Some results on the representations for generalized Drazin inverse of 2X2 matrix

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Abstract: Some representations for the Drazin inverse of a 2×2 block matrix $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are Drazin invertible, in terms of the Drazin inverses of A and D are developed under the assumptions that $(I-D^{\pi})CA^{\pi}B=0$ and $AA^{\pi}B(I-D^{\pi})=0$ with the generalized Schur complement $S=D-CA^{d}B$ neither nonsingular nor zero.

Keywords: Block matrix; Generalized Drazin inverse; Schur complement

1 Introduction

Let X and Y be complex Banach spaces. Denote by $B(X,Y)$ the set of all bounded linear operators from X to Y. An element $T\in B(X)$ whose spectrum $\sigma(T)$ consists of the set $\{0\}$ is said to be quasi-nilpotent [1]. It is clear that T is quasi-nilpotent if and only if the spectral radius $\gamma(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}=0$. For $T\in B(X)$, the concept of the generalized Drazin inverse (for short GD-inverse) in a Banach algebra was introduced by Koliha [2], which is the unique (if exists) element $T^{d}\in B(X)$ such that $TT^{d}=T^{d}T$, $T^{d}TT^{d}=T^{d}$, $T-T^{2}T^{d}$ is quad quasi-nilpotent. If there exists an integer k such that $(T-T^{2}T^{d})^{k}=0$, then the least such k is the index of T, denoted by ind$(T)=k$. Otherwise, we say ind$(T)=\infty$. If T is generalized Drazin invertible, then the spectral idempotent T^{π} of T corresponding to $\{0\}$ is given by $T^{\pi}=I-TT^{d}$. The operator matrix form of T with respect to the space decomposition $X=N(T^{\pi})\oplus R(T^{\pi})$ is given by $T=T_1\oplus T_2$, where T_1 is invertible and T_2 is quasi-nilpotent.

In recent years, the study of GD-inverse has been of leading interest to many researchers (see [3]-[6]). This is because such inverses are useful tools for several applications. And properties of the Drazin inverse and its applications to singular differential equations and singular difference equations, to Markov chains and iterative methods, to structured matrices, and to perturbation bounds for the relative eigenvalue problem can be found in (see [3,7-9,10,11,12] and [13]). In [14], Campbell and Meyer posed the following open problem: find an explicit representation for the Drazin inverse of a 2×2 block matrix $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in terms of the blocks of the partition, where the blocks A and D are assumed to be square matrices. The motivation for this open problem is the desire to find general expressions for the solutions to the second-order system of the differential equations $Ex''(t)+Fx'(t)+Gx(t)=0$, where the matrix E is singular. The detailed discussion of the importance of the problem together with the prerequisite mathematical definitions needed for its statement can be found in [7]. Finding an explicit representation for the

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GD-inverse of a general 2×2 block matrix in terms of A^d and D^d with arbitrary A, B, C and D appears to be difficult. The generalized Schur complement $S = D - CA^d B$ plays an important role in the representations for M^d. However, to the best of our knowledge, it is still an open problem to find an explicit formula for M^d if A^d exists and the generalized Schur complement $S = D - CA^d B \neq \mathbf{0}$ is not invertible.

This paper is devoted to the GD-inverse of 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (1)$$

Where $A \in B(X)$ and $D \in B(Y)$ are GD-invertible. We will obtain some explicit GD-inverse formulae for a 2×2 operator matrix M under the conditions that $S = D - CA^d B \neq \mathbf{0}$ is not invertible. Our results do not appear in the literature and some resent results are extended with simplified proof. Moreover, we consider some applications of our results to obtain GD-inverse of various structured matrices.

2 Key lemmas and preliminaries

First we present some additive results for the GD-inverse of $P+Q$, which are closely connected to the GD-inverse of a 2×2 operator matrix.

Lemma 2.1 Let P and $Q \in \mathcal{B}(X)$ be GD-invertible.

