A Note on Periodic Solutions of Singular Hamiltonian Systems

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Abstract

In this paper, we apply the Minimax Method of Bahri- Rabinowitz ([8]) with Cerami-Palais-Smale conditin to study the existence of new periodic solutions with a prescribed energy for a class of singular second order Hamiltonian systems without any symmetry,our result generalize a result of Tanaka [20].A new point is proving $(\text{CPS})^+$ condition under no symmetry.

Key Words: Singular Newtonian equations, periodic solutions, Bahri-Rabinowitz's Minimax Method,Cerami-Palais-Smale conditin.

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1. Introduction

In 1975 and 1977, Gordon ([13,14]) firstly used variational methods to study periodic solutions of planar 2-body type problems, in 1980's and 1990's, Ambsosetti-Coti Zelati ,Bahri–Rabinowitz,Greco etc([1-8,10-12,15,18-20],...) studied 2-body type problems in $R^N (N \geq 3)$.

Let $\Omega = \mathbb{R}^N - \{0\} (N \in \mathbb{N}, N \geq 3)$, and let $V \in C^1(\mathbb{R} \times \Omega, \mathbb{R})$. The above authors studied the existence of periodic solutions $t \mapsto u(t) \in \Omega$, with a prescribed period, of the following second-order differential equations:

$$\ddot{u} = -V'(t, u) \tag{1.1}$$

there '$\cdot$'$=d/dt$, and $V'(t, \cdot)$ denotes the gradient of the function $V(t, \cdot)$ defined on Ω. Suppose that $V(t, x)$ is T-periodic in t and:

Condition (V_1). There exists a neighborhood $\mathcal{N}$ of 0 and a function $U \in C^1(\Omega, \mathbb{R})$ such that:
(i) $\lim\limits_{s \to 0} U(x) = -\infty$;
(ii)$-V(t, x) \geq |U'(x)|^2$ for every $x \in \mathcal{N} - \{0\}$ and $t \in [0, T]$.
Moreover

(iii)$\lim_{x\to 0} V(t,x) = -\infty$.

About the behaviour of $V(t,x)$ at the infinity, they suppose that one of the following conditions holds.
Condition (V_2). $\lim_{|x|\to\infty} V(t,x) = \lim_{|x|\to\infty} V'(t,x) = 0$ (uniformly) and $V(t,x) < 0$ for every $t \in [0,T], x \in \Omega$.
Condition (V_3). There exist $c_1, M_1, R_1, \nu > 0$ such that, for every $t \in [0,T]$ and $x \in \mathbb{R}^N$ with $|x| \geq R_1$:
(i)$|V'(t,x)| \leq M_1$;
(ii)$V(t,x) \geq c_1|x|^\nu$.
Condition (V_4). There exist $c_1, R_1 > 0, \theta > \frac{1}{2}, \nu > 1$ such that, for every $t \in [0,T], |x| \geq R_1$:
(i)$\theta(V'(t,x)|x) \leq V(t,x)$;
(ii)$V(t,x) \geq c_1|x|^\nu$.

Set $K = \{x \in \Omega | V'(t,x) = 0 \text{ for every } t \in [0,T]\}$; They got the following results.
Theorem 1.1(Greco [15]) If (V_1) and one of $(V_2) - (V_4)$ hold, and moreover $K = \emptyset$, then there at least one nonconstant T-periodic C^2 solution.
Theorem 1.2(Bahri-Rabinowitz [8],Greco [15]) Suppose that $\partial V/\partial t \equiv 0$, so $V(t,x) \equiv V(x)$. Moreover suppose the following.
Condition (V_5). K is compact (or empty).

Then, if (V_1) and one of $(V_2) - (V_4)$ hold, there exist infinitely many nonconstant T-periodic C^2 solutions.
In [5],[6], Ambrosetti-Coti Zelati studied the periodic solutions of a fixed energy $h \in R$ for Hamiltonian systems with singularity V:

$$\frac{1}{2}|\dot{q}|^2 + V(q) = h \tag{1.2}$$

They got the following Theorems:
Theorem 1.3(Ambrosetti-Coti Zelati[5,6]) Suppose $V \in C^1(R^n\backslash\{0\}, R)$ satisfies:
$(V1)$ $V(-q) = V(q)$
$(V2)$ $\exists \alpha \in [1,2)$, s.t.

$$V'(u) \cdot u \geq -\alpha V(u) > 0$$

$(V3)$ $\exists \delta \in (0,2), r > 0$, s.t.

$$V'(u) \cdot u \leq -\delta V(u), 0 < |u| < r$$

$(V4)$

$$V(q) \to 0, |u| \to +\infty$$

Then the system (1.1)-(1.2) has at least a non-constant weak periodic solution for any $h < 0$.

