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Conformal invariance of curvature perturbation

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We show that in the single component situation all perturbation variables in the comoving gauge are conformally invariant to all perturbation orders. Generally we identify a special time slicing, the uniform-conformal transformation slicing, where all perturbations are again conformally invariant to all perturbation orders. We apply this result to the δN formalism, and show its conformal invariance.

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I. INTRODUCTION

A large class of extensions of general relativity is described in the context of scalar-tensor theories of gravity [1], where a space-time metric $g_{\mu\nu}$ is coupled to a scalar field ϕ, with the other matter contents such as fermion fields being minimally coupled to gravity. By an appropriate conformal transformation

$$g_{\mu\nu} \to \overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} , \tag{1.1}$$

we can move to another frame. A prime example is the Einstein frame, where after the conformal transformation (1.1) the gravitational action becomes the Einstein-Hilbert one. These conformal frames are mathematically equivalent, so one can work in any frame as long as mathematical manipulations are concerned. It is, however, not explicitly clear whether these conformally related frames also enjoy physical equivalence [2].

In cosmology, we encounter various frames of the metric which are related by conformal transformation, such as Jordan frame, Einstein frame, string frame and so on. The physical equivalence between these frames is especially important, since the cosmological observations with improving sensitivity can probe gravitational theories on the largest observable scales [3]. Further, some models of the early universe which accounts for the primordial perturbations incorporate non-minimal coupling between gravity and a scalar field, such as the recently proposed standard model Higgs inflation [4]. Usually one moves from the original, Jordan frame to the Einstein frame by a conformal transformation, and then computes perturbations there. The invariance of the vector and tensor perturbations, and that of the scalar perturbation in the comoving gauge, are shown up to second order [5], but it is not clear if this invariance holds fully non-perturbatively. Regarding precise upcoming experiments that may detect non-linear signatures such as non-Gaussianity, it is important to clarify the full non-linear invariance of the perturbations [6].

In this article, we study the conformal invariance of the cosmological perturbation. Our particular attention will be given to the curvature perturbation on super-horizon scales. We will show that in the single component case all perturbations in the comoving slicing are conformally invariant to fully non-linear order. Once this is given, we can see that in the context of the δN formalism it is a matter of gauge transformation between different coordinate systems which impose different slicing conditions.

As a temporal gauge condition, we take the conformal transformation factor Ω to be unperturbed to full non-linear order. This may be called as the uniform-conformal-transformation slicing, or, uniform Ω slicing (UΩS). It is convenient to decompose Ω into the background and perturbation as $\Omega = \Omega_0 e^{\omega}$, then the U$\Omega$S means $\omega = 0$ as the slicing condition. Thus we have

$$\Omega = \Omega_0(t) \,. \tag{1.2}$$

Under this slicing condition it is obvious, almost a tautology, that all perturbed quantities are naturally invariant under the conformation transformation. This result applies even in multiple component situation: see (3.8) and (3.9) for relations implied by this slicing condition to non-linear orders of perturbation. Note that we may consider the action given by

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} f(\phi, R) - \frac{1}{2}\omega(\phi) g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} - V(\phi) \right] . \tag{1.3}$$

Here, Ω is a function of either the field ϕ for $f = F(\phi)R$-type gravity or a function of F or R for $f(R)$-type gravity, where

$$F \equiv \frac{\partial f}{\partial R} \,. \tag{1.4}$$

In these cases, the UΩS corresponds to the uniform-field slicing or the uniform-F slicing, respectively, and we may call it as the UFS.

II. SINGLE FIELD CASE

First, we consider the single field $F(\phi)R$-type gravity in which we can show the invariance, especially that of the curvature perturbation. As for our spatial metric, we write

$$\gamma_{ij} = a^2 e^{2\mathcal{R}} \widetilde{\gamma}_{ij} \,, \tag{2.1}$$

where $\widetilde{\gamma}_{ij}$ includes vector and tensor perturbations. We choose $\widetilde{\gamma}_{ij}$ such that it remains invariant under the conformal transformation as described in detail in Appendix A. Then, under the conformal transformation (1.1), (2.1) becomes

$$\overline{\gamma}_{ij} = \Omega^2 a^2 e^{2\mathcal{R}} \widetilde{\gamma}_{ij} = a^2 \Omega_0^2 \exp\left[2(\mathcal{R}+\omega)\right] \widetilde{\gamma}_{ij} \equiv a^2 \Omega_0^2 \exp\left(2\overline{\mathcal{R}}\right) \widetilde{\gamma}_{ij} \,, \tag{2.2}$$

so that the spatial curvature perturbation in the conformally transformed frame becomes simply

$$\overline{\mathcal{R}} = \mathcal{R} + \omega \,. \tag{2.3}$$

Thus, in the UΩS in which $\omega = 0$, we have $\overline{\mathcal{R}} = \mathcal{R}$. Further, as spelled out in Appendix A, the vector and tensor perturbations in $\widetilde{\gamma}$ defined with respect to a fixed fiducial background metric are also conformally invariant to fully non-linear order.

