Perturbed stellar motions around the rotating black hole in Sgr A* for a generic orientation of its spin axis

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ABSTRACT

Empirically determining the averaged variations of the orbital parameters of the stars orbiting the Supermassive Black Hole (SBH) hosted by the Galactic Centre (GC) in Sgr A* is, in principle, a valuable tool to put on the test the General Theory of Relativity (GTR), in regimes far stronger than those tested so far, and certain key predictions of it like the no-hair theorems. We analytically work out the long-term variations of all the six osculating Keplerian orbital elements of a test particle orbiting a non-spherical, rotating body with quadrupole moment Q_2 and angular momentum $\boldsymbol{S}$ for a generic spatial orientation of its spin axis $\hat{\boldsymbol{k}}$. This choice is motivated by the fact that, basically, we do not know the position in the sky of the spin axis of the SBH in Sgr A* with sufficient accuracy. We apply our results to S2, which is the closest star discovered so far having an orbital period $P_{\rm b} = 15.98$ yr, and to a hypothetical closer star X with $P_{\rm b} = 0.5$ yr. Our calculations are quite general, not being related to any specific parameterization of $\hat{\boldsymbol{k}}$, and can be applied also to astrophysical binary systems, stellar planetary systems, and planetary satellite geodesy in which different reference frames, generally not aligned with the primary's rotational axis, are routinely used.

Subject headings: Classical general relativity; Experimental tests of gravitational theories; Black holes; Physics of black holes; Satellite orbits; Harmonics of the gravity potential field
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1. Introduction

There is nowadays wide consensus (Genzel et al. 1996; Schödel et al. 2002; Ghez et al. 2008) that the Galactic Center (GC) hosts a Supermassive Black Hole (SBH) (Wollman et al. 1977; Falcke et al. 1993) whose position coincides with that of the radio-source Sagittarius A* (Sgr A*) (Balick & Brown 1974; Reid et al. 2007) at $d = 8.28 \pm 0.44$ kpc from us (Gillessen et al. 2009a); for a popular overview of such an object, see, e.g., Melia (2007). The Galactic SBH is surrounded by a number of recently detected[1] main-sequence stars of spectral class B (Paumard et al. 2006; Gillessen et al. 2009a). Their relatively fast orbital motions, characterized by orbital periods $P_{\rm b} \gtrsim 16$ yr, high eccentricities $e \gtrsim 0.2$, random orientations i of their orbits in the sky and average distances from the SBH[2] $\overline{r} \gtrsim 2 \times 10^4 r_g$, allowed to dynamically infer a mass of about $M \approx 4 \times 10^6 M_\odot$ (Ghez et al. 2008; Gillessen et al. 2009a,b) for it.

The direct access to such S/S0 stars, and of other closer objects which may hopefully be discovered in the future, has induced several researchers to investigate various predictions[3] that the General Theory of Relativity (GTR) directly makes for their orbital motions along with other competing effects from standard Newtonian gravity which may mask the relativistic ones (Jaroszyński 1998; Fragile & Mathews 2000; Rubilar & Eckart 2001; Weinberg et al. 2005; Kraniotis 2007; Nucita et al. 2007; Will 2008; Preto & Saha 2009; Kannan & Saha 2009; Merritt et al. 2010; Iorio 2011a; Sadeghian & Will 2011). In fact, although the currently known stars, in a strict sense, cannot probe the strong field regime of GTR because of their relatively large distance from the SBH, on the other hand they yield a unique opportunity to put on the test GTR in the strongest field regime ever probed so far. Indeed, even in the double binary pulsar PSR J0737-3039A/B (Burgay et al. 2003; Lyne et al. 2004) $r_g/\overline{r}$ is one order of magnitude smaller than for S2, which is the closest SBH star discovered so far (Gillessen et al. 2009a).

In this paper we analytically work out the averaged variations of all the six standard osculating Keplerian parameters of a test particle caused by the rotation of the central objected endowed with angular momentum $\boldsymbol{S} = S\hat{\boldsymbol{k}}$ and quadrupole moment Q_2. Note that

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[1]They have been revealed and tracked in the near infrared since 1992 at the 8.2-m Very Large Telescope (VLT) on Cerro Paranal, Chile and the 3.58-m New Technology Telescope (NTT) on La Silla, Chile (Eckart & Genzel 1996), and since 1995 at the Keck 10-m telescope on Mauna Kea, Hawaii (Ghez et al. 1998). They are dubbed SN, or S0-N in the Keck nomenclature, where N is a progressive order number.

[2]Here r_g denotes the Schwarzschild radius.

[3]Concerning several effects related to propagating electromagnetic waves in connection with the stellar orbital motions like, e.g., relativistic reshifts, see Zucker et al. (2006); Angelil & Prasenjit (2010); Angelil et al. (2010); Angelil & Prasenjit (2011).

the stars orbiting the SBH can safely be considered test particles: their masses are about $m \lesssim 10^{-5}M$, and relativistic corrections to their internal structures are assumed to be too small to yield noticeable effects on their orbital motions. No assumptions about any specific spatial orientation for $\hat{\boldsymbol{k}}$ are made. Thus, our calculations are not restricted to a particular reference frame, and are valid also for different scenarios like, e.g., stellar planetary systems and planetary satellite geodesy in which natural and/or artificial test bodies are employed for testing GTR. Moreover, in order to keep our calculations as general as possible, we will not adopt any particular representation for $\hat{\boldsymbol{k}}$ in terms of specific angular variables in the sense that we will refer it to the global reference frame adopted; for a different approach, see Will (2008) in which $\hat{\boldsymbol{k}}$ is referred to the orbital plane of each star. Concerning the SBH in Sgr A*, the orientation of its spin axis is substantially unknown, despite the attempts by different groups (Meyer et al. 2007; Broderick et al. 2009, 2011) to constrain it using different parameterizations which yielded quite loose bounds. A strategy to partially overcome such an obstacle have been recently put forward by Hioki & Maeda (2009); it is based on the possible observation the apparent shape of the shadow cast by the BH on the plane of the sky, and would allow to measure S and the angle $i^{'}$ between $\hat{\boldsymbol{k}}$ and the line-of-sight.