(1) (see [15]) If $PQ = QP$, then $P+Q$ is GD-invertible if and only if $I + P^d Q$ is GD-invertible, in this case we have

$$(P+Q)^d = P^d (I + P^d Q)^d QQ^d + (I - QQ^d)\left[\sum_{n=0}^{\infty} (-Q)^n (P^d)^n\right] P^d + Q^d \left[\sum_{n=0}^{\infty} (Q^d)^n (-P)^n\right](I - PP^d).$$

(2) (see [16]) If $PQ = 0$, then $P+Q$ is GD-invertible and

$$(P+Q)^d = (I - QQ^d)\left[\sum_{n=0}^{\infty} Q^n (P^d)^n\right] P^d + Q^d \left[\sum_{n=0}^{\infty} (Q^d)^n P^n\right](I - PP^d).$$

If the generalized Schur complement $S = D - CA^d B$ is equal to 0, then M is GD-invertible. The following result is due to Wei in [17].

Lemma 2.2 (see [17]) Let $W = AA^d + A^d BCA^d$ and AW be GD-invertible. If $CA^\pi = 0$, $A^\pi B = 0$ and the generalized Schur complement $S = D - CA^d B$ is equal to 0, then M is GD-invertible and

$$M^d = \begin{pmatrix} I \\ CA^d \end{pmatrix} \left[(AW)^d\right]^2 A \begin{pmatrix} I & A^d B \end{pmatrix}.$$

Let $S_1 = D^\pi \sum_{n=0}^{\infty} D^n C (A^d)^{n+2}$ and $T_1 = \sum_{n=0}^{\infty} (D^d)^{n+2} CA^n A^\pi$. Recently, the Drazin inverse of a 2×2 block matrix has been studied by Hartwig and Cvetković-Ilić (see [18]).

Lemma 2.3 (see [18]) Let M be defined as in (1). If $BC = 0, BD = 0$, then M is GD-invertible and

$$M^d = \begin{pmatrix} A^d & (A^d)^2 B \\ S_1 + T_1 - D^d C A^d & D^d + (D^d T_1 + S_1 A^d) B - D^d (D^d C + C A^d) A^d B \end{pmatrix}.$$

Proof. Since A and D are GD-invertible, we have

$$\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^d & (A^d)^2 B \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (D^d)^2 C & D^d \end{pmatrix}.$$

Note that $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$ and $\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} = 0$. The result can be obtained directly by using the second result in Lemma 2.1. ■

Throughout this paper, we use some notations. Let A and D be GD-invertible,

$$X_1 = \mathcal{N}(A^\pi),\ X_2 = \mathcal{R}(A^\pi),\ Y_1 = \mathcal{N}(D^\pi) \text{ and } Y_2 = \mathcal{R}(D^\pi).$$

Let $S = D - CA^d B$ and $I_0 = I \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \oplus I$ be the invertible operator from $X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$ onto $X_1 \oplus Y_1 \oplus X_2 \oplus Y_2$. Then M as an operator on $X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$ has the following operator matrix form

$$M = \begin{pmatrix} A_1 & 0 & B_1 & B_3 \\ 0 & A_2 & B_4 & B_2 \\ C_1 & C_3 & D_1 & 0 \\ C_4 & C_2 & 0 & D_2 \end{pmatrix} = I_0 \begin{pmatrix} A_1 & B_1 & 0 & B_3 \\ C_1 & D_1 & C_3 & 0 \\ 0 & B_4 & A_2 & B_2 \\ C_4 & 0 & C_2 & D_2 \end{pmatrix} I_0^{-1}, \tag{2}$$

where A_1, D_1 are invertible, A_2, D_2 are quasi-nilpotent. Denote by

$$A_0 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},\ B_0 = \begin{pmatrix} 0 & B_3 \\ C_3 & 0 \end{pmatrix},\ C_0 = \begin{pmatrix} 0 & B_4 \\ C_4 & 0 \end{pmatrix} \text{ and } D_0 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}. \tag{3}$$

Therefore

$$M = I_0 \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} I_0^{-1} \text{ and } M^d = I_0 \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}^d I_0^{-1}. \tag{4}$$