In the single component case, the UΩS implies $\delta\phi = 0$, the uniform field gauge or the comoving gauge for $F(\phi)R$-type gravity: in most scalar-tensor theories of gravity, the conformal transformation is a function of ϕ, $\Omega = \Omega(\phi)$. This result has an important implication that, for $F(\phi)R$ gravity [7] all perturbation variables in the uniform field gauge are conformally invariant. In (2.2), we have explicitly shown the invariance to all perturbation orders, and also have shown that why the invariance is natural in the UΩS which is the same as the uniform field gauge, or equivalently comoving gauge, in the single component case.

Now, we provide a complementary point of view using the δN formalism [8–10] in the universe dominated by a single scalar field ϕ. Since we will be interested in super-horizon scales in terms of the δN formalism, we restrict ourselves to zeroth order in $\epsilon \equiv k/(aH)$, where k is the comoving wavenumber of a characteristic perturbation of our interest. We consider a point on a comoving slice on which $\delta\phi = 0$. Hereafter we use the subscript c to denote the quantities evaluated on the comoving slices, and likewise f and $\overline{f}$ on the flat and $\overline{\text{flat}}$ slices (see below). When the universe is dominated by a single component ϕ, the comoving slice is the same as the UΩS, thus $\overline{\mathcal{R}}_c = \mathcal{R}_c$. That is, the comoving curvature perturbation is the same and is thus conformally invariant.

Until this point we have not yet related the perturbation in the number of e-folds δN to $\mathcal{R}_c$. We can compute N in the standard manner. From (2.1), the expansion scalar $\theta = 3H$, viz. the "local" Hubble parameter is given by

$$H = \frac{1}{\mathcal{N}}\frac{d}{dt}\log\left(ae^{\mathcal{R}}\right) , \tag{2.4}$$

where $\mathcal{N}$ is the lapse function. From this, the number of e-folds N is given by the integral of H along the proper time $d\tau = \mathcal{N}dt$ as

$$N = \int_i^e H d\tau = \log\left[\frac{a(t_e)}{a(t_i)}\right] + \mathcal{R}(t_e) - \mathcal{R}(t_i) \equiv N_0 + \mathcal{R}(t_e) - \mathcal{R}(t_i)\,. \tag{2.5}$$

Here, t_i and t_e denote the initial and the final moments respectively.

Now, we consider the conformally transformed frame. We can take similar steps to obtain $\overline{H} = d\log\left(a\Omega e^{\mathcal{R}}\right)/(\mathcal{N}\Omega dt)$ and $d\overline{\tau} = \mathcal{N}\Omega dt$, so that the number of e-folds in this frame $\overline{N}$ becomes

$$\overline{N} = \int_i^e \overline{H} d\overline{\tau} = \log\left[\frac{(a\Omega_0)(t_e)}{(a\Omega_0)(t_i)}\right] + \overline{\mathcal{R}}(t_e) - \overline{\mathcal{R}}(t_i)\,. \tag{2.6}$$

Note that the background is *different* from N_0 as

$$\overline{N}_0 \equiv \log\left[\frac{(a\Omega_0)(t_e)}{(a\Omega_0)(t_i)}\right] = N_0 + \log\left[\frac{\Omega_0(t_e)}{\Omega_0(t_i)}\right] \equiv N_0 + \Delta_0\,. \tag{2.7}$$

Here, we make a specific choice for the initial and final moments where we evaluate the number of e-folds and the curvature perturbation. As a reference point, we set the final moments in both frames to be identical on a comoving slice. That is, $\phi_c(t = t_e) = \phi_0(t_e)$. Thus, we have $\omega(t_e) = \omega_c(t_e) = 0$ and in both frames $\mathcal{R}(t_e) = \overline{\mathcal{R}}(t_e) = \mathcal{R}_c(t_e)$. This is the conformal invariance of $\mathcal{R}_c$ we have seen above. Meanwhile, in the frame without overbar, we set the initial moment to be flat so that $\mathcal{R}(t_i) = \mathcal{R}_f(t_i) = 0$, and in the other frame "$\overline{\text{flat}}$" in the sense that $\overline{\mathcal{R}}(t_{\overline{f}}) = \overline{\mathcal{R}}_f(t_{\overline{f}}) = (\mathcal{R}+\omega)_{\overline{f}}(t_{\overline{f}}) = 0$. Then, in the frame without overbar, we have

$$N - N_0 \equiv \delta N = \mathcal{R}_c(t_e)\,, \tag{2.8}$$

and in the frame with overbar,

$$\overline{N} - \overline{N}_0 \equiv \delta\overline{N} = \mathcal{R}_c(t_e)\,. \tag{2.9}$$

That is, in both frames, $\delta N = \delta\overline{N} = \mathcal{R}_c$. Despite of the different background numbers of e-folds, their perturbations are invariant in both frames.