Theorem 1.4(Ambrosetti-Coti Zelati[5,6]) Suppose $V \in C^1(R^n\backslash\{0\}, R)$satisfies$(V1), (V3), (V4)$ and

$(V2')$ $\exists \alpha \in (0, 2)$, s.t.

$$V'(u) \cdot u \geq -\alpha V(u) > 0$$

$(V5)$ $V \in C^2(R^n\backslash\{0\}, R)$ and

$$3V'(u) \cdot u + (V''(u)u, u) > 0$$

Then the system (1.1)-(1.2) has at least a non-constant weak periodic solution for any $h < 0$.

Tanaka K. ([20]) studied some non-symmetrical cases, he combined and generalized the arguments of Ambrosetti-Coti Zelati ([5]) and Bahri-Rabinowitz ([8]) and got

Theorem 1.5(Tanaka[20]) Let $V \in C^1(R^n\backslash\{0\}, R)(n \geq 3)$ satisfies (V_2) and $(V2')$ and $(V3)$, then there exists at least one generalized solution of (1.1) and (1.2).

Furthermore, he got some results on the collision times.

In this paper, we observe that the Saddle Point Method of Bahri-Rabinowitz [8] works under Cerami-Palais-Smale conditin which is weaker than the original Palais-Smale condition,and we get the following result:

Theorem 1.6 Suppose $V(x) \in C^1(R^n\backslash\{0\}, R), (n \geq 3)$ satisfies $(V2')$ and $(V3)$ and $(V_2)'$:

$$\lim_{|x|\to\infty} V'(x) = 0, \tag{1.3}$$

Then $\forall h < 0$, there exists a nonconstant weak periodic solution for autonomous systems $(1.1) - (1.2)$.

2 A Few Important Lemmas

Definition 2.1([17]) Let X is a Banach space, $f \in C^1(X, R)$, if $\{x_n\} \subset X$ s.t.

$f(x_n) \to C, f'(x_n) \to 0$, then $\{x_n\}$ has a strongly convergent subsequence.

Then we call f satisfies $(PS)_C$ condition.

Lemma 2.2([5]) Let $f(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2 dt \int_0^1 (h - V(u))dt$ and $\widetilde{u} \in \Lambda_0 = \{u \in H^1, u(t) \neq 0, \forall t\}$ be such that $f'(\widetilde{u}) = 0$ and $f(\widetilde{u}) > 0$. Set

$$\frac{1}{T^2} = \frac{\int_0^1 (h - V(\widetilde{u}))dt}{\frac{1}{2}\int_0^1 |\dot{\widetilde{u}}|^2 dt} \tag{2.1}$$

Then $\widetilde{q}(t) = \widetilde{u}(t/T)$ is a non-constant T-periodic solution for (1.1)-(1.2)in Section 1.

By the above Lemma , we have

Lemma 2.3([5]) If $\bar{u} \in E$ is a critical point of $f(u)$ and $f(\bar{u}) > 0$, then

$\bar{q}(t) = \bar{u}(t/T)$ is a non-constant T-periodic solution of (1.1)-(1.2) in Section 1.

Lemma 2.4(Sobolev-Rellich-Kondrachov[16],[22])

$$W^{1,2}(R/TZ, R^n) \subset C(R/TZ, R^n)$$

and the imbedding is compact.

Lemma 2.5(Eberlein-Shmulyan [21]) A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence.

