Generalizations on Rabinowitz's Theorems

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Abstract We use the famous Benci-Rabinowitz's Saddle Point Theorem ([3])with Cerami-Palais-Smale condition to study the existence of new periodic solutions with a fixed period for second order Hamiltonian systems under weaker conditions than Rabinowitz's original conditions in his pioneer paper([11]),the key point of our proof is proving Cerami-Palais-Smale condition,which is difficult since no symmetry for the potential. We use Rabinowitz's Saddle Point Theorem to study periodic solution for sub-quadratic second order Hamiltonian systems.

Key Words: Second order Hamiltonian systems, Periodic solutions, Saddle Point Theorems.

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1. Introduction

In 1978, Rabinowitz firstly used variational methods in large to study the periodic solutions for second order Hamiltonian systems with super-quadratic potential.

$$\ddot{q} + V'(q) = 0 \tag{1.1}$$

He proved that

Theorem 1.0([11]) Suppose V satisfies

(V_1) $V \in C^1(R^n, R)$

(V_2) There exist constants $\mu > 2, r_0 > 0$ such that

$$0 < \mu V(x) \leq V'(x) \cdot x, \quad \forall |x| \geq r_0,$$

(V_3) $V(x) \geq 0, \quad \forall x \in R^n,$

(V_4) $V(x) = o(|x|^2)$, as $|x| \to 0$.

Then for any $T > 0$, (1.1) has a non-constant T-periodic solution.

In the last 30 years, there were many works for (1.1), we can refer ([3]-[9],[12,13]), and the references there. In this paper, we try to generalize the result of Rabinowitz,we get the following Theorem:

Theorem 1.1 Suppose V satisfies

$(V1)$ $V \in C^1(R^n, R)$

$(V2)$ There exist constants $\mu_1 > 2, \mu_2 \in R$ such that

$$V'(x) \cdot x \geq \mu_1 V(x) + \mu_2, \qquad \forall x \in R^n,$$

$(V3)$ There are $a_1 > 0, a_2 \in R$ such that

$$V(x) \geq a_1 |x|^{\mu_1} + a_2, \qquad \forall x \in R^n,$$

$$V'(x) \cdot x - 2V(x) \to +\infty, |x| \to +\infty$$

$(V4)$

$$V(x) \leq A|x|^2, |x| \to 0.$$

Then for any $T < (\frac{2}{A})^{1/2}\pi$, (1.1) has a non-constant T-periodic solution.

Remark Comparing Rabinowitz's Theorem1.0,the biggest difference is that we didn't assume the potential V is nonnegative,our conditions $V(2) - V(4)$ are weaker.

For sub-quadratic second order Hamiltonian system,we can get

Theorem 1.2 Suppose V satisfies

$(V1)$ $V \in C^1(R^n, R)$.

$(V2')$ There exist constants $\mu_1 < 2, \mu_2 \in R$ such that

$$V'(x) \cdot x \leq \mu_1 V(x) + \mu_2, \qquad \forall x \in R^n.$$

$(V3')$

$$V'(x) \cdot x - 2V(x) \to -\infty, |x| \to +\infty.$$

$(V4')$

$$V(x) \leq A|x|^2 + a.$$

$(V5)$

$$V(x) \to +\infty, |x| \to +\infty.$$

Then for any $T < (\frac{2}{A})^{1/2}\pi$, (1.1) has a T-periodic solution.

2. Some Lemmas

In order to prove Theorem 1.1, we define functional

$$f(q) = \frac{1}{2}\int_0^T |\dot{q}|^2 dt - \int_0^T V(q)dt, \qquad \forall q \in H^1 \tag{2.1}$$

where

$$H^1 = W^{1,2}(R/TZ, R^n). \tag{2.2}$$

Lemma 2.1([11,12]) Let $\widetilde{q} \in H^1$ be such that $f'(\widetilde{q}) = 0$

Then $\widetilde{q}(t)$ is a T-periodic solution for (1.1).

Lemma2.2(Sobolev-Rellich-Kondrachov, Compact Imbedding Theorem,[5],[6],[8],[13])

$$W^{1,2}(R/TZ, R^n) \subset C(R/TZ, R^n)$$

and the imbedding is compact.