Notice that the curvature perturbation on a comoving slice $\mathcal{R}_c$ and the field fluctuation on a flat slice $\delta\phi_f$ are related by [see Eq. (287) in Ref. [11]]

$$-\mathcal{R}_c = \frac{H}{\dot{\phi}_0}\delta\phi_f - \frac{H}{\dot{\phi}_0^2}\delta\phi_f\delta\dot{\phi}_f + \frac{H^2}{2\dot{\phi}_0^3}\frac{d}{dt}\left(\frac{\dot{\phi}_0}{H}\right)\delta\phi_f^2 + \cdots\,. \tag{2.10}$$

Here we remind that we are working at zeroth order in $k/(aH)$. This relation holds at any arbitrary time. Meanwhile, in the δN formalism, by construction there is no field fluctuation on the final comoving slice where we evaluate the curvature perturbation. Thus, we are to relate the curvature perturbation on the *final* comoving slice $\mathcal{R}_c(t_e)$ to the field fluctuations on the *initial* flat slice $\delta\phi_f(t_i)$. Furthermore, using the fact that we are interested in the dynamics along a given background trajectory, we can show that up to second order in $\delta\phi$,

$$-\delta N = \frac{H}{\dot{\phi}_0}\delta\phi - \frac{H}{\dot{\phi}_0^2}\delta\phi\delta\dot{\phi} + \frac{H^2}{2\dot{\phi}_0^3}\frac{d}{dt}\left(\frac{\dot{\phi}_0}{H}\right)\delta\phi^2 + \cdots\,. \tag{2.11}$$

We give detailed steps to find this relation in Appendix B. Once evaluated on a flat slice, this should be identical to $\mathcal{R}_c(t_e)$ as we have shown in (2.8) [or (2.9) as well] and is in perfect agreement with (2.10). Thus, we conclude that $\mathcal{R}_c(t_e) = \mathcal{R}_c(t_i)$, i.e. the curvature perturbation is conserved on large scales[1].

Given the fully non-perturbative invariance of $\mathcal{R}_c$ in two frames, we can obtain a useful perturbative relation as follows. For this, it is convenient to insert another comoving slice common to

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[1] This could be regarded as an alternative proof of the conservation of the non-linear curvature perturbation as shown in Ref. [10] for a perfect fluid, and in Ref. [12] for a generic scalar field. However there is an important difference that here the separate universe approach is assumed from the beginning, while its validity is explicitly shown in Refs. [10, 12].

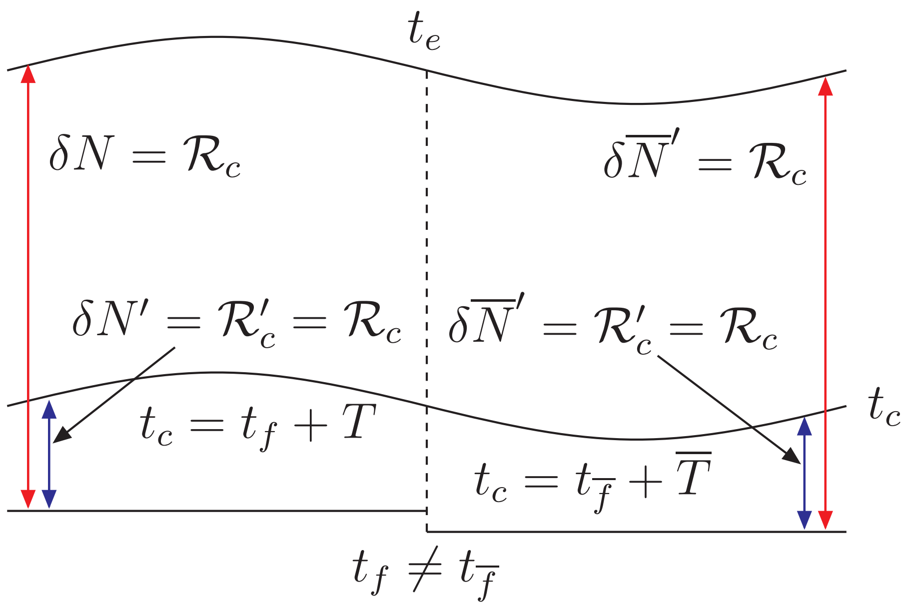

FIG. 1. A schematic figure showing the different slices which are connected by gauge transformations. Since $\mathcal{R}_c$ is conserved on large scales, $\mathcal{R}_c(t_e) = \mathcal{R}_c(t_c)$. The comoving slice on which time is denoted by t_c is common to both frames. Their initial slices are set in such a way that in one frame we set the initial slice to satisfy $\mathcal{R}(t_f) = 0$. In the other frame, on the other hand, we demand $\overline{\mathcal{R}}(t_{\overline{f}}) = (\mathcal{R} + \omega)\,(t_{\overline{f}}) = 0$. We call the former and the latter to be "flat" and "$\overline{\text{flat}}$", and the quantities evaluated on these slices by a subscript f and $\overline{f}$ respectively.

the both frames between the initial and the final moments, and we denote by t_c the time on this slice. Since we are interested in the large scale limit where $\mathcal{R}_c$ is conserved, the perturbations in the number of e-folds between the initial moments and t_c in two frames, $\delta N'$ and $\delta\overline{N}'$, are identical to $\mathcal{R}_c$. This is schematically shown in Fig. 1.

