis of nonlinear partial differential equations. Especially, via any subgroup of the symmetry group, the original equation can be reduced to an equation with fewer independent variables. For more contents on this topic, you can refer to the books [1-3].About the optimal systems, a lot of work has done by many famous experts [1,2,4,5].Up to now, several methods have been developed to construct optimal systems. Here, we will use Olver's method [2] ， which only depends on the fragments of the theory of Lie algebra. We will discuss this on the latter.

In this paper, we will consider Benjamin-Bona-Mahoney equation [6]

$$u_t - u_{xxx} + uu_x = 0 \tag{1}$$

This equation is used in the analysis of the following type [6]:surface waves of long wave length in liquids, hydromagnetic waves in cold plasma,acoustic-gravity waves in compressible fluids.

The paper is organized as follows.In Sec.1,group analysis of Eq.(1) is made,the set of infinitesimal generators are obtained.In Sec.2,One-dimensional optimal system and the adjoint representations table are constructed. We obtained its symmey reduced equations via optimal system in Sec.3.At last,a brief conclusion are made in Sec.4.

1 Group analysis of Benjamin-Bona-Mahoney equation

We write the invariance condition for Eq(1) as

$$X^{(2)}[u_t - u_{xxx} + uu_x] = 0 \tag{1.1}$$

Here, $X^{(2)}$ is the second prolongation of the infinitesimal operator

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u \tag{1.2}$$

Obtained via the following prolongation formulas:

$$X^{(2)} = X + \phi^t \frac{\partial}{\partial u_t} + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} \tag{1.3}$$

Where

$$\begin{aligned}
\phi^t &= D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt} \\
\phi^x &= D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{tx} \\
\phi^{xx} &= D_x^{\,2}(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{txx} \\
\phi^{tt} &= D_t^{\,2}(\phi - \xi u_x - \tau u_t) + \xi u_{xtt} + \tau u_{ttt} \\
\phi^{xt} &= D_x D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xxt} + \tau u_{ttx}
\end{aligned} \tag{1.4}$$

The operator D_t and D_x denote the total derivatives with respect to t and x :

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots$$
$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots \qquad (1.5)$$

Substituting (2.3) in (2.1), we obtain the following determining equation:

$$\phi u_x + \phi_x u + \phi^t - \phi^{xxx} = 0 \qquad (1.6)$$

Substituting(2.4)and making $u_t = u_{xxx} - uu_{xx}$,then the left side of（2.6）is an equation about u , u_x , u_{xx} , u_{xxx} , uu_{xx} … ,making all the coefficients equation to zero, we get a PDE system:

$$\tau_u = 0$$
$$\tau_{uu} = 0$$
$$\tau_{ux} = 0$$
$$\tau_{uuu} = 0$$
$$\tau_{xuu} = 0$$
$$\tau_{xx} = 0$$
$$\tau_x = 0$$
$$\tau_{xxu} + \xi u - \tau_u u = 0$$
$$-4\tau_x - \tau_t + 3\xi_x + \tau_{xxx} - 3\tau_u + 3\xi_u = 0$$
$$-2\tau_u + \xi_{uu} - \tau_{uu} u = 0$$
$$-3\tau_x + 3\xi_{xu} - \phi_{uu} - 3\tau_{xu} u = 0$$
$$-\tau_{xx} - \phi_{xu} + \xi_{xx} = 0$$
$$-\tau_{uuu} + \xi_{uuu} - \tau_{uu} = 0$$
$$-3\tau_{xuu} u - \phi_{uuu} + 3\xi_{xuu} - 6\tau_{xu} = 0$$
$$-\tau_{xxu} - \phi_{xxu} + \xi_{xxu} - \tau_{xx} = 0$$
$$\tau_t u - \tau_{xxx} u - u\xi_x + \tau_x u^2 - 3\phi_{xxu} + \phi - \xi_t + \xi_{xxx} = 0 \qquad (1.7)$$
$$u\phi_x + \phi_t - \phi_{xxx} = 0$$