The GTR prediction for the standard 1PN periastron precession, which is the analogous of the well known Mercury's perihelion precession of 42.98 arscec cty^{-1} and is independent of $\hat{\boldsymbol{k}}$, amounts to about[4]

$$\dot{\omega}_{\rm S2}^{\rm (1PN)} = 45 \pm 10 \text{ arcsec yr}^{-1} \tag{1}$$

for S2. The result of eq. (1), computed in a frame with the SBH at its origin, corresponds to a precession of $\dot{\xi}_{\rm S2} = 27 \pm 6$ microarcseconds per year (μas yr^{-1} in the following) as seen from the Earth. At present, it is still undetectable from the currently available direct astrometric measurements in terms of right ascension α and declination δ which barely cover just one full orbital period of S2. Indeed, according to Table 1 of Gillessen et al. (2009a), the present-day error in the periastron is $\sigma_{\omega} = 0.81$ deg $= 2916$ arcsec over about 16 yr, from which an uncertainty of about $\sigma_{\dot{\omega}} \simeq 182$ arcsec yr^{-1} in the periastron precession may naively be inferred: it corresponds to a limiting accuracy of $\sigma_{\dot{\xi}} = 110$ μas yr^{-1} in monitoring angular rates as seen from the Earth. As we will see, the sizes of the other precessions of S2 due to S and Q_2 may be smaller by about 2 and $4-5$ orders of magnitude, respectively for a moderate rotation of the SBH. Concerning future perspectives, according to Eisenhauer et al. (2009) future astrometric measurements of S2 may bring its 1PN periastron rate into the measurability domain; indeed, the periastron advance would indirectly be inferred from the corresponding apparent position shift in the recorded orbit. Moreover, the ASTrometric and phase-Referenced Astronomy (ASTRA) project (Eisner et al. 2010), to be applied to the Keck interferometer, should be able to

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[4]The quoted uncertainty comes from the errors in the parameters of both the SBH and S2 entering the GTR formula: they are displayed in Table 1.

monitor stellar orbits with an accuracy of (Pott et al. 2008) $\sigma_{\Delta\xi} \approx 30$ μas as seen from the Earth. The GRAVITY instrument (Gillessen et al. 2010), devoted to enhance the capabilities of the VLT interferometer (VLTI), aims to reach an accuracy of $\sigma_{\Delta\xi} \approx 10$ μas (Gillessen et al. 2010) in measuring astrometric shifts $\Delta\xi$ as seen from the Earth, which, among other things, would allow to explore the innermost stable circular orbits around the SBH (Vincent et al. 2011).

About testing GTR in the SBH scenario, we make the following general considerations. In order to meaningfully compare theoretical predictions for a given effect to its empirically determined counterpart, we need to know some of the key ambient parameters entering the predictions independently from the effects themselves we are just looking for. In the specific case, the mass M, the spin $\boldsymbol{S}$ and the quadrupole Q_2 of the SBH should be known, if possible, independently of the precessions we are going to consider. Concerning the SBH mass M, the values which we presently have for it can be thought as inferred from the third Kepler law used in conjunction with the directly measured orbital period $P_{\rm b}$, and the semi-major axis a empirically determined by modeling the recorded stellar orbit in the plane of the sky with an ellipse (see Fig. 2 of Gillessen et al. (2009a)). Such a determination of M would be, in principle, "imprinted" by GTR itself since it implies a correction to the third Kepler law, but it is far too small with respect to the present-day accuracy in determining $P_{\rm b}$. Indeed, it is $\sigma_{P_{\rm b}} \simeq 10^{-1}$ yr (Ghez et al. 2008; Gillessen et al. 2009b), while the 1PN GTR correction to the Keplerian orbital period is (Damour & Deruelle 1986; Soffel 1989) $\Delta P_{\rm b}^{(1{\rm PN})} \propto (3\pi/c^2)\sqrt{GMa} \simeq 10^{-3}$ yr for S2. The same holds also if M is straightforwardly inferred, in a perhaps less transparent manner, as a solve-for quantity from multi-parameter global fits of all the stars' data: modeling[5] or not GTR at 1PN level has not yet statistically significant influence in its estimated values, as shown by Table 2 of Gillessen et al. (2009a). The quadrupole moment Q_2 of the SBH in Sgr A* may be measured, e.g., by means of imaging observations with Very Long Baseline Interferometry (VLBI) in the strong field regime; see Johannsen (2011); Bambi (2011) for recent reviews and other proposals. In regard to the spin $\boldsymbol{S}$ of Sgr A*, one tries to gain independent information about S from the interpretation of some measured Quasi-Periodic Oscillations (QPOs) in the X-ray spectrum of the electromagnetic radiation emitted by the gas orbiting in the accretion disk close to its inner edge (Kato et al. 2010; Genzel et al. 2003). More recent observations conducted with the Millimeter Very Long Baseline Interferometry (mm-VLBI), probing the immediate vicinity of the horizon, have been able to get information on S (Broderick et al. 2009, 2011). In interpreting such measurements, the validity of the Kerr (1963) metric as predicted by GTR is assumed, thus inferring S from, say, the radius of the inner edge extracted from the X-ray diagnostics. It is worth pointing out that the mere fact of obtaining a good

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[5]When such an approach is followed to test GTR, it is intended that different dynamical models, with and without GTR, are fitted to the same data sets to see if statistically significant differences occur in the solve-for estimated parameters.