Then, from the frame without overbar, suppressing the dependence of N and N_0 on $\phi_0(t_c)$ which is common, for arbitrary $t < t_c$ we have[2]

$$\mathcal{R}_c(t_c) = N[\phi_f(t)] - N_0[\phi_0(t)] = \sum_n \frac{N_0^{(n)}[\phi_0(t)]}{n!}[\delta\phi_f(t)]^n\,, \tag{2.12}$$

where the superscript (n) denotes n-th derivative and the right hand side is evaluated on the flat slice. Likewise, we can write another relation,

$$\mathcal{R}_c(t_c) = \sum_n \frac{N_0^{(n)}[\phi_0(t)] + \Delta_0^{(n)}[\phi_0(t)]}{n!}\left[\overline{\delta\phi}_{\overline{f}}(t)\right]^n\,. \tag{2.13}$$

Note that we have the same background field ϕ_0, since it can be redefined in one frame to absorb the background conformal transformation factor Ω_0. Then, essentially now we have three different

[2] Here, we have implicitly made a stronger assumption than (2.11) that $\dot{\phi}_0$ can be written as a function of ϕ_0, so that we have a single degree of freedom in the phase space.

gauges conditions: $\mathcal{R}+\omega=0$, $\mathcal{R}=0$ and $\delta\phi=0$, which correspond to $\overline{\text{flat}}$, flat and comoving gauges respectively. We need a specific conformal transformation $\Omega(\phi)$ only to define the first gauge. Thus, using appropriate coordinate transformation laws, we can freely move between different gauges and relate one to another. For example, we consider the coordinate transformation from t_f and $t_{\overline{f}}$ to t_c,

$$t_c = t_f + T(t_f, \boldsymbol{x}) = t_{\overline{f}} + \overline{T}(t_{\overline{f}}, \boldsymbol{x}) \,. \tag{2.14}$$

Then, we only have to write the field fluctuations in terms of the translation T or $\overline{T}$. Since on the comoving slice there is no field fluctuation by construction, we have

$$\phi_0(t_c) = \phi_0[t_f + T(t_f, \boldsymbol{x})] \equiv \phi_f(t_f, \boldsymbol{x}) = \phi_0(t_f) + \delta\phi_f(t_f, \boldsymbol{x}) \,, \tag{2.15}$$

and a similar relation holds for $t_{\overline{f}}$. Then, using (2.14) we can write

$$t_{\overline{f}} = t_f + \left[T(t_f, \boldsymbol{x}) - \overline{T}(t_{\overline{f}}, \boldsymbol{x})\right] \equiv t_f + \mathcal{T}(t_f, \boldsymbol{x}) \,, \tag{2.16}$$

where $t_{\overline{f}}$ dependence of $\overline{T}$ on the right-hand side can be removed by iteratively using this relation. Then, from (2.15), we can find the linear relation between $\overline{\delta\phi}(t_f, \boldsymbol{x})$ and $\delta\phi(t_f, \boldsymbol{x})$ as

$$\overline{\delta\phi}_f(t_f, \boldsymbol{x}) = \delta\phi_f(t_f, \boldsymbol{x}) - \dot{\phi}_0(t_f, \boldsymbol{x})\mathcal{T}(t_f, \boldsymbol{x}) + \cdots \,. \tag{2.17}$$

This linear relation is in agreement with the well-known linear gauge transformation law [13] as it should be. Note that in fact it does not matter that we necessarily restrict ourselves to the transformation between two flat gauges, and hence we may drop all the subscript f. General coordinate transformations at arbitrary time would work as well.

III. MULTI-FIELD CASE

In this section, we consider the multiple component situation. We consider an action,

$$S = \int dx^4 \sqrt{-g} \left[\frac{1}{2} f\left(\phi^K, R\right) - \frac{1}{2} G_{IJ}\left(\phi^K\right) g^{\mu\nu} \phi^I_{;\mu} \phi^J_{;\nu} - V\left(\phi^K\right) \right] \,. \tag{3.1}$$

Under a conformal transformation (1.1) with the field redefinition $\Omega^2 \equiv F \equiv e^{\sqrt{2/3}\psi}$, (3.1) becomes

$$S = \int d^4x \sqrt{-\overline{g}} \left[\frac{1}{2}\overline{R} - \frac{1}{2}\overline{g}^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - \frac{1}{2F} G_{IJ} \overline{g}^{\mu\nu} \phi^I_{,\mu}\phi^J_{,\nu} - \overline{V} \right] \,, \tag{3.2}$$

where

$$\overline{V} \equiv \frac{1}{4F^2}\left(2V - f + RF\right) \,, \tag{3.3}$$

with $F \equiv \partial f/\partial R$. We have $\psi = \psi\left(\phi^K, R\right)$ in general. We consider two cases:

(I) The case with $f = f(R)$. In this case we have $\psi = \psi(R)$, which is an additional scalar field minimally coupled to gravity in the Einstein frame. An equivalent formulation in terms of a Brans-Dicke type scalar field Φ is given in Appendix C, with its relation to the canonical scalar field ψ in the Einstein frame.