A role of I_0 is to re-arrange the blocks of a 4×4 matrix M. For example,

$$\begin{aligned} I_0 \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix} I_0^{-1} &= \begin{pmatrix} 0 & AA^d BD^\pi \\ DD^d CA^\pi & 0 \end{pmatrix}, \\ I_0 \begin{pmatrix} 0 & 0 \\ C_0 & 0 \end{pmatrix} I_0^{-1} &= \begin{pmatrix} 0 & A^\pi BDD^d \\ D^\pi CAA^d & 0 \end{pmatrix}, \\ I_0 \begin{pmatrix} 0 & 0 \\ 0 & D_0 \end{pmatrix} I_0^{-1} &= \begin{pmatrix} AA^\pi & A^\pi BD^\pi \\ D^\pi CA^\pi & DD^\pi \end{pmatrix}. \end{aligned} \tag{5}$$

Note that AA^π and DD^π are quasi-nilpotent. The following lemma gives some equivalent conditions such that D_0 is quasi-nilpotent.

Lemma 2.4 Let D_0 be defined as in (3). If one of the following conditions holds, then D_0 is quasi-nilpotent.

$$\begin{array}{ll} (1)\ A^{\pi}ABD^{\pi}=0, D^{\pi}CA^{\pi}BD^{\pi}=0, & (2)\ A^{\pi}BDD^{\pi}=0, A^{\pi}BD^{\pi}CA^{\pi}=0, \\ (3)\ D^{\pi}CAA^{\pi}=0, D^{\pi}CA^{\pi}BD^{\pi}=0, & (4)\ D^{\pi}DCA^{\pi}=0, A^{\pi}BD^{\pi}CA^{\pi}=0, \\ (5)\ D^{\pi}CA^{\pi}=0, & (6)\ A^{\pi}BD^{\pi}CA^{\pi}=0, D^{\pi}CA^{\pi}BD^{\pi}=0, D^{\pi}CAA^{\pi}=D^{\pi}DCA^{\pi}, \\ (7)\ A^{\pi}BD^{\pi}=0, & (8)\ A^{\pi}BDD^{\pi}=0, D^{\pi}CA^{\pi}BD^{\pi}=0, A^{\pi}BDD^{\pi}=A^{\pi}ABD^{\pi}. \end{array} \tag{6}$$

Proof: Note that A_2 and D_2 are two quasi-nilpotent operators. If $AA^{\pi}BD^{\pi}=0$ and $D^{\pi}CA^{\pi}BD^{\pi}=0$, then $A_2B_2=0$ and $C_2B_2=0$. By Lemma 2.1, we have

$$D_0^d=\left(\begin{pmatrix} A_2 & 0 \\ C_2 & 0 \end{pmatrix}+\begin{pmatrix} 0 & B_2 \\ 0 & D_2 \end{pmatrix}\right)^d=0.$$

Hence D_0 is quasi-nilpotent. Similarly, we can show that if any condition holds in (6), then D_0 is quasi-nilpotent. ■

As for the 2×2 matrix A_0 in (3), we have the following result.

Lemma 2.5 Let A_0 be defined as (3). If $\Gamma = DD^dSDD^d+D^{\pi}$ is invertible, then A_0 is invertible and

$$R=:I_0\begin{pmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}I_0^{-1}=\begin{pmatrix} A^d+A^dB\Gamma^{-1}D^dDCA^d & -A^dB\Gamma^{-1}D^dD \\ -\Gamma^{-1}D^dDCA^d & \Gamma^{-1}D^dD \end{pmatrix}. \tag{7}$$

Proof: By (2), we have

$$\begin{aligned} S=D-CA^dB&=\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}-\begin{pmatrix} C_1 & C_3 \\ C_4 & C_2 \end{pmatrix}\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_3 \\ B_4 & B_2 \end{pmatrix} \\ &=\begin{pmatrix} D_1-C_1A_1^{-1}B_1 & -C_1A_1^{-1}B_3 \\ -C_4A_1^{-1}B_1 & D_2-C_4A_1^{-1}B_3 \end{pmatrix}. \end{aligned} \tag{8}$$

If $\Gamma = DD^dSDD^d+D^{\pi}$ is invertible, then $S_1=D_1-C_1A_1^{-1}B_1$ is invertible. Hence, A_0 is invertible and

$$A_0^{-1}=\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}^{-1}=\begin{pmatrix} A_1^{-1}+A_1^{-1}B_1S_1^{-1}C_1A_1^{-1} & -A_1^{-1}B_1S_1^{-1} \\ -S_1^{-1}C_1A_1^{-1} & S_1^{-1} \end{pmatrix}.$$

Using rearrangement effect of I_0, we get (7). ■

3 Some results related to the generalized Schur complement

Let M be defined as in (1) such that A and D are GD-invertible, $S=D-CA^dB$ be the generalized Schur complement. Denote by $\Gamma = DD^dSDD^d+D^{\pi}$. In this section, we will give some expressions for M^d according to the R in (7).