Lemma 2.3(Eberlein-Shmulyan [14]) A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence.

Lemma 2.4([1],[8],[15]) Let $q \in W^{1,2}(R/TZ, R^n)$ and $q(0) = q(T) = 0$

We have Friedrics-Poincare's inequality:

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt$$

Let $q \in W^{1,2}(R/TZ, R^n)$ and $\int_0^T q(t)dt = 0$, then

(i) We have Poincare-Wirtinger's inequality

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt$$

(ii) We have Sobolev's inequality

$$\max_{0 \leq t \leq T} |q(t)| = \|q\|_\infty \leq \sqrt{\frac{T}{12}} \left(\int_0^T |\dot{q}(t)|^2 dt\right)^{1/2}$$

We define the equivalent norm in $H^1 = W^{1,2}(R/TZ, R^n)$

$$\|q\|_{H^1} = \left(\int_0^T |\dot{q}|^2 dt\right)^{1/2} + |q(0)|$$

Definition2.1([4]) Let X is a Banach space,$\{q_n\} \subset X$ satisfy

$$f(q_n) \to C, \qquad (1 + \|q_n\|)f'(q_n) \to 0. \tag{2.3}$$

Then we call $\{q_n\}$ satisfy Cerami-Palais-Smale condition.

Lemma 2.5(Benci-Rabinowitz [3],[5], Generalized Mountain-Pass Lemma) Let X be a Banach space, $f \in C(X, R)$. Let $X = X_1 \bigoplus X_2$, $\dim X_1 < +\infty$,X_2 is closed in X. Let

$$\begin{aligned}
& B_a = \{x \in X | \|x\| \leq a\}, \\
& S = \partial B_\rho \cap X_2, \rho > 0, \\
Q \quad = \quad & \{x_1 + se | (x_1, s) \in X_1 \times R^1, s \geq 0, \|x_1\|^2 + s^2 \leq R^2\}, \\
& \partial Q = (B_R \cap X_1) \cup (\partial B_R \cap (X_1 \bigoplus R^+ e)), R > \rho,
\end{aligned}$$

where $e \in X_2, \|e\| = 1$,

$$\partial B_R \cap (X_1 \bigoplus R^+ e) \quad = \quad \{x_1 + se | (x_1, s) \in X_1 \times R^+, \|x_1\|^2 + s^2 = R^2\}$$

If

$$f|_S \geq \alpha,$$

and

$$f|_{\partial Q} \leq \beta < \alpha,$$

Then $C = \inf_{\phi\in\Gamma}\sup_{x\in Q} f(\phi(x)) \geq \alpha$,if $f(q)$ satisfies $(CPS)_C$ on $[\beta, \alpha]$,then C is a critical value for f.

Lemma 2.6(Rabinowitz's Saddle Point Theorem [12], Mawhin-Willem [8]) Let X be a Banach space and let $f \in C^1(X, R)$, let $X = X_1 \bigoplus X_2$ with

$$\mathrm{dim X_1} < +\infty$$

and

$$\sup_{S_R^1} f < \inf_{X_2} f,$$

where $S_R^1 = \{u \in X_1 | |u| = R\}$.

Let $B_R^1 = \{u \in X_1, |u| \leq R\}, M = \{g \in C(B_R^1, X) | g(s) = s,\ s \in S_R^1\}$

$$C = \inf_{g\in M} \max_{s\in B_R^1}(g(s))$$

Then $C \geq \inf_{X_2} f$, if f satisfies $(PS)_C$ condition, C is a critical value of f.

3. The Proof of Theorems 1.1 and 1.2

Lemma 3.1 If $(V1)-(V3)$ in Theorem1.1 hold, then $f(q)$ satisfies the $(Cerami-Palais-Smale)$ condition on H^1.