Lemma 2.6([16],[22]) Let $q \in W^{1,2}([0,T],R^n)$ and $q(0) = q(T) = 0$,then we have Friedrichs-Poincare's inequality:

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt$$

We define the equivalent norm in $H^1 = W^{1,2}(R/Z, R^n)$

$$\|u\| = \|u\|_{H^1} = \left(\int_0^1 |\dot{u}|^2 dt\right)^{1/2} + |u(0)|$$

3 The Proof of Theorem 1.6

Lemma 3.1

Under the assumptions of Theorem1.6,$f(u)$ satisfies $(CPS)^+$ condition,that is : If $\{u_n\} \subset \Lambda_0$ satisfy

$$0 < C \leq f(u_n) \leq d, \quad (1 + \|u_n\|)f'(u_n) \to 0. \tag{3.1}$$

Then $\{u_n\}$ has a strongly convergent subsequence.

Proof We claim $\int_0^1 |\dot{u}_n|^2 dt$ is bounded. In fact, by $f(u_n) \leq d$, we have

$$-\frac{1}{2}\|\dot{u}_n\|_{L^2}^2 \cdot \int_0^1 V(u_n)dt \leq d - \frac{h}{2}\|\dot{u}_n\|_{L^2}^2 \tag{3.2}$$

By $(V2')$ we have

$$\begin{aligned} < f'(u_n), u_n > &= \|\dot{u}_n\|_{L^2}^2 \cdot \int_0^1 (h - V(u_n) - \frac{1}{2} < V'(u_n), u_n >)dt \\ &\leq \|\dot{u}_n\|_{L^2}^2 \int_0^1 [h - (1 - \frac{\alpha}{2})V(u_n)]dt \end{aligned} \tag{3.3}$$

By (3.2) and (3.3) we have

$$\begin{aligned} < f'(u_n), u_n > &\leq h\|\dot{u}_n\|_{L^2}^2 + (1 - \frac{\alpha}{2})(2d - h\|\dot{u}_n\|_{L^2}^2) \\ &= (\frac{\alpha}{2}h)\|\dot{u}_n\|_{L^2}^2 + C_1 \end{aligned} \tag{3.4}$$

Where $C_1 = 2(1 - \frac{\alpha}{2})d, 0 < \alpha < 2, h < 0$. So $\|\dot{u}_n\|_2 \leq M_1$.

We notice that

$$\begin{aligned} f'(u_n) \cdot (u_n - u_n(0)) &= \int_0^1 |\dot{u}_n|^2 dt \int_0^1 (h - V(u_n))dt \\ &\quad -\frac{1}{2}\int_0^1 |\dot{u}_n|^2 dt \int_0^1 < V'(u_n), u_n - u_n(0))dt \\ &= 2f(u_n) - \frac{1}{2}\int_0^1 |\dot{u}_n|^2 dt \cdot \int_0^1 < V'(u_n), u_n - u_n(0)) \end{aligned}$$

Then we claim $|u_n(0)|$ is bounded.

Otherwise, there a subsequence, still denoted by u_n s.t. $|u_n(0)| \to +\infty$,since $\|\dot{u}_n\| \leq M_1$,then

$$\min_{0\leq t\leq 1} |u_n(t)| \quad \geq \quad |u_n(0)| - \|\dot{u}_n\|_2 \to +\infty, \text{as n} \to +\infty$$

Then by (V_2) we have

$$V'(u_n) \to 0,$$

By Friedrics-Poincare's inequality ,we have

$$\int_0^1 |\dot{u_n}(t)|^2 dt \geq \pi^2 \int_0^1 |u_n(t) - u_n(0)|^2 dt,$$

Hence

$$\int_0^1 V'(u_n)(u_n - u_n(0))dt \to 0$$

From (3.1) we have

$$\|f'(u_n)\| \to 0, \|f'(u_n)\|\|u_n\| \to 0.$$

Hence

$$f'(u_n) \cdot (u_n - u_n(0)) \to 0.$$

So $f(u_n) \to 0$, this is a contradication, hence $u_n(0)$ is bounded, and $\|u_n\| = \|\dot{u}_n\|_{L^2} + |u_n(0)|$ is bounded.

By the embedding theorem, $\{u_n\}$ has a weakly convergent subsequence which strongly converges to $u \in H^{1,2}$.

Furthermore, it's similar to Ambrosetti-Coti Zelati ([5]), the weakly convergent subsequence is also strongly convergent to $u \in W^{1,2}$.

Proof of Theorem 1.6

Once we proved Lemma3.1,the rest for the proof of Theorem1.6 is similar to the paper of Bahri-Rabinowitz([8]),here we omit the details.

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