Theorem 3.1 Let M be defined as in (1) such that one of the conditions in (6) is satisfied. Let $\Gamma = DD^dSDD^d+D^{\pi}$ be invertible and R have the form as in (7).

(1) If

$$A^dABD^{\pi}C=0,\ A^dABD^{\pi}D=0,\ D^dDCA^{\pi}B=0,\ D^dDCA^{\pi}A=0, \tag{9}$$

then M is GD-invertible and

$$M^d=\left[R+\sum_{n=0}^{\infty}\begin{pmatrix} AA^{\pi} & A^{\pi}BD^{\pi} \\ D^{\pi}CA^{\pi} & DD^{\pi} \end{pmatrix}^n\begin{pmatrix} 0 & A^{\pi}B \\ D^{\pi}C & 0 \end{pmatrix}R^{n+2}\right]\left[I+R\begin{pmatrix} 0 & BD^{\pi} \\ CA^{\pi} & 0 \end{pmatrix}\right].$$

(2) If

$$A^{\pi}B(I-D^{\pi})C = BD^{\pi}C(I-A^{\pi}),\quad D^{\pi}C(I-A^{\pi})B = CA^{\pi}B(I-D^{\pi}),$$
$$A^{\pi}AB(I-D^{\pi}) = A^{\pi}BD(I-D^{\pi}),\quad D^{\pi}DC(I-A^{\pi}) = D^{\pi}CA(I-A^{\pi}), \tag{10}$$

then M is GD-invertible and

$$M^{d} = \left[I-\begin{pmatrix} 0 & A^{\pi}B \\ D^{\pi}C & 0 \end{pmatrix}R\right]\left[R+\sum_{n=0}^{\infty}R^{n+2}\begin{pmatrix} 0 & BD^{\pi} \\ CA^{\pi} & 0 \end{pmatrix}\begin{pmatrix} A^{\pi}A & A^{\pi}BD^{\pi} \\ D^{\pi}CA^{\pi} & DD^{\pi} \end{pmatrix}^{n}\right].$$

Proof: Since one of the conditions in (6) is satisfied, we have D_0 is quasi-nilpotent. Lemma 2.5 implies that A_0 is invertible.

(1) From (9), we obtain $B_0C_0 = 0$ and $B_0D_0 = 0$. By (4), (5) and Lemma 2.3 and 2.4, we have

$$\begin{aligned} M^{d} &= I_0\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}^{d} I_0^{-1} = I_0\begin{pmatrix} A_0^{d} & (A_0^{d})^{2}B_0 \\ \sum_{n=0}^{\infty}D_0^{d}C_0(A_0^{d})^{n+2} & \sum_{n=0}^{\infty}D_0^{d}C_0(A_0^{d})^{n+3}B_0 \end{pmatrix} I_0^{-1} \\ &= I_0\left\{\begin{pmatrix} A_0^{d} & 0 \\ 0 & 0 \end{pmatrix}+\sum_{n=0}^{\infty}\begin{pmatrix} 0 & 0 \\ 0 & D_0^{n} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C_0 & 0 \end{pmatrix}\times\begin{pmatrix} (A_0^{d})^{n+2} & 0 \\ 0 & 0 \end{pmatrix}\right\}\left[I+\begin{pmatrix} A_0^{d} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix}\right]I_0^{-1} \\ &= \left[R+\sum_{n=0}^{\infty}\begin{pmatrix} AA^{\pi} & A^{\pi}BD^{\pi} \\ D^{\pi}CA^{\pi} & DD^{\pi} \end{pmatrix}^{n}\begin{pmatrix} 0 & A^{\pi}BDD^{d} \\ D^{\pi}CAA^{d} & 0 \end{pmatrix}R^{n+2}\right]\times\left[I+R\begin{pmatrix} 0 & AA^{d}BD^{\pi} \\ DD^{d}CA^{\pi} & 0 \end{pmatrix}\right]. \end{aligned}$$