fit of the Kerr (1963) metric to a certain empirically determined quantity like, e.g., the X-ray spectrum, getting a reasonable value for S as a least-square adjustable parameter, cannot be considered in itself as a test of the rotation-related predictions of GTR, also because other competing mechanisms to explain, say, the QPOs, whose physics is still rather disputed, exist. Independent empirical determinations of different effects connected with $\boldsymbol{S}$ are required, and the stellar orbital precessions would be just what we need. The greater the number of precessions empirically determined, the greater the number of GTR tests which can be performed. In principle, more than five[6] precessions are required since M, S, Q_2 and two components of $\hat{\boldsymbol{k}}$ must be determined; thus, the need for more than one star is apparent. Such a number of necessary orbital rates may be reduced if some of the aforementioned parameters are somehow reliably obtained from other sources. Of course, also the accuracy with which the precessions can be determined plays a role, in the sense that the previous reasoning holds in the ideal case in which all the three dynamical effects considered are detectable. Basically, it is the same logic behind the usual tests in the binary systems hosting at least one active radio-pulsar (Damour 2009). Indeed, in that case the interpretation of just two empirically determined post-Keplerian effects[7] in terms of their 1PN-GTR predictions is not sufficient since it only allows to obtain the masses m_1 and m_2 of the system, which are a priori unknown. Genuine tests of GTR are made when more than two post-Keplerian parameters are empirically determined, and the additional ones are interpreted with GTR by using in their analytical predictions just the previously obtained values for m_1 and m_2 (Damour 2009).

The plan of the paper is as follows. In Section 2 we review basic facts of standard perturbation theory which will be applied in Section 3 to Q_2 (Section 3.1) and $\boldsymbol{S}$ (Section 3.2). In Section 3.4 it is briefly remarked that also gravitational waves with ultra-low frequency traveling from the outside could be constrained by the orbital precessions of the stars in Sgr A*. In Section 3.3 we compare our results to those obtained by Will (2008). Numerical evaluations of the effects worked out in Section 3 are presented in Section 4. Section 5 is devoted to summarizing our findings and to the conclusions.

2. Overview of the method adopted

Here we deal with a generic perturbing acceleration $\boldsymbol{A}$ induced by a given dynamical effect which can be considered as small with respect to the main Newtonian monopole $A_{\rm Newton} = GM/r^2$, where G is the Newtonian constant of gravitation and r is the relative particle-body distance.

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[6]See also Will (2008).

[7]In the binary pulsar systems the effects which can, actually, be inferred from the data are not limited just to the post-Keplerian periastron precession.

First, $\boldsymbol{A}$ has to be projected onto the radial, transverse and normal orthogonal unit vectors $\hat{\boldsymbol{R}}, \hat{\boldsymbol{T}}, \hat{\boldsymbol{N}}$ of the co-moving frame of the test particle orbiting the central object. Their components, in cartesian coordinates of a reference frame centered in the primary, are (Montenbruck & Gill 2000)

$$\hat{\boldsymbol{R}} = \begin{pmatrix} \cos\mathit{\Omega}\cos u \ -\cos i\sin\mathit{\Omega}\sin u \\ \sin\mathit{\Omega}\cos u + \cos i\cos\mathit{\Omega}\sin u \\ \sin i\sin u \end{pmatrix} \tag{2}$$

$$\hat{\boldsymbol{T}} = \begin{pmatrix} -\sin u\cos\mathit{\Omega} - \cos i\sin\mathit{\Omega}\cos u \\ -\sin\mathit{\Omega}\sin u + \cos i\cos\mathit{\Omega}\cos u \\ \sin i\cos u \end{pmatrix} \tag{3}$$

$$\hat{\boldsymbol{N}} = \begin{pmatrix} \sin i\sin\mathit{\Omega} \\ -\sin i\cos\mathit{\Omega} \\ \cos i \end{pmatrix}. \tag{4}$$

In eq. (2)-eq. (4), $\mathit{\Omega}, \omega, i$ are the longitude of the ascending node[8], the argument of pericenter, reckoned from the line of the nodes, and the inclination of the orbital plane to the reference $\{xy\}$ plane, respectively. In this specific case, we will choose the unit vector $\hat{\boldsymbol{\rho}}$ of the line-of-sight, pointing from the object to the observer, to be directed along the positive z axis, so that the $\{xy\}$ plane coincides with the usual plane of the sky which is tangential to the celestial sphere at the position of the BH. With such a choice, corresponding to the frame actually used in data reduction (Eisenhauer et al. 2005; Ghez et al. 2008), i is the inclination of the orbital plane to the plane of the sky ($i = 90$ deg corresponds to edge-on orbits, while $i = 0$ deg implies face-on orbits), and $\mathit{\Omega}$ is an angle in it counted from the reference x direction; it is such a node which is actually determined from the observations (Ghez et al. 2008; Gillessen et al. 2009a,b), and, in general, it is not referred to the SBH's equator. Moreover, $u \doteq f + \omega$ is the argument of latitude. Subsequently, the projected components of $\boldsymbol{A}$ have to be evaluated onto the Keplerian ellipse

$$r = \frac{p}{1 + e\cos f},\ p \doteq a(1 - e^2), \tag{5}$$

where p is the semilatus rectum and a, e are the semi-major axis and the eccentricity, respectively. The cartesian coordinates of the Keplerian motion in space are (Montenbruck & Gill 2000)

$$\begin{aligned} x &= r\left(\cos\mathit{\Omega}\cos u \ -\cos i\sin\mathit{\Omega}\sin u\right), \\ y &= r\left(\sin\mathit{\Omega}\cos u + \cos i\cos\mathit{\Omega}\sin u\right), \\ z &= r\sin i\sin u. \end{aligned} \tag{6}$$

[8]It is an angle in the reference $\{xy\}$ plane from the reference x direction to the line of the nodes which is the intersection of the orbital plane with the reference plane $\{xy\}$.