(II) The case with $f = F\left(\phi^K\right) R$. In this case we have $\psi = \psi\left(\phi^K\right)$. Hence there is no additional scalar field. We have

$$S = \int d^4x \sqrt{-\overline{g}} \left[\frac{1}{2}\overline{R} - \frac{1}{2}\overline{G}_{IJ}\overline{g}^{\mu\nu}\phi^I_{,\mu}\phi^J_{,\nu} - \overline{V} \right] , \tag{3.4}$$

where

$$\overline{G}_{IJ} \equiv \frac{1}{F}G_{IJ} + \psi_{,I}\psi_{,J} . \tag{3.5}$$

We see that the effect of the conformal transformation is to change the non-linear sigma coupling term and the potential. These conformal transformation properties are presented in Appendix A of Ref. [7].

When the universe is dominated by a single component, we have shown the conformal invariance of all perturbations in the comoving slicing, which is equivalent to the UΩS. In the multi-component situation the UΩS is different from the uniform field slicing $\delta\phi^I = 0$ of a chosen field component I or the collective comoving slicing which sets ${T^0}_i = 0$ in the energy-momentum tensor (or $u_i = 0$ in the collective fluid four-vector. It should be noted however that it is in general impossible to uniquely define the comoving slicing in the multi-component situation because $u_{[\mu;\nu]}$ may not be zero). Consequently, the curvature perturbation in the gauge where $\delta\phi^I = 0$ is not conformally invariant: denoting the gauge $\delta\phi^I = 0$ by a subindex I, to linear order, we have

$$\overline{\mathcal{R}}_I = \mathcal{R}_I + \sum_{J\neq I} \frac{\Omega_{,J}\delta\phi^J_I}{\Omega} , \tag{3.6}$$

where

$$\delta\phi^J_I = \delta\phi^J - \dot{\phi}^J_0 \frac{\delta\phi^I}{\dot{\phi}^I_0} = -\frac{\dot{\phi}^J_0}{H}\left(\mathcal{R}_J - \mathcal{R}_I\right) , \tag{3.7}$$

so that clearly $\overline{\mathcal{R}}_I \neq \mathcal{R}_I$.

Nevertheless, even in the multiple component situation we can take UΩS with $\omega \equiv 0$ to arbitrary non-linear order. Under this slicing condition we have $\overline{\mathcal{R}} = \mathcal{R}$, and similarly for all the other metric perturbation variables [14]. Let us denote quantities in UΩS by suffix ω, e.g. $\mathcal{R}_\omega$.

For definiteness, let us consider the meaning of UΩS in the two cases of our interest. Since $\omega \equiv 0$ implies $\delta F = 0 = \delta\psi$, we have the following:

(I) For $F = F(R)$, we have

$$\delta F = F_{,R}\delta R + \frac{1}{2!}F_{,RR}\delta R^2 + \cdots = 0\,, \tag{3.8}$$

thus $\delta R = 0$.

(II) For $F = F\left(\phi^K\right)$ we have

$$\delta F = F_{,I}\delta\phi^I + \frac{1}{2!}F_{,IJ}\delta\phi^I\delta\phi^J + \cdots = 0\,. \tag{3.9}$$

This condition should be imposed among all the fields that appear in F.

Here we note that, in the spirit of the δN formalism, the final comoving hypersurface should be chosen such that there remains only a single adiabatic mode in the perturbation at and after that epoch. Once the universe has entered this stage, we also have the conservation of the comoving curvature perturbation $\mathcal{R}_c$. At this stage, the only dynamical degree of freedom coupled to gravity should be encoded in the function F, where the UΩS coincides with the comoving slicing (as discussed below) on super-horizon scales, and hence $\mathcal{R}_\omega = \mathcal{R}_c$ =constant[3]. All the other degrees of freedom should either be died out by then or be purely isocurvature at and after that epoch in the sense that they have no coupling to gravity whatsoever.

Conversely, at a stage during which multiple components of matter or fields have their individual dynamics, $\mathcal{R}_\omega$ is not conserved yet. At this stage, whether some quantities are conformally invariant or not is not really important. It is similar to the matter of the gauge choice. Some quantities may be conserved in a certain gauge, which may be a mathematically useful fact but has no physical/observational significance. In the present case, $\mathcal{R}_\omega$ and all the other quantities defined in UΩS are conformally invariant. But this would not help us much to understand the physics. It becomes useful and meaningful only after $\mathcal{R}_\omega$ becomes to be conserved.

Now we address the relation between the comoving slicing condition and the UΩS in the case of $f(R)$ gravity,

$$S = \int d^4x\sqrt{-g}\frac{1}{2}f(R)\,. \tag{3.10}$$

In the case of $f(R)$-gravity the comoving slicing *differs* from the UFS in general even at linear order. In the case of a minimally coupled scalar field we have $T^0{}_i = \phi^{;0}\phi_{,i}$, thus $\delta\phi \equiv 0$ implies

[3] Here we assume that F and consequently Ω depend non-trivially on the remaining dynamical degree of freedom. If it is not the case, then Ω will be simply a constant, and the conformal invariance is trivial.