(2) From (10), we obtain $B_0C_0 = 0, C_0B_0 = 0$ and $D_0C_0 = C_0A_0$. Let $P_0 = \begin{pmatrix} A_0 & B_0 \\ 0 & D_0 \end{pmatrix}$ and $Q_0 = \begin{pmatrix} 0 & 0 \\ C_0 & 0 \end{pmatrix}$. Then $P_0Q_0 = Q_0P_0$, $Q_0^{d} = 0$ and $P_0^{d} = \begin{pmatrix} A_0^{d} & \sum_{n=0}^{\infty}(A_0^{d})^{n+2}B_0D_0^{n} \\ 0 & 0 \end{pmatrix}$. By Lemma 2.1(1), we obtain that

$$(P_0+Q_0)^{d} = P_0^{d}-Q_0(P_0^{d})^{2} = \begin{pmatrix} A_0^{d} & \sum_{n=0}^{\infty}(A_0^{d})^{n+2}B_0D_0^{n} \\ -C_0(A_0^{d})^{2} & -C_0A_0^{d}\sum_{n=0}^{\infty}(A_0^{d})^{n+2}B_0D_0^{n} \end{pmatrix}.$$

Hence,

$$\begin{aligned} M^{d} &= I_0\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}^{d} I_0^{-1} = I_0\begin{pmatrix} A_0^{d} & \sum_{n=0}^{\infty}(A_0^{d})^{n+2}B_0D_0^{n} \\ -C_0(A_0^{d})^{2} & -C_0A_0^{d}\sum_{n=0}^{\infty}(A_0^{d})^{n+2}B_0D_0^{n} \end{pmatrix} I_0^{-1} \\ &= \left[I-\begin{pmatrix} 0 & A^{\pi}B \\ D^{\pi}C & 0 \end{pmatrix}R\right]\cdot R\cdot\left[I+\sum_{n=0}^{\infty}R^{n+1}\begin{pmatrix} 0 & BD^{\pi} \\ CA^{\pi} & 0 \end{pmatrix}\begin{pmatrix} A^{\pi}A & A^{\pi}BD^{\pi} \\ D^{\pi}CA^{\pi} & DD^{\pi} \end{pmatrix}^{n}\right]. \end{aligned}$$

Let us use Theorem 3.1 to analyze some interesting special cases. If we replace the conditions in item (1) with the stronger conditions $(I-A^{\pi})BD^{\pi}=0$ and $(I-D^{\pi})CA^{\pi}=0$ the results can be reduced as:

Corollary 3.1 Let M be defined as in (1) such that one of the conditions in (6) be satisfied. Let $\Gamma = DD^dSDD^d + D^{\pi}$ be invertible and R have the form as in (7). If $(I-A^{\pi})BD^{\pi}=0$, $(I-D^{\pi})CA^{\pi}=0$, then M is GD-invertible and

$$M^d = R+\sum_{n=0}^{\infty}\begin{pmatrix} AA^{\pi} & BD^{\pi} \\ CA^{\pi} & DD^{\pi}\end{pmatrix}^n \begin{pmatrix} 0 & A^{\pi}B \\ D^{\pi}C & 0\end{pmatrix} R^{n+2}.$$

If the invertibility of Γ is replaced by the stronger condition $S=D-CA^dB$ invertible, we can get following corollary. It is worth pointing out the result (1) in next corollary has been given in [19], but to the best of out knowledge, the result (2) in next corollary hasn't been given yet by others.

Corollary 3.2 Let M be defined as in (1) such that $S=D-CA^dB$ is invertible and $R=\begin{pmatrix} A^d+A^dBS^{-1}CA^d & -A^dBS^{-1} \\ -S^{-1}CA^d & S^{-1}\end{pmatrix}$.