The cartesian components of the velocity can be obtained as

$$\begin{aligned} v_x &= \frac{\partial x}{\partial f}\frac{df}{dt}, \\ v_y &= \frac{\partial y}{\partial f}\frac{df}{dt}, \\ v_z &= \frac{\partial z}{\partial f}\frac{df}{dt}, \end{aligned} \tag{7}$$

in which df/dt is given by (Roy 2005)

$$dt = \frac{(1-e^2)^{3/2}}{n(1+e\cos f)^2}df, \tag{8}$$

where $n \doteq \sqrt{GM/a^3}$ is the Keplerian mean motion related to the orbital period by $n = 2\pi/P_{\rm b}$. Thus, they are

$$\begin{aligned} v_x &= -\frac{an[\cos\mathit{\Omega}(\sin u + e\sin\omega) + \cos i\sin\mathit{\Omega}(\cos u + e\cos\omega)]}{\sqrt{1-e^2}}, \\ v_y &= \frac{an[-\sin\mathit{\Omega}(\sin u + e\sin\omega) + \cos i\cos\mathit{\Omega}(\cos u + e\cos\omega)]}{\sqrt{1-e^2}}, \\ v_z &= \frac{an\sin i(\cos u + e\cos\omega)}{\sqrt{1-e^2}}. \end{aligned} \tag{9}$$

Then, A_R, A_T, A_N are to be plunged into the right-hand-sides of the Gauss equations for the variations of the osculating Keplerian orbital elements. They are (Roy 2005; Soffel 1989)

$$\begin{aligned} \frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}}\left[A_R e\sin f + A_T\left(\frac{p}{r}\right)\right], \\ \frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na}\left\{A_R\sin f + A_T\left[\cos f + \frac{1}{e}\left(1-\frac{r}{a}\right)\right]\right\}, \\ \frac{di}{dt} &= \frac{1}{na\sqrt{1-e^2}}A_N\left(\frac{r}{a}\right)\cos u, \\ \frac{d\mathit{\Omega}}{dt} &= \frac{1}{na\sqrt{1-e^2}\sin i}A_N\left(\frac{r}{a}\right)\sin u, \\ \frac{d\omega}{dt} &= -\cos i\frac{d\mathit{\Omega}}{dt} + \frac{\sqrt{1-e^2}}{nae}\left[-A_R\cos f + A_T\left(1+\frac{r}{p}\right)\sin f\right], \\ \frac{d\varpi}{dt} &= 2\sin^2\left(\frac{i}{2}\right)\frac{d\mathit{\Omega}}{dt} + \frac{\sqrt{1-e^2}}{nae}\left[-A_R\cos f + A_T\left(1+\frac{r}{p}\right)\sin f\right], \\ \frac{d\mathcal{M}}{dt} &= n - \frac{2}{na}A_R\left(\frac{r}{a}\right) - \frac{(1-e^2)}{nae}\left[-A_R\cos f + A_T\left(1+\frac{r}{p}\right)\sin f\right], \end{aligned} \tag{10}$$

where[9] $\varpi \doteq \omega + \Omega$ is the longitude of pericenter, and $\mathcal{M}$ is the mean anomaly.

The right-hand-sides of eq. (13), computed for the perturbing accelerations of the dynamical effect considered, have to be inserted into the analytic expression of the time variation $d\Psi/dt$ of the osculating Keplerian orbital element Ψ of interest which, then, must be averaged over one orbital revolution by means of eq. (8).

3. Calculation of the long-term orbital effects

3.1. The long-term precessions caused by the quadrupole mass moment of the central body for an arbitrary orientation of its spin axis

The acceleration experienced by a test particle orbiting an oblate central mass rotating about a generic direction $\boldsymbol{\hat{k}}$ is

$$\boldsymbol{A}^{(Q_2)} = -\frac{3Q_2}{2r^4}\left\{\left[1 - 5\left(\boldsymbol{\hat{r}}\cdot\boldsymbol{\hat{k}}\right)^2\right]\boldsymbol{\hat{r}} + 2\left(\boldsymbol{\hat{r}}\cdot\boldsymbol{\hat{k}}\right)\boldsymbol{\hat{k}}\right\}, \tag{11}$$

where Q_2 is the quadrupole moment of the body, with $[Q_2] = \mathrm{L}^5\mathrm{T}^{-2}$. A dimensionless quadrupole parameter J_2 can be introduced by posing $Q_2 \rightarrow GM\mathcal{R}_e^2 J_2$, where $\mathcal{R}_e$ is the equatorial radius of the rotating body. According to the "no-hair" or uniqueness theorems of GTR (Chrusciel 1994; Heusler 1998), an electrically neutral BH is completely characterized by its mass M and angular momentum S only. As a consequence, all the multipole moments of its external spacetime are functions of M and S (Geroch 1970; Hansen 1974). In particular, the quadrupole moment of the BH is

$$Q_2 = -\frac{S^2 G}{c^2 M}. \tag{12}$$

The spatial orientation of the BH's spin axis can be considered as unknown. Thus, looking for a more direct connection with actually measurable quantities, we will retain a generic orientation for $\boldsymbol{\hat{k}}$ in the ongoing calculation, i.e. we will not align it to any of axes of the reference frame used. After having computed the $R-T-N$ components of eq. (11) by means of eq. (2)-eq. (4) as

$$\begin{aligned} A_R^{(Q_2)} &= \boldsymbol{A}^{(Q_2)}\cdot\boldsymbol{\hat{R}}, \\ A_T^{(Q_2)} &= \boldsymbol{A}^{(Q_2)}\cdot\boldsymbol{\hat{T}}, \\ A_N^{(Q_2)} &= \boldsymbol{A}^{(Q_2)}\cdot\boldsymbol{\hat{N}}, \end{aligned} \tag{13}$$

[9] It is a "dogleg" angle.

to be evaluated onto the unperturbed Keplerian ellipse, it is possible to obtain

$$\frac{da}{dt} = \frac{de}{dt} = 0, \tag{14}$$

for the semi-major axis and the eccentricity, as in the standard calculations in which $\hat{\boldsymbol{k}}$ is usually aligned with the z axis.

Instead, the inclination i undergoes a long-term variation given by

$$\frac{di}{dt} = -\frac{3Q_2}{2\sqrt{GMa^7}\left(1-e^2\right)^2}\mathfrak{I}\left(\Omega, i; \hat{\boldsymbol{k}}\right), \tag{15}$$

with

$$\mathfrak{I}\left(\Omega, i; \hat{\boldsymbol{k}}\right) \doteq \left(\hat{k}_x \cos\Omega + \hat{k}_y \sin\Omega\right)\left[\hat{k}_z \cos i + \sin i\left(\hat{k}_x \sin\Omega - \hat{k}_y \cos\Omega\right)\right]. \tag{16}$$

If $\hat{k}_x = \hat{k}_y = 0$, as in the usual calculation (Roy 2005), i stays constant.