Proof Let $\{q_n\} \subset H^1$ satisfy

$$f(q_n) \to C, \qquad (1 + \|q_n\|)f'(q_n) \to 0. \tag{3.1}$$

Then we claim $\{q_n\}$ is bounded. In fact,by $f(q_n) \to C$, we have

$$\frac{1}{2}\|\dot{q}_n\|_{L^2}^2 - \int_0^T V(q_n)dt \to C \tag{3.2}$$

By $(V2)$ we have

$$\begin{aligned} < f'(q_n), q_n > &= \|\dot{q}_n\|_{L^2}^2 - \int_0^T (< V'(q_n), q_n >)dt \\ &\leq \|\dot{q}_n\|_{L^2}^2 - \int_0^T [\mu_2 + \mu_1 V(q_n)]dt \end{aligned} \tag{3.3}$$

By (3.2) and (3.3) we have

$$< f'(q_n), q_n > \ \leq \ a\|\dot{q}_n\|_{L^2}^2 + C_1 + \delta, n \to +\infty \tag{3.4}$$

Where $C_1 = C\mu_1 - T\mu_2 + \delta, \delta > 0, a = 1 - \frac{\mu_1}{2} < 0.$

By $f'(q_n) \to 0$, there exist $C_2 > 0$ and $C_3 > 0$ such that

$$| < f'(q_n), q_n > | \leq C_2 + C_3\|q_n\| = C_2 + C_3(\|\dot{q}_n\|_{L^2} + |q_n(0)|) \tag{3.5}$$

By (3.4) and (3.5) we have

$$-(C_2 + C_3\|q_n\|) \leq \left(1 - \frac{\mu_1}{2}\right) \|\dot{q}_n\|^2_{L^2} + C_1 + \delta, \tag{3.6}$$

If $\|\dot{q}_n\|_{L^2}$ is unbounded, then since $\mu_1 > 2, |q_n(0)|$ must be unbounded and there exists subsequence, still denoted by $\{q_n\}$, and $b > 0$, s.t.

$$|q_n(0)| \geq b\|\dot{q}_n\|^2_{L^2}, \tag{3.7}$$

By Newton-Leibniz formula and Cauchy-Schwarz inequality,we have

$$\begin{aligned} |q_n(t)| &\geq |q_n(0)| - \|\dot{q}_n\|_2 \\ &\geq b\|\dot{q}_n\|_2^2 - \|\dot{q}_n\|_2 \to +\infty, \quad \text{as n} \to +\infty \end{aligned} \tag{3.8}$$

$$\min_{0 \leq t \leq 1} |q_n(t)| \to +\infty, \quad \text{as n} \to +\infty \tag{3.9}$$

We notice that

$$f'(q_n)q_n = \int_0^T |\dot{q}_n|^2 dt - \int_0^T < V'(q_n), q_n > dt \tag{3.10}$$

$$= 2f(q_n) + \int_0^T [2V(q_n) - < V'(q_n), q_n >] dt \tag{3.11}$$

By (V3)and (3.1),this is a contradiction,so $\|\dot{q}_n\|_{L^2} \leq M_1$. Then we claim $|q_n(0)|$ is also bounded. Otherwise, there a subsequence, still denoted by q_n, s.t. $|q_n(0)| \to +\infty$,since $\|\dot{q}_n\| \leq M_1$,then

$$\min_{0 \leq t \leq 1} |q_n(t)| \geq |q_n(0)| - \|\dot{q}_n\|_2 \to +\infty, \text{as n} \to +\infty \tag{3.12}$$

Similar to the above proof,(3.11) is a contradiction.

So $\|u_n\| = \|\dot{u}_n\|_{L^2} + |u_n(0)|$ is bounded.

By the embedding theorem, $\{q_n\}$ has a weakly convergent subsequence which uniformly converges to $q \in H^{1,2}$.

Hence by $V \in C^1$ we have

$$V(q_n) \to V(q), < V'(q_n), q_n > \to < V'(q), q >, \tag{3.13}$$

Furthermore, it's standard step for the rest proof , the weakly convergent subsequence is also strongly convergent to $q \in H^{1,2}$.