(1) (see [19] for matrix case) If $CA^{\pi}B=0$ and $CA^{\pi}A=0$, then M is GD-invertible and

$$M^d = \left[I+\begin{pmatrix} 0 & \sum_{n=0}^{\infty}A^nA^{\pi}B \\ 0 & 0\end{pmatrix}R^{n+1}\right]\cdot R\cdot\left[I+R\begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0\end{pmatrix}\right].$$

(2) If $A^{\pi}BC=0$, $CA^{\pi}B=0$, $A^{\pi}AB=A^{\pi}BD$, then M is GD-invertible and

$$M^d = \left[I-\begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0\end{pmatrix}R\right]\cdot R\cdot\left[I+\sum_{n=0}^{\infty}R^{n+1}\begin{pmatrix} 0 & 0 \\ CA^{\pi}A^n & 0\end{pmatrix}\right].$$

Proof. If $S=D-CA^dB$ is invertible, we set $Y_1=Y, Y_2=\{0\}$. So B_2, B_3, C_2, C_4 and D_2 are equal to zero in (4). All these are equivalent to that $BD^{\pi}=0, D^{\pi}C=0$ and $D^{\pi}D=0$ in Theorem 3.1. Hence the results are clear. ■

Next, let $S=D-CA^dB$, $S_0=DD^dSDD^d$ and $W_0=AA^d+A^dBDD^dCA^d$. By (8), if

$$S_0=DD^dSDD^d=\begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} D_1-C_1A_1^{-1}B_1 & -C_1A_1^{-1}B_3 \\ -C_4A_1^{-1}B_1 & D_2-C_4A_1^{-1}B_3\end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}=0,$$

then $D_1-C_1A_1^{-1}B_1=0$. By Lemma 2.2, we have A_0 is GD-invertible and

$$A_0^d=\begin{pmatrix} I \\ C_1A_1^{-1}\end{pmatrix}\left[(A_1+B_1C_1A_1^{-1})^d\right]^2A_1\begin{pmatrix} I & A_1^{-1}B_1\end{pmatrix}.$$

A direct calculation shows that

$$R_0 =: I_0\begin{pmatrix} A_0^d & 0 \\ 0 & 0\end{pmatrix}I_0^{-1}=\begin{pmatrix} I \\ D^dDCA^d\end{pmatrix}\left[(AW_0)^d\right]^2A\begin{pmatrix} I & A^dBDD^d\end{pmatrix}. \tag{11}$$

So we can deduce some results parallel with Theorem 3.1.

Theorem 3.2 Let M be defined as in (1) such that one of the conditions in (6) is satisfied. Let $S_0=DD^dSDD^d=0$ and R_0 be defined as in (11).

(1) If (9) holds, then M is GD-invertible and M^d has the representation as in Theorem 3.1(1), where one need replace R by R_0.

(2) If (10) holds, then M is GD-invertible and M^d has the representation as in Theorem 3.1(2), where one needs to replace R by R_0.

We remark that the conditions in Theorem 3.2 are weaker than the conditions used in papers [19] and [17]. Similar to Corollaries 3.1-3.2, various special results are easily derived from Theorem 3.2. We leave it to readers who are interested in.

4 Block triangular matrices

In this section, we apply results obtained in Theorems 3.1 and 3.2 to 2×2 block triangular matrix. By the proof of Lemma 2.5, if $(I-A^{\pi})B(I-D^{\pi})=0$, then $B_1=0$ and R in (7) reduces as

$$R = I_0\left[\begin{pmatrix} A_1^{-1} & 0 \\ -D_1^{-1}C_1A_1^{-1} & D_1^{-1}\end{pmatrix}\oplus 0\right]I_0^{-1}=\begin{pmatrix} A^d & 0 \\ -D^dCA^d & D^d\end{pmatrix} \tag{12}$$

and

$$R^n=\begin{pmatrix} (A^d)^n & 0 \\ -\sum_{i=0}^{n-1}(D^d)^{i+1}C(A^d)^{n-i} & (D^d)^n\end{pmatrix}. \tag{13}$$

By Theorem 3.1, we have following results.