$T^0{}_i = 0$ to fully non-linear order which is the comoving slicing condition. Whereas in the case of pure $f(R)$ gravity, we have [see Eq. (3) in Ref. [7]]

$$G_{\mu\nu} = \frac{1}{F}\left[\frac{1}{2}(f - RF)\, g_{\mu\nu} + F_{;\mu\nu} - g_{\mu\nu}F^{;\sigma}{}_{;\sigma}\right] . \tag{3.11}$$

If we define the right hand side as an effective $T_{\mu\nu}$, to the linear order we have [see Eqs. (18) and (B2) in Ref. [7]]

$$T^0{}_i = -\frac{1}{aF}\left(\delta\dot{F} - H\delta F - \dot{F}\alpha\right)_{,i} , \tag{3.12}$$

where $g_{00} \equiv -a^2(1 + 2\alpha)$. Thus, the UFS condition $\delta F = 0$ does not imply $T^0{}_i = 0$ in general in the original frame.

However, one can show that the UFS condition coincides with the comoving slicing condition $T^0{}_i = 0$ on super-horizon scales to full non-linear order provided that the decaying mode becomes negligible and there remains only a single adiabatic mode. One can show this with the help of the field equation for F given by the trace of (3.11). One finds that it is identical to the background field equation on super-horizon scales, i.e. at each spatial point $\boldsymbol{x}$ where a point means a Hubble size region, if one replaces t by τ. Thus the general solution to full non-linear order is given by $F = F\left(\tau(t, \boldsymbol{x}), \boldsymbol{x}\right)$. Then after the decaying mode has disappeared, $\partial F/\partial\tau$ becomes a function of F itself similar to the case of standard slow-roll inflation. This implies that on UFS on which $F = F_0(t)$, $\partial F/\partial\tau$ is also a function of only t. Then on UFS we have

$$T^0{}_i \propto n^\mu F_{;\mu\nu}P^\nu{}_i = (F_{;\mu}n^\mu)_{;\nu}P^\nu{}_i = \frac{\partial}{\partial x^i}\left(\frac{\partial F}{\partial\tau}\right) = 0 , \tag{3.13}$$

where n^μ is the unit normal to the uniform F slice, $P^\nu{}_\mu = \delta^\nu{}_\mu + n_\mu n^\nu$ is the induced metric on the uniform F slice, and $\partial/\partial\tau = \partial/(\mathcal{N}\partial t)$ is the proper time along x^i =constant [the shift vector is $\mathcal{O}(\epsilon)$ on super-horizon scales]. In fact, this leads to the conservation of the curvature perturbation on the comoving slices [12].

It may be instructive to spell out how this happens at linear order. In the UFS the perturbed field equation gives [7]

$$\dot{\mathcal{R}} = \left(H + \frac{\dot{F}}{2F}\right)\alpha . \tag{3.14}$$

Thus the fact that there remains only a single adiabatic mode implies $\alpha = \mathcal{O}(\epsilon^2)$, and hence $\dot{\mathcal{R}} = 0$ at leading order on super-horizon scales. Also the fact $\alpha = \mathcal{O}(\epsilon^2)$ implies $\tau = t$ at leading order. Thus the field F in UFS is actually given by $F\left(\tau(t, \boldsymbol{x}), \boldsymbol{x}\right) = F(t, 0) = F_0(t)$.

In the above, we have considered the case when the field F dominates the universe. In general, we may have other fields or matter which may have non-trivial background dynamics. In this multi-component situation, we have no conformal invariance of the comoving curvature perturbation (if it can ever be sensibly defined). In this situation, we can still define a conformally invariant curvature perturbation, $\mathcal{R}_\omega$ in UΩS. As we discussed, however, $\mathcal{R}_\omega$ will not be conserved if the multi-component matter or fields are still dynamical. It becomes to be conserved when the universe becomes dominated by a single adiabatic degree of freedom. Assuming this degree of freedom is encoded in F, we have the equivalence between UΩS and the comoving slicing, and hence recover the conformal invariance of $\mathcal{R}_c$, namely $\mathcal{R}_\omega = \mathcal{R}_c$ =constant.

IV. SUMMARY

To summarize, we have studied the non-perturbative conformal invariance of the cosmological perturbations. When the universe is dominated by a single component, the UΩS and the comoving slicing coincide and the comoving curvature perturbation $\mathcal{R}_c$ is conformally invariant fully non-perturbatively. Consistent with the conformal invariance of $\mathcal{R}_c$, we have shown that the δN formalism gives identical results irrespective of the choice of conformal frames.

When the universe is dominated by a multiple matter or field components, the comoving curvature perturbation is no longer conformally invariant. However, at and after the universe has settled down to a unique evolutionary trajectory, i.e. in an era when there remains only a single adiabatic mode, we recover the conservation of the comoving curvature perturbation on super-horizon scales and so it is conformally invariant.

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Appendix A: Conformal decomposition of the spatial metric

Here we give the decomposition of the spatial metric γ_{ij} in a conformally invariant way up to the determinant of γ_{ij}.