Solving (2.7) by maple, one can have:

$$\phi = -\frac{2}{3} c_1 u + c_3 \quad , \quad \tau = c_1 t + c_2 \quad , \quad \xi = \frac{1}{3} c_1 x + c_3 t + c_4 \qquad (1.8)$$

c_1, c_2, c_3, c_4 are arbitrary constants. Hence, the corresponding vector can be written as:

$$X = (\frac{1}{3} c_1 x + c_3 t + c_4)\frac{\partial}{\partial x} + (c_1 t + c_2)\frac{\partial}{\partial t} + (-\frac{2}{3} c_1 u + c_3)\frac{\partial}{\partial u} \qquad (1.9)$$

Therefore, we can say that the symmetry algebra of Eq.(1) is generated by the four vector fields：

$$X_1=\frac{\partial}{\partial x}\quad,\quad X_2=\frac{\partial}{\partial t}\quad,\quad X_3=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\quad,\quad X_4=\frac{1}{3}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{2}{3}u\frac{\partial}{\partial u} \tag{1.10}$$

2 Optical system

The commutation relations between these vector fields are listed in table 1.

Tab.1 交换子关系表
Tab.1 Commutator Table

$[X_i,X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	$\frac{1}{3}X_1$
X_2	0	0	X_1	X_2
X_3	0	0	0	$-\frac{2}{3}X_3$
X_4	$-\frac{1}{3}X_1$	$-X_2$	$\frac{2}{3}X_3$	0

From the above commutator table, we can see that the operators $X_i(i=1\cdots4)$ form a Lie algebra. The Lie algebra spanned by $X_i(i=1\cdots4)$ generates the symmetry group of Eq.(1). To go on with the classification of group invarient solutions,we have to compute the adjoint representation using Lie series in conjunction with the above commutator table.

For instance,there is

$$\begin{aligned}Ad(\exp(\varepsilon X_1))X_2&=X_2-\varepsilon[X_1,X_2]+\frac{1}{2!}[X_1,[X_1,X_2]-\frac{1}{3!}[X_1,[X_1,[X_1,X_2]]]+\cdots\\&=X_2-\varepsilon 0\\&=X_2\end{aligned} \tag{2.1}$$

We do not write this in detail .In this manner, we construct the table.

Tab.2 伴随关系表
Tab.2 Adjoint Table

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4-\frac{1}{3}\varepsilon X_1$
X_2	X_1	X_2	$X_3-\varepsilon X_1$	$X_4-\varepsilon X_2$
X_3	X_1	X_2	X_3	$X_4+\frac{2}{3}\varepsilon X_3$
X_4	$\frac{2}{3}\varepsilon X_1$	$(1-\varepsilon)X_2$	$(1+\frac{1}{3}\varepsilon)X_3$	X_4

Following Ovsiannikov[1,7],one calls two subalgebras v_1 and v_2 of a given Lie algebra equivalent if one can find an element g in the Lie group so that $Adg(v_1)=v_2$,where Adg is the adjoint representation of g on v. Given a nonzero vector

$$Y=aX_1+bX_2+cX_3+dX_4 \tag{2.2}$$

Our task is to simplify as many of the coefficients a , b , c , d as possible through judicious application of adjoint maps to Y .

Case1.

Suppose first that $d\neq 0$ .Scaling Y necessary, we can assume that $d=1$ .Referring to table 2, we can make the coefficient of X_1 , X_2 vanish:

$$Y'=cX_3+X_4 \tag{2.3}$$

Case2.

Suppose that $d=0$, Scaling Y necessary, we can assume that $c\neq 0, c=1$ .At this time, $Y=aX_1+bX_2+cX_3$ .Referring to table 2,we can make the coefficient of X_1 vanish:

$$Y''=bX_2+X_3 \tag{2.4}$$

Case3.

Suppose that $d=0$, $c=0$, we assume that $b\neq 0, b=1$ .Then

$$Y'''=aX_1+X_2 \tag{2.5}$$

Otherwise $b=0$, $Y''''=X_1$ (2.6)