Theorem 4.1 Let M be defined as in (1) such that $(I-A^{\pi})B(I-D^{\pi})=0$ and one of the conditions in (6) is satisfied.

(1) If (9) holds, then M is GD-invertible and

$$M^d=\Big[\sum_{n=0}^{\infty}\begin{pmatrix} A^{\pi}A^n & 0 \\ D^{\pi}\sum_{i=0}^{n-1}D^iCA^{n-i-1}A^{\pi} & D^nD^{\pi}\end{pmatrix}\times\begin{pmatrix} -A^{\pi}B\sum_{i=0}^{n+1}(D^d)^{i+1}C(A^d)^{n-i+2} & A^{\pi}B(D^d)^{n+2} \\ D^{\pi}C(A^d)^{n+2} & 0\end{pmatrix}$$
$$+\begin{pmatrix} A^d & 0 \\ -D^dCA^d & D^d\end{pmatrix}\Big]\times\begin{pmatrix} I & A^dBD^{\pi} \\ D^dCA^{\pi} & I-D^dCA^dBD^{\pi}\end{pmatrix}$$

(2) If (10) holds, then M is GD-invertible and

$$M^d=\begin{pmatrix} I+A^{\pi}BD^dCA^d & -A^{\pi}BD^d \\ -D^{\pi}CA^d & I\end{pmatrix}\times\Big[\begin{pmatrix} A^d & 0 \\ -D^dCA^d & D^d\end{pmatrix}$$
$$+\sum_{n=0}^{\infty}\begin{pmatrix} 0 & (A^d)^{n+2}BD^{\pi} \\ (D^d)^{n+2}CA^{\pi} & -\sum_{i=0}^{n+1}(D^d)^{i+1}C(A^d)^{n-i+2}BD^{\pi}\end{pmatrix}\times\begin{pmatrix} A^{\pi}A^n & 0 \\ D^{\pi}\sum_{i=0}^{n-1}D^iCA^{n-i-1}A^{\pi} & D^nD^{\pi}\end{pmatrix}\Big].$$

Proof: (1) If (9) holds, by Theorem 3.1, (12) and (13}), we have

$$M^d = \Big[R + \sum_{n=0}^{\infty} \begin{pmatrix} A^\pi A & 0 \\ D^\pi C A^\pi & DD^\pi \end{pmatrix}^n \begin{pmatrix} 0 & A^\pi B \\ D^\pi C & 0 \end{pmatrix} R^{n+2}\Big] \Big[I + R \begin{pmatrix} 0 & BD^\pi \\ CA^\pi & 0 \end{pmatrix}\Big]$$

$$= \Big[\begin{pmatrix} A^d & 0 \\ -D^d C A^d & D^d \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} A^\pi A^n & 0 \\ D^\pi \sum_{i=0}^{n-1} D^i C A^{n-i-1} A^\pi & D^n D^\pi \end{pmatrix} \begin{pmatrix} 0 & A^\pi B \\ D^\pi C & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} (A^d)^{n+2} & 0 \\ -\sum_{i=0}^{n+1} (D^d)^{i+1} C (A^d)^{n-i+2} & (D^d)^{n+2} \end{pmatrix} \begin{pmatrix} I & A^d B D^\pi \\ D^d C A^\pi & I - D^d C A^d B D^\pi \end{pmatrix}$$

$$= \Big[\begin{pmatrix} A^d & 0 \\ -D^d C A^d & D^d \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} A^\pi A^n & 0 \\ D^\pi \sum_{i=0}^{n-1} D^i C A^{n-i-1} A^\pi & D^n D^\pi \end{pmatrix} \times$$

$$\begin{pmatrix} -A^\pi B \sum_{i=0}^{n+1} (D^d)^{i+1} C (A^d)^{n-i+2} & A^\pi B (D^d)^{n+2} \\ D^\pi C (A^d)^{n+2} & 0 \end{pmatrix}\Big] \begin{pmatrix} I & A^d B D^\pi \\ D^d C A^\pi & I - D^d C A^d B D^\pi \end{pmatrix}.$$