Concerning the node Ω, its long-term variation is

$$\frac{d\Omega}{dt} = \frac{3Q_2}{4\sqrt{GMa^7}\left(1-e^2\right)^2}\mathfrak{O}\left(\Omega, i; \hat{\boldsymbol{k}}\right), \tag{17}$$

with

$$\begin{aligned}\mathfrak{O}\left(\Omega, i; \hat{\boldsymbol{k}}\right) &\doteq 2\hat{k}_z \cos 2i \csc i\left(\hat{k}_x \sin\Omega - \hat{k}_y \cos\Omega\right) + \\ &+ \cos i\left[\hat{k}_x^2 + \hat{k}_y^2 - 2\hat{k}_z^2 + \left(\hat{k}_y^2 - \hat{k}_x^2\right)\cos 2\Omega - 2\hat{k}_x\hat{k}_y \sin 2\Omega\right].\end{aligned} \tag{18}$$

Notice that $\hat{k}_x = \hat{k}_y = 0$ in eq. (18) yields the standard secular precession (Roy 2005) with

$$\mathfrak{O}\left(i\right) = -2\cos i. \tag{19}$$

The long-term change of the argument of pericenter ω is a little more cumbersome. It is

$$\frac{d\omega}{dt} = \frac{3Q_2}{16\sqrt{GMa^7}\left(1-e^2\right)^2}\mathfrak{o}\left(\Omega, i; \hat{\boldsymbol{k}}\right), \tag{20}$$

with

$$\begin{aligned}\mathfrak{o}\left(\Omega, i; \hat{\boldsymbol{k}}\right) &\doteq 8 - 11\hat{k}_x^2 - 11\hat{k}_y^2 - 2\hat{k}_z^2 + \left(\hat{k}_y^2 - \hat{k}_x^2\right)\cos 2\Omega - \\ &- 2\hat{k}_z\left(\cot i - 5\cos 3i \csc i\right)\left(\hat{k}_y \cos\Omega - \hat{k}_x \sin\Omega\right) - 2\hat{k}_x\hat{k}_y \sin 2\Omega + \\ &+ 5\cos 2i\left[2\hat{k}_z^2 - \hat{k}_x^2 - \hat{k}_y^2 + \left(\hat{k}_x^2 - \hat{k}_y^2\right)\cos 2\Omega + 2\hat{k}_x\hat{k}_y \sin 2\Omega\right].\end{aligned} \tag{21}$$

In the case $\hat{k}_x = \hat{k}_y = 0$ eq. (21) reduces to

$$\mathfrak{o}\,(i) = 2\,(3 + 5\cos 2i) = 4\left(4 - 5\sin^2 i\right), \tag{22}$$

which yields the standard expression for the secular precession of the pericenter (Roy 2005).

The longitude of the pericentre ϖ experiences a long-term variation given by

$$\frac{d\varpi}{dt} = \frac{3Q_2}{16\sqrt{GMa^7}\left(1-e^2\right)^2}\mathfrak{V}\left(\mathit{\Omega}, i; \hat{\boldsymbol{k}}\right), \tag{23}$$

with

$$\begin{aligned}
\mathfrak{V}\left(\mathit{\Omega}, i; \hat{\boldsymbol{k}}\right) &\doteq 8 - 11\hat{k}_x^2 - 11\hat{k}_y^2 - 2\hat{k}_z^2 + \left(\hat{k}_x^2 + \hat{k}_y^2 - 2\hat{k}_z^2\right)(4\cos i - 5\cos 2i) - \\
&- 4\left(\hat{k}_x^2 - \hat{k}_y^2\right)(3 + 5\cos i)\sin^2\left(\tfrac{i}{2}\right)\cos 2\mathit{\Omega} - \\
&- 2\hat{k}_y\hat{k}_z\sec\left(\tfrac{i}{2}\right)\left[\sin\left(\tfrac{3i}{2}\right) + 5\sin\left(\tfrac{5i}{2}\right)\right]\cos\mathit{\Omega} + \\
&+ 2\hat{k}_x\hat{k}_z\sec\left(\tfrac{i}{2}\right)\left[\sin\left(\tfrac{3i}{2}\right) + 5\sin\left(\tfrac{5i}{2}\right)\right]\sin\mathit{\Omega} - \\
&- 8\hat{k}_x\hat{k}_y\sin^2\left(\tfrac{i}{2}\right)(3 + 5\cos i)\sin 2\mathit{\Omega}.
\end{aligned} \tag{24}$$

For $\hat{k}_x = \hat{k}_y = 0$ eq. (24) reduces to

$$\mathfrak{V}\,(i) = 2\,[3 - (4\cos i - 5\cos 2i)] = 4(4 - 5\sin^2 i - 2\cos i), \tag{25}$$

which yields the usual expression for the secular rate of ϖ (Roy 2005).

Finally, the long-term change of the mean anomaly $\mathcal{M}$ is

$$\frac{d\mathcal{M}}{dt} = -\frac{3Q_2}{16\sqrt{GMa^7\left(1-e^2\right)^3}}\mathfrak{M}\left(\mathit{\Omega}, i; \hat{\boldsymbol{k}}\right), \tag{26}$$

with

$$\begin{aligned}
\mathfrak{M}\left(\mathit{\Omega}, i; \hat{\boldsymbol{k}}\right) &\doteq -8 + 9\hat{k}_x^2 + 9\hat{k}_y^2 + 6\hat{k}_z^2 + 3\left(\hat{k}_x^2 + \hat{k}_y^2 - 2\hat{k}_z^2\right)\cos 2i + \\
&+ 6\left(\hat{k}_x^2 - \hat{k}_y^2\right)\sin^2 i\cos 2\mathit{\Omega} + \\
&+ 12\left[\hat{k}_z\sin 2i\left(\hat{k}_y\cos\mathit{\Omega} - \hat{k}_x\sin\mathit{\Omega}\right) + \hat{k}_x\hat{k}_y\sin^2 i\sin 2\mathit{\Omega}\right].
\end{aligned} \tag{27}$$

Also in this case, for $\hat{k}_x = \hat{k}_y = 0$ the standard secular precession (Roy 2005) is recovered since eq. (27) reduces to

$$\mathfrak{M}\left(i\right) = -2\left(1 + 3\cos 2i\right) = -4\left(2 - 3\sin^2 i\right). \tag{28}$$

Incidentally, we remark that the field of applicability of eq. (14)-eq. (27) is not limited just to the BH arena, being them generally valid also for astrophysical binary systems, stellar planetary systems, and planetary satellite geodesy. In particular, they could be useful when satellite-based tests of GTR are performed or designed.