As a fiducial background, we take the flat metric, δ_{ij}. We introduce a traceless matrix C'_{ij}, and decompose it as

$$C'_{ij} = \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\Delta\right) E + F_{(i,j)} + h_{ij} \equiv C_{ij} - \frac{1}{3}\delta_{ij}\Delta E \,, \tag{A.1}$$

where the spatial indices are to be raised or lowered by the flat metric δ_{ij}, and F_i and h_{ij} satisfy ${F^i}_{,i} = {h^i}_i = {h^i}_{j,i} = 0$.

We write the spatial metric as

$$\gamma_{ij} = a^2(t) e^{2H_L} \widetilde{\gamma}'_{ij} \,, \tag{A.2}$$

where

$$\widetilde{\gamma}'_{ij} = \left(e^{2C'}\right)_{ij} \,. \tag{A.3}$$

Then, we define $\mathcal{R}$ as the sum of the trace contributions on the exponent,

$$\mathcal{R} \equiv H_L - \frac{1}{3}\Delta E \,, \tag{A.4}$$

and introduce

$$\widetilde{\gamma}_{ij} = \left(e^{2C}\right)_{ij} \,. \tag{A.5}$$

It is then clear that under the conformation transformation,

$$\gamma_{ij} \to \overline{\gamma}_{ij} = \Omega^2 \gamma_{ij} \,, \tag{A.6}$$

$\widetilde{\gamma}'_{ij}$ hence $\widetilde{\gamma}_{ij}$ is conformally invariant. In particular, for perturbations on a given background with $\Omega = \Omega_0 e^{\omega}$, the conformal transformation affects only H_L, or equivalently $\mathcal{R}$ as

$$\overline{H}_L = H_L + \omega \quad \leftrightarrow \quad \overline{\mathcal{R}} = \mathcal{R} + \omega \,. \tag{A.7}$$

We can extend the above to the non-flat fiducial metric in a similar way by covariantizing the derivatives, with the definitions of all the variables including that of $\mathcal{R}$, (A.4), as well as their conformal transformation properties remain the same.

Appendix B: Trajectory in phase space and δN

In this appendix, we provide a detailed derivation of (2.11). In many occasions, it is usually assumed that the number of e-folds N is only a function of ϕ and the dependence on $\dot{\phi}$ is neglected. But as can be read from (2.10) then we lose another independent degree of freedom in the phase space and cannot find $\delta\dot{\phi}$ dependence of δN.

To find the correct dependence of δN on $\delta\dot{\phi}$, we proceed as follows. In the phase space of ϕ, we consider that ϕ and $\dot{\phi}$ are functions of two coordinates, N and λ. Being the number of e-folds, N describes where we are on a given background trajectory, while λ labels which background trajectory we follow. Then, formally we can write $\delta\phi$ as

$$\delta\phi = \frac{\partial\phi}{\partial N}\delta N + \frac{\partial\phi}{\partial\lambda}\delta\lambda + \frac{1}{2}\frac{\partial^2\phi}{\partial N^2}\delta N^2 + \frac{\partial^2\phi}{\partial N\partial\lambda}\delta N\delta\lambda + \frac{1}{2}\frac{\partial^2\phi}{\partial\lambda^2}\delta\lambda^2 + \cdots \, . \tag{B.1}$$

We can do the same for $\delta\dot{\phi}$, but using $dN = Hdt$ we have

$$\frac{\delta\dot{\phi}}{H} = \frac{\partial^2\phi}{\partial N^2}\delta N + \frac{\partial^2\phi}{\partial N\partial\lambda}\delta\lambda + \cdots \, . \tag{B.2}$$

Multiplying by δN and using this relation to replace $\delta N\delta\lambda$ term in (B.1), we have

$$\delta\phi = \frac{\partial\phi}{\partial N}\delta N + \frac{\delta\dot{\phi}}{H}\delta N - \frac{1}{2}\frac{\partial^2\phi}{\partial N^2}\delta N^2 + \cdots \, , \tag{B.3}$$

where we have omitted higher order derivative terms as well as pure $\delta\lambda$ terms, because now we only consider the perturbation *along* a given trajectory.

Therefore, from (B.3) we can write δN up to second order as

$$\frac{\dot{\phi}_0}{H}\left(1 + \frac{\delta\dot{\phi}}{\dot{\phi}_0}\right)\delta N = \delta\phi + \frac{1}{2H}\frac{d}{dt}\left(\frac{\dot{\phi}_0}{H}\right)\delta N^2 \, . \tag{B.4}$$

Perturbatively expanding, we can find δN as

$$-\delta N = \frac{H}{\dot{\phi}_0}\delta\phi - \frac{H}{\dot{\phi}_0^2}\delta\phi\delta\dot{\phi} + \frac{H^2}{2\dot{\phi}_0^3}\frac{d}{dt}\left(\frac{\dot{\phi}_0}{H}\right)\delta\phi^2 + \cdots \, , \tag{B.5}$$

where we have reversed the time order, $\delta N \to -\delta N$, because in the context of the δN formalism it is defined to be the variation of N due to the variation of the *initial* conditions. Thus we can correctly extract the $\delta\dot{\phi}$ dependence of δN. Note that we could have found this dependence by considering $\dot{\phi}$ as an independent degree of freedom in the phase space. Once the trajectory follows the attractor, $\dot{\phi}_0$ can be written as a function of ϕ_0 and we may follow the naive expansion $\delta N = N'\delta\phi + N''\delta\phi^2/2 + \cdots$.