From (2.3),(2.4),(2.5),(2.6),we can get the optimal system of one-dimensional subalgebras of L_4 to be those spanned by:

$$Y^1=X_1=\frac{\partial}{\partial x},$$

$$Y^2=X_2=\frac{\partial}{\partial t},$$

$$Y^3=X_1+X_2=\frac{\partial}{\partial x}+\frac{\partial}{\partial t},$$

$$Y^4=-X_1+X_2=-\frac{\partial}{\partial x}+\frac{\partial}{\partial t},$$

$$Y^5=X_3=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u},$$

$$Y^6=X_4=\frac{1}{3}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{2}{3}u\frac{\partial}{\partial u},$$

$$Y^7=X_3+X_2=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}+\frac{\partial}{\partial t},$$

$$Y^8=X_3-X_2=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}-\frac{\partial}{\partial t},$$

$$Y^9=X_3+X_4=(t+\frac{1}{3}x)\frac{\partial}{\partial x}+(1-\frac{2}{3}u)\frac{\partial}{\partial u}+t\frac{\partial}{\partial t},$$

$$Y^{10}=-X_3+X_4=(\frac{1}{3}x-t)\frac{\partial}{\partial x}-(1+\frac{2}{3}u)\frac{\partial}{\partial u}+t\frac{\partial}{\partial t}$$

3 Reduction equations

In this section, we will do the symmetry reductions for every element in the optimal system by solving the characteristic equations. We take $Y^7 = X_3 + X_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial t}$ for example.

Its characteristic equation reads:

$$\frac{dx}{t} = \frac{dt}{1} = \frac{du}{1} \qquad (3.1)$$

By $\frac{dx}{1} = \frac{dt}{1}$, we can get invariant $\lambda = x - \frac{t^2}{2}$ (3.2)

By $\frac{du}{1} = \frac{dt}{1}$, we can get $u = t + c$, c is a constant. So, u can be taken in the form

$$u = t + f(\lambda) \qquad (3.3)$$

Substituting (3.3) into Eq.(1),we can get the reduction equation $1 - f''' + ff' = 0$ (3.4)

For concise, we list the reduced equation in the following table.

Tab.3 约化方程列表

Tab.3 the list of reduction equation

N	Y	Invariants λ	Equation
1	Y^1	$\lambda = t$	$f' = 0$
2	Y^2	$\lambda = x$	$-f''' + ff' = 0$
3	Y^3	$\lambda = x - t$	$-f' - f''' + ff' = 0$
4	Y^4	$\lambda = x + t$	$f' - f''' + ff' = 0$
5	Y^5	$\lambda = t$	$\frac{f'}{t} = 0$
6	Y^6	$\lambda = \frac{x^3}{t}$	$f^{-1} f' + x^{-2} f^{-2} + 12x^{-2} = 0$
7	Y^7	$\lambda = x - \frac{t^2}{2}$	$1 - f''' + ff' = 0$
8	Y^8	$\lambda = x + \frac{t^2}{2}$	$-1 - f''' + ff' = 0$
9	Y^9	$\lambda = \frac{2}{3}xt - \frac{t^2}{2}$	$\frac{8}{27}t^2 f''' + \frac{24}{81} f' f'' f^{-1} + t^{-\frac{2}{3}} f^{\frac{2}{3}} f' + \frac{2}{3}xt^{-1} f' - \frac{32}{243}t^{-2} f^{-2} f'^3 - ft^{-2} = 0$
10	Y^{10}	$\lambda = \frac{2}{3}xt + \frac{t^2}{2}$	$\frac{8}{27}t^2 f''' + \frac{24}{81} f' f'' f^{-1} + t^{-\frac{2}{3}} f^{\frac{2}{3}} f' - \frac{2}{3}xt^{-1} f' - \frac{32}{243}t^{-2} f^{-2} f'^3 + ft^{-2} = 0$

4 Conclusion

We have performed Lie symmetry analysis for the Benjamin-Bona-Mahoney equation.Optimal system and reduced equations are also derived.From table 3,we can see that:for some ODEs(reduced equations),we can easily get the solutions of them,so can get the solutions of Eq.(1). But, in general, we can not obtain the exact and explicit solutions for the nonlinear differential equations(ODES),so we cannot get the solution of Eq.(1).At this time,in order to get the solution of Eq.(1),we can use other mathods to solve reduced equations,such as equation the power series method[8,9] , $(\frac{G'}{G})$ -expansion method[10,11]and so on. We will deal with this problem in the future.

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对称群分析和最优系统

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摘要：在这篇文章中，首先给出 Benjamin-Bona-Mahoney 方程的对称群分析，得到该方程的李对称和伴随表示。其次，给出了一维子代数的最优系统。最后在最优系统基础上得到方程约化。

关键词： Benjamin-Bona-Mahoney 方程；对称群；最优系统；约化方程。
中图分类号：0221.62

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