3.2. The Lense-Thirring long-term precessions for a generic orientation of the spin axis of the central body

According to GTR, the gravitomagnetic Lense-Thirring acceleration felt by a test particle moving with velocity $\boldsymbol{v}$ around a rotating body with angular momentum $\boldsymbol{S} = S\hat{\boldsymbol{k}}$ at large distance from it is

$$\boldsymbol{A}^{(\mathrm{LT})} = -2\left(\frac{\boldsymbol{v}}{c}\right) \times \boldsymbol{B}_g. \tag{29}$$

In eq. (29) the gravitomagnetic field $\boldsymbol{B}_g$, far from the central object where the Kerr (1963) metric reduces to the Lense-Thirring one, is

$$\boldsymbol{B}_g = -\frac{GS}{cr^3}\left[\hat{\boldsymbol{k}} - 3\left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}}\right)\hat{\boldsymbol{r}}\right]. \tag{30}$$

Concerning S, the existence of the horizon in the Kerr (1963) metric implies a maximum value for the angular momentum of a spinning BH (Bardeen et al. 1972; Melia et al. 2001), so that $S = \chi_g S_{\max}$, with

$$S_{\max} = \frac{M^2 G}{c}. \tag{31}$$

If $\chi_g > 1$, the Kerr (1963) metric would have a naked singularity without a horizon. Thus, closed timelike curves could be considered, implying a causality violation (Chandrasekhar 1983). Although not yet proven, the cosmic censorship conjecture (Penrose 1969) asserts that naked singularities cannot be formed via the gravitational collapse of a body. If the limit of eq. (31) is actually reached or not by astrophysical BHs depends on their accretion history (Bardeen 1970). In the case of Sgr A*, it may be $\chi_g \approx 0.44 - 0.52$ (Genzel et al. 2003; Kato et al. 2010) or even less (Broderick et al. 2009, 2011). Contrary to BHs, no theoretical constraints on the value of χ_g exist for stars. For main-sequence stars, χ_g depends sensitively on the stellar mass, and can be much larger than unity (Kraft 1968, 1970; Dicke 1970; Gray 1982). The case of compact stars was recently treated by Lo & Lin (2011), showing that for neutron stars with $M \gtrsim 1 M_\odot$ it should be $\chi_g \lesssim 0.7$, independently of the Equation Of State (EOS) governing the stellar matter. Hypotethical quark stars may have $\chi_g > 1$, strongly depending on the EOS and the stellar mass (Lo & Lin 2011).

In the standard derivations of the Lense-Thirring effect (Lense & Thirring 1918) existing in literature the reference $\{x, y\}$ plane was usually chosen coincident with the equatorial plane of the rotating mass. In principle, the Lense-Thirring orbital precessions for a generic orientation of $\boldsymbol{S}$ could be worked out with the Gauss equations in the same way as done for Q_2. Anyway, they were recently worked out (Iorio 2010), in a different framework, with the less cumbersome Lagrange planetary equations (Roy 2005). For the sake of convenience, we display here the final result

$$\begin{aligned}
\frac{da}{dt} &= 0,\\
\frac{de}{dt} &= 0,\\
\frac{di}{dt} &= \frac{2GS\left(\hat{k}_x\cos\Omega+\hat{k}_y\sin\Omega\right)}{c^2a^3(1-e^2)^{3/2}},\\
\frac{d\Omega}{dt} &= \frac{2GS\left[\hat{k}_z+\cot i\left(\hat{k}_y\cos\Omega-\hat{k}_x\sin\Omega\right)\right]}{c^2a^3(1-e^2)^{3/2}},\\
\frac{d\omega}{dt} &= -\frac{GS\left[6\hat{k}_z\cos i+(3\cos 2i-1)\csc i\left(\hat{k}_y\cos\Omega-\hat{k}_x\sin\Omega\right)\right]}{c^2a^3(1-e^2)^{3/2}},\\
\frac{d\varpi}{dt} &= -\frac{GS\left\{4\left[\hat{k}_z\cos i+\sin i\left(\hat{k}_x\sin\Omega-\hat{k}_y\cos\Omega\right)\right]-2\left[\hat{k}_z\sin i+\cos i\left(\hat{k}_y\cos\Omega-\hat{k}_x\sin\Omega\right)\right]\tan(i/2)\right\}}{c^2a^3(1-e^2)^{3/2}},\\
\frac{d\mathcal{M}}{dt} &= 0;
\end{aligned} \tag{32}$$

Notice that eq. (32) yields[10] just the usual Lense-Thirring secular rates (Lense & Thirring 1918; Soffel 1989) for $\hat{k}_x = \hat{k}_y = 0$. Contrary to such a scenario, the inclination i experiences a long-term gravitomagnetic change for an arbitrary orientation of $\boldsymbol{S}$: it is independent of the inclination i itself.