Appendix C: $f(R)$ gravity in Brans-Dicke form

It is known that $f(R)$ gravity can be cast into the form of a Brans-Dicke type theory. Here let us recapitulate it.

We can rewrite the $f(R)$ gravity action, (3.10), by introducing an auxiliary scalar field as

$$S = \int d^4x\sqrt{-g}\frac{1}{2}\left[f(s) + \lambda(R - s)\right] . \tag{C.1}$$

Then the variation with respect to λ gives $R = s$, and we recover the original action. On the other hand, if we take the variation with respect to s, we obtain the constraint,

$$\frac{df(s)}{ds} - \lambda = 0 . \tag{C.2}$$

Using this to eliminate λ from the above action, we obtain

$$S = \int d^4x\sqrt{-g}\frac{1}{2}\left[\frac{df(s)}{ds}R + f(s) - s\frac{df(s)}{ds}\right] . \tag{C.3}$$

If we further introduce a scalar field Φ by $\Phi \equiv df/ds$, then the above action is rewritten again as

$$S = \int d^4x\sqrt{-g}\frac{1}{2}\left[\Phi R - I(\Phi)\right] , \tag{C.4}$$

where $I(\Phi) \equiv s\, df(s)/ds - f(s)$. This is a Brans-Dicke scalar-tensor theory with a non-vanishing potential $V = I(\Phi)/2$, with the Brans-Dicke parameter $w_{\rm BD} = 0$.

The field equations are

$$R = \frac{dI(\Phi)}{d\Phi} , \tag{C.5}$$

$$\Phi\, G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}I(\Phi) + g_{\mu\nu}\nabla^\gamma\nabla_\gamma\Phi - \nabla_\mu\nabla_\nu\Phi = 0 . \tag{C.6}$$

This shows that Φ is a dynamical field despite the absence of an apparent kinetic term in (C.4).

We have not performed a conformal transformation so far. Hence the metric is still the original metric $g_{\mu\nu}$. Therefore the $f(R)$ action (3.10) and the $F(\Phi)R$ action (C.4) are completely equivalent to each other.

With the conformal transformation of $\overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, with $\Omega^2 = \Phi = e^{\sqrt{2/3}\,\psi}$, the action (C.4) can be transformed to the Einstein frame,

$$S = \int d^4x\sqrt{-\overline{g}}\left[\frac{1}{2}\overline{R} - \frac{1}{2}\overline{g}^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - \overline{V}(\psi)\right] . \tag{C.7}$$

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[1] See e.g. T. Damour and G. Esposito-Farese, Class. Quant. Grav. **9**, 2093 (1992) ;
Y. Fujii and K. Maeda, "The scalar-tensor theory of gravitation," *Cambridge, USA: Univ. Pr. (2003) 240 p*

[2] N. Deruelle and M. Sasaki, arXiv:1007.3563 [gr-qc].

[3] Y. S. Song and K. Koyama, JCAP **0901**, 048 (2009) [arXiv:0802.3897 [astro-ph]].

[4] F. L. Bezrukov and M. Shaposhnikov, Phys. Lett. B **659**, 703 (2008) [arXiv:0710.3755 [hep-th]].

[5] N. Makino and M. Sasaki, Prog. Theor. Phys. **86**, 103 (1991) ;
S. Koh, J. Korean Phys. Soc. **49**, S787 (2006) [arXiv:astro-ph/0510030] ;
N. Sugiyama and T. Futamase, Phys. Rev. D **81**, 023504 (2010).

[6] See also T. Chiba and M. Yamaguchi, JCAP **0810**, 021 (2008) [arXiv:0807.4965 [astro-ph]] for a different proof of the conformal equivalence.

[7] J. Hwang and H. Noh, Phys. Rev. D **65**, 023512 (2002) [arXiv:astro-ph/0102005].

[8] A. A. Starobinsky, JETP Lett. **42**, 152 (1985) [Pisma Zh. Eksp. Teor. Fiz. **42**, 124 (1985)] ;

[9] M. Sasaki and E. D. Stewart, Prog. Theor. Phys. **95**, 71 (1996) [arXiv:astro-ph/9507001] ;
M. Sasaki and T. Tanaka, Prog. Theor. Phys. **99**, 763-782 (1998) [arXiv:gr-qc/9801017].

[10] D. H. Lyth, K. A. Malik and M. Sasaki, JCAP **0505**, 004 (2005) [arXiv:astro-ph/0411220].

[11] H. Noh and J. Hwang, Phys. Rev. D **69**, 104011 (2004) [arXiv:astro-ph/0305123].

[12] A. Naruko, M. Sasaki, Class. Quant. Grav. **28**, 072001 (2011). [arXiv:1101.3180 [astro-ph.CO]].

[13] See e.g. H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984) ;
V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992).

[14] See e.g. J. O. Gong and H. M. Lee, arXiv:1105.0073 [hep-ph].