Similar to the proof of (1), we can prove (2). So the details are omitted. ■

If $(I - D^\pi) C (I - A^\pi) = 0$, then $C_1 = 0$ and R in (7) reduces as

$$R \;=\; I_0 \left[\begin{pmatrix} A_1^{-1} & -A_1^{-1} B_1 D_1^{-1} \\ 0 & D_1^{-1} \end{pmatrix} \oplus 0 \right] I_0^{-1} = \begin{pmatrix} A^d & -A^d B D^d \\ 0 & D^d \end{pmatrix}.$$

Under the assumptions of Theorem 4.1(1) with "$A^d A B D^\pi C = 0$, $A^d A B D^\pi D = 0$, $D^d D C A^\pi B = 0$, $D^d D C A^\pi A = 0$"(i.e., (9)) replaced by" $B D^\pi C A A^d = 0$, $D D^\pi C A A^d = 0$, $C A^\pi B D D^d = 0$, $A A^\pi B D D^D = 0$", the further result may be proved in much the same way as Theorem 4.1(1).

Theorem 4.2 Let M be defined as in (1) such that $(I - D^\pi) C (I - A^\pi) = 0$ and one of the conditions in (6) is satisfied. If (14) holds, then M is GD-invertible and

$$M^d = \begin{pmatrix} I & A^\pi B D^d \\ D^\pi C A^d & I - D^\pi C A^d B D^d \end{pmatrix} \times \Big[\begin{pmatrix} A^d & -A^d B D^d \\ 0 & D^d \end{pmatrix}$$

$$+ \sum_{n=0}^{\infty} \begin{pmatrix} -\sum_{i=0}^{n+1} (A^d)^{i+1} B (D^d)^{n-i+2} C A^\pi & (A^d)^{n+2} B D^\pi \\ (D^d)^{n+2} C A^\pi & 0 \end{pmatrix} \times \begin{pmatrix} A^\pi A^n & 0 \\ D^\pi \sum_{i=0}^{n-1} D^i C A^{n-i-1} A^\pi & D^n D^\pi \end{pmatrix}\Big].$$

In finite dimensional space, X. Li and Y. Wei (see [20]) gave a representation of M^d under the conditions that $A A^\pi B = 0$, $C A^\pi B = 0$, $DC(I - A^\pi) = 0$, $BC(I - A^\pi) = 0$. In this case, $0 = BC(I - A^\pi) = B D^\pi C A A^d$ and (14) holds.

Moreover, we have $0 = DC(I - A^\pi) = (I - D^\pi) C (I - A^\pi)$ and Lemma 2.4(1) holds. Hence, the result in [20] still holds in infinite space and it is just one special case of Theorem 4.2.

Corollary 4.1 (see [20] for the matrix case) If

$$A A^\pi B = 0,\quad DC(I - A^\pi) = 0,\quad C A^\pi B = 0,\quad BC(I - A^\pi) = 0,$$

then M is GD-invertible and

$$M^d=\begin{pmatrix} I & A^{\pi}BD^d \\ CA^d & I-CA^dBD^d \end{pmatrix}\times\left[\begin{pmatrix} A^d & -A^dBD^d \\ 0 & D^d \end{pmatrix}\right.$$

$$\left.+\sum_{n=0}^{\infty}\begin{pmatrix} -\sum_{i=0}^{n+1}(A^d)^{i+1}B(D^d)^{n-i+2}CA^{\pi} & (A^d)^{n+2}BD^{\pi} \\ (D^d)^{n+2}CA^{\pi} & 0 \end{pmatrix}\times\begin{pmatrix} A^{\pi}A^n & 0 \\ D^{\pi}\sum_{i=0}^{n-1}D^iCA^{n-i-1}A^{\pi} & D^nD^{\pi} \end{pmatrix}\right].$$

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2X2 算子矩阵的广义 Drazin 逆的表示

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摘要：设 2X2 算子矩阵的对角元是 Drazin 可逆的，在一定的条件下，我们给出 2X2 算子矩阵的 Drazin 逆的表示. 我们的结论推广了现有的一些结论，把原有的关于矩阵的 Drazin 逆的表示仅仅限定在Schur 补可逆或为零的情形推广到更一般的广义Schur 补即可奇异也可不为零的情形.

关键词：块矩阵；广义 Drazin 逆；Schur 补

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