3.3. A comparison with a different approach

Will (2008) refers $\hat{\boldsymbol{k}}$ to the orbital plane of a generic star by choosing as orthonormal vectors $\boldsymbol{e}_p, \boldsymbol{e}_q, \boldsymbol{h}$: $\boldsymbol{e}_p$ is directed along the line of the nodes, $\boldsymbol{e}_q$ lies in the orbital plane perpendicularly to $\boldsymbol{e}_p$, and $\boldsymbol{h}$ is directed along the orbital angular momentum. They are

$$\boldsymbol{e}_p = \begin{pmatrix}\cos\Omega\\ \sin\Omega\\ 0\end{pmatrix} \tag{33}$$

[10]Concerning ϖ, by posing $i/2 \doteq \zeta$ it can be shown that $-2\cos i+\sin i\tan(i/2) = 1-3\cos i$.

$$\boldsymbol{e}_q = \begin{pmatrix} -\cos i \sin \mathit{\Omega} \\ \cos i \cos \mathit{\Omega} \\ \sin i \end{pmatrix} \quad (34)$$

$$\boldsymbol{h} = \begin{pmatrix} \sin i \sin \mathit{\Omega} \\ -\sin i \cos \mathit{\Omega} \\ \cos i \end{pmatrix}. \quad (35)$$

Note that eq. (33) and eq. (34) can be obtained from eq. (2) and eq. (3), respectively, by posing $u \rightarrow 0$, while eq. (35) coincides with eq. (4). Thus, one has

$$\begin{aligned} \hat{k}_x &= \hat{k}_p \cos \mathit{\Omega} + \left(\hat{k}_h \sin i - \hat{k}_q \cos i\right) \sin \mathit{\Omega}, \\ \hat{k}_y &= \hat{k}_p \sin \mathit{\Omega} - \left(\hat{k}_h \sin i - \hat{k}_q \cos i\right) \cos \mathit{\Omega}, \\ \hat{k}_z &= \hat{k}_h \cos i + \hat{k}_q \sin i, \end{aligned} \quad (36)$$

with

$$\begin{aligned} \hat{k}_p &= \sin \alpha \cos \beta, \\ \hat{k}_q &= \sin \alpha \sin \beta, \\ \hat{k}_h &= \cos \alpha, \end{aligned} \quad (37)$$

having introduced the polar angles α and β in the orbital frame.

Inserting eq. (36) and eq. (37) into the equations of Section 3.1 and Section 3.2 allows to obtain eq. (2a), eq. (2b), eq. (2c) of Will (2008) after some algebra.

3.4. Stellar orbital perturbations caused by ultra-low frequency gravitational waves

The stars orbiting the SBH in Sgr A* could also be used, in principle, as probes for detecting or constraining plane gravitational waves of ultra-low frequency ($\nu \approx 10^{-8} - 10^{-10}$ Hz or less) impinging on the system from the outside. Indeed, the passage of such waves through the orbits of the closest stars would cause long-term variations of all their Keplerian orbital elements, apart from the semi-major axis a. They have recently been worked out by Iorio (2011b) for general orbital configurations, i.e. without making a-priori assumptions on their inclinations and eccentricities of the perturbed test particle, and arbitrary directions of incidence for the wave. Conversely, gravitational waves can be generated within the stellar system of Sgr A*, as discussed by Freitag (2003).

4. Numerical evaluations

In Table 1 we quote the relevant physical and orbital parameters for the SBH-S2 system. The orbital period of S2 is $P_{\rm b} = 15.98$ yr, so that the astrometric measurements currently available cover a full revolution of it.

Table 1: Relevant physical and orbital parameters of the SBH-S2 system in Sgr A* (first row), and their uncertainties (second row). The Keplerian orbital elements of S2 were retrieved from Table 1 of (Gillessen et al. 2009a). The figure for χ_g comes from Genzel et al. (2003), while the one for the gravitational parameter $\mu \doteq GM$ is from a multi-star fit yielding $\mu = (4.30 \pm 0.50) \times 10^6 \mu_\odot$ (Gillessen et al. 2009a). The quoted value in m for the semi-major axis of S2 was obtained by multiplying its angular value $a = 0.1246 \pm 0.0019$ arcsec (Gillessen et al. 2009a) by the distance of the SBH $d = 8.28 \pm 0.44$ kpc (Gillessen et al. 2009a). For the angular momentum and the quadrupole moment of the SBH we used $S = \chi_g(M^2G)/c$ and $Q_2 = -(S^2G)/(c^2M) = -\chi_g^2(G^3M^3)/c^4$. The orbital period of S2 is $P_{\rm b} = 15.98$ yr $= 5.04 \times 10^8$ s. The figures for S and Q_2, obtained in the hypothesis that GTR is correct, strongly depends on χ_g, which is, at present, highly uncertain: for example, Kato et al. (2010) yield $\chi_g = 0.44 \pm 0.08$, while for Broderick et al. (2009, 2011) it could be even smaller. We will use them to indicatively give order-of-magnitude evaluations of the additional orbital precessions which would occur because of S and Q_2 according to GTR.

μ (m^3 s^{-2})	S (kg m^2 s^{-1})	Q_2 (m^5 s^{-2})	χ_g	a (m)	e	i (deg)	Ω (deg)
5.70×10^{26}	8.46×10^{54}	-6.22×10^{45}	0.52	1.54×10^{14}	0.8831	134.87	226.53
6.6×10^{25}	4.66×10^{54}	6.58×10^{45}	0.26	8×10^{12}	0.0034	0.78	0.72

The quadrupole-induced precessions of eq. (15)-eq. (27) are all linear combinations of the products of the components of $\hat{\boldsymbol{k}}$ plus, sometimes, a term independent of $\hat{\boldsymbol{k}}$: they can be cast into the form

$$\frac{d\Psi}{dt} = D_0\left(Q_2, a, e, i, \Omega\right) + \frac{1}{2}\sum_{s,l} D_{sl}\left(Q_2, a, e, i, \Omega\right)\hat{k}_s\hat{k}_l,\ \ s, l = x, y, z,\ \ \Psi = i, \Omega, \omega, \mathcal{M}. \quad (38)$$

The numerical values of the coefficients D_0 and $D_{sl} = D_{ls}$ for S2, in μas yr^{-1}, are quoted in Table 2. The largest effects occur for ω and $\mathcal{M}$ because of D_0, which is of the order of ≈ 1 milliarcsec yr^{-1} (mas yr^{-1}). The other terms are damped by the square of the components of $\hat{\boldsymbol{k}}$. Moreover, partial mutual cancelation may occur depending on the orientation of the SBH spin axis.