Now we prove **Theorem 1.1.** In Benci-Rabinowitz's Saddle Point Theorem, we take

$$X_1 = R^n, X_2 = \{u \in W^{1,2}(R/TZ, R^n), \int_0^T udt = 0\}$$

$$S = \left\{ u \in X_2 | \left(\int_0^T |\dot{u}_2|^2 dt \right)^{1/2} = \rho \right\},$$

$$\partial Q = \{u_1 \in R^n | |u_1| \leq R\} \cup$$

$$\left\{ u = u_1 + se, u_1 \in R^n, e \in X_2, \|e\| = 1, s > 0, \|u\| = (|u_1(0)|^2 + s^2)^{1/2} = R > \rho \right\}.$$

When $q \in X_2$,by Sobolev's inequality,$\int_0^T |\dot{q}|^2 dt \to 0$ implies $max|q(t)| \to 0$.So when$\int_0^T |\dot{q}|^2 dt \to 0$, $(V4)$ implies

$$V(q) \leq A|q|^2$$

When $q \in X_2$,we have Poincare-Wirtinger inequality, so when

$$\rho = [\int_0^T |\dot{q}|^2 dt]^{\frac{1}{2}} \to 0$$

We have

$$f(q) \geq \frac{1}{2} \int_0^T |\dot{q}|^2 dt - A \int_0^T |q|^2 dt$$
$$\geq [\frac{1}{2} - A(2\pi)^{-2} T^2] \rho^2,$$

On the other hand, if $q \in X_1$,then we have

$$f(q) = -\int_0^T V(q) dt \to -\infty, |q| = R \to +\infty,$$

if

$$q \in \left\{ q = u_1 + se, u_1 \in R^n, e \in X_2, \|e\| = 1, s > 0, \|u\| = (|u_1(0)|^2 + s^2)^{1/2} = R > \rho \right\},$$

then by $(V3)$ and Jensen's inequality,we have

$$f(q) = \frac{1}{2}s^2 - \int_0^T V(u_1 + se) dt$$
$$\leq \frac{1}{2}s^2 - \int_0^T (a|u_1 + se|^{\mu_1} + b) dt$$
$$\leq \frac{1}{2}s^2 - [aT^{1-\frac{\mu_1}{2}} (\int_0^T |u_1 + se|^2 dt)^{\frac{\mu_1}{2}} + bT]$$
$$= \frac{1}{2}s^2 - aT^{1-\frac{\mu_1}{2}} [T|u_1|^2 + s^2 \int_0^T |e(t)|^2 dt]^{\frac{\mu_1}{2}} - bT$$
$$\to -\infty, s \to +\infty (R \to +\infty).$$

The rest of the proof for Theorem 1.1 is obvious.

Using Rabinowitz's Saddle Point Theorem and similar methods to Theorem 1.1,we can prove Theorm1.2,here we omit it

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References

[1] R.A.Adams and J.F.Fournier, Sobolev spaces, Second Edition, Academic Press, 2003.

[2] A.Ambrosetti and P.Rabinowitz,Dual variational methods in critical point thery and applications,J.Funct. Anal.14(1973),349-381.

[3] V.Benci and P.Rabinowitz,Critical point theorem for indefinite functionals,Inv.Math.52(1979),241-273.

[4] Cerami G.,Un criterio di esistenza per i punti critici so variete illimitate, Rend. dell academia di sc.lombardo112(1978),332-336.

[5] Chang K.C.,Critical point theory and applications,Shanghai Academic Press,1986.

[6] J.Jost and X.Jost,Calculus of variations,Cambridge Univ. Press,1998.

[7] Y.Long, Index theory for symplectic paths with applications, Birkhauser Verlag, 2002.

[8] J. Mawhin and M.Willem, Critical point theory and Hamiltonian system, Springer, Berlin ,1989.

[9] L.Nirenberg,Variational and toplogical methods in nonlinear problems,Bull.AMS,New Series4(1981),267-302.

[10] R.Palais,S.Smale,A generalized Morse theory,BAMS70(1964),165-171.

[11] P.H.Rabinowitz, Periodic solutions of Hamiltonian systems, Comm. Pure Appl. Math. 31(1978), 157-184.

[12] P.H.Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS Reg. Conf. Ser. in Math. 65, AMS, 1986.

[13] M.Struwe ,Variational methods, Springer, Berlin, 1990.

[14] K.Yosida, Functional analysis, 5th ed., Springer, Berlin, 1978.

[15] W.P.,Ziemer,Weakly differentiable functions,Springer,1989.

[16] Zhang S.Q.,Notes on Rabinowitz's Saddle Point Theorems,Preprint,2008.