The Lense-Thirring precessions of eq. (32) are all linear combinations of the components

Table 2: Coefficients of the quadrupole precessions of S2, in μas yr^{-1}, according to Table 1. GTR was assumed for Q_2, with $\chi_g = 0.52$.

	D_0	D_{x^2}	D_{y^2}	D_{z^2}	D_{xy}	D_{xz}	D_{yz}
i	0	406	-406	0	43	558	588
Ω	0	427	384	-810	-809	-5	5
ω	-1149	1568	1584	294	293	-1254	1189
$\mathcal{M}$	-539	595	616	406	405	-587	556

of $\hat{\boldsymbol{k}}$: they can be cast into the form

$$\frac{d\Psi}{dt} = \sum_j C_j\left(S, a, e, i, \Omega\right)\hat{k}_j,\ \ j = x, y, z,\ \ \Psi = i, \Omega, \omega. \tag{39}$$

The numerical values of the coefficients C_j for S2, in arcsec yr^{-1}, are listed in Table 3. They are of the order of about 10^{-1} arcsec yr^{-1}, i.e. orders of magnitude larger than the

Table 3: Coefficients of the Lense-Thirring precessions of S2, in arcsec yr^{-1}, according to Table 1. In particular, $\chi_g = 0.52$ was used for the spin of the SBH.

	C_x	C_y	C_z
i	-0.14	-0.15	0
Ω	-0.15	0.14	0.21
ω	0.11	-0.10	0.45

quadrupole precessions of Table 2: also in this case, partial mutual cancelations may occur depending on $\hat{\boldsymbol{k}}$, thus impacting the detectability of the gravitomagnetic rates.

The figures of Table 2 and Table 3 can be compared with the present-day accuracies in empirically determining the orbital precessions of S2 listed in Table 4. They are of the order of $10^2 - 10^3$ arcsec yr^{-1}. The Lense-Thirring precessions of S2 (Table 3) are about 3 orders of magnitude smaller than the current accuracy, while the quadrupole effects of Table 2 are negligibly small.

By considering a fictitious star X with, say, the same orbital parameters of S2 apart from the semi-major axis a, assumed to be one order of magnitude smaller so that its orbital period would just be $P_\mathrm{b} = 0.5$ yr, it turns out that its 1PN GTR periastron precession would be as large as 4 deg yr^{-1}, while its Lense-Thirring and quadrupole precessions would be of the order of about $\approx 10^2$ arcsec yr^{-1} and ≈ 1 arcsec yr^{-1}, respectively.

Table 4: Naive evaluations of the uncertainties in the secular variations of the S2 osculating Keplerian orbital elements, in arcsec yr^{-1}, obtained by dividing the errors in the elements from Table 1 of Gillessen et al. (2009a) by a time interval $\Delta T \approx P_{\rm b}$. Concerning the mean anomaly, its uncertainty was evaluated from that of the time of periastron passage $t_{\rm p}$, released by Gillessen et al. (2009a), according to the expression for the mean anomaly at the epoch of periastron passage $\mathcal{M}_0 = -nt_{\rm p}$; also the errors coming from a and μ through n were taken into account.

$\sigma_{\dot i}$ (arcsec yr^{-1})	$\sigma_{\dot\Omega}$ (arcsec yr^{-1})	$\sigma_{\dot\omega}$ (arcsec yr^{-1})	$\sigma_{\dot{\mathcal{M}}}$ (arcsec yr^{-1})
176	163	182	1203

If, as expected, angular shifts of $\Delta\xi \approx 10$ μas, as seen from the Earth, will really become measurable in future thanks to GRAVITY and ASTRA, this would imply an accuracy of the order of $\Delta\Psi \approx (d/a)\,\Delta\xi = 16$ arcsec for S2, and 160 arsec for a star one order of magnitude closer to the SBH. If such targets will be discovered, their Lense-Thirring shifts should become detectable after some years, while the Q_2-induced perturbations would still remain hard to be measured, eve for $e \approx 0.9$.

5. Summary and conclusions

We analytically worked out the long-term, i.e. averaged over one full revolution, variations of all the six osculating Keplerian orbital elements of a test particle orbiting a non-spherical, spinning body endowed with angular momentum $\boldsymbol{S}$ and quadrupole moment Q_2 for a generic spatial orientation of its spin axis $\hat{\boldsymbol{k}}$. We did not restrict ourselves to any specific orbital configuration. Concerning $\hat{\boldsymbol{k}}$, we referred it to the reference frame actually used in order to make easier and more direct a comparison with the effectively determined quantities. Our results can be extended also to other scenarios like astrophysical binaries, stellar planetary systems and planetary satellite geodesy. Here we applied our results to the stars orbiting the SBH in Sgr A*: those identified so far are moving along highly elliptical trajectories with periods $P_{\rm b} \geq 16$ yr. The current level of accuracy in empirically determining the precessions of the angular orbital elements of S2, having $P_{\rm b} = 16$ yr, can be evaluated to be of the order of $\approx 10^2 - 10^3$ arcsec yr^{-1}. The predicted 1PN GTR periastron precession of S2, which is independent of the orientation of the spin axis of the SBH, is 40 ± 10 arcsec yr^{-1}. The predicted GTR spin and quadrupole-induced precessions of S2 are of the order of $\approx 10^{-1}$ arcsec yr^{-1} and $\approx 10^2 - 10^3 \mu$as yr^{-1}, respectively: they depend on $\hat{\boldsymbol{k}}$, and partial cancelations among their components may occur, thus reducing their magnitude. Concerning hypothetical stars with orbital periods of less than 1 yr, not yet discovered, the 1PN GTR periastron precessions would be as large as some deg yr^{-1}, while the S and Q_2 effects would be of the order of $\approx 10^2$ arcsec yr^{-1} and ≈ 1 arcsec yr^{-1}, respectively. Planned improvements of the infrared telescopes used so far aim to reach an accuracy level of ≈ 10 μas at best in measuring angular shifts as seen from the Earth corresponding to stellar orbital shifts of about $1.6 \times 10^1 - 10^2$ arcsec for S2 and stars closer than it by one order of magnitude, respectively.

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