Common fixed points theorem for two multivalued mappings in cone metric spaces

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Abstract: In this paper, a new generalized contractive condition is introduced in metric space. By the condition and without the normality of the cone, the existence of common fixed points of multivalued mappings satisfying generalized contractive conditions in cone metric spaces is proved. These results extend some of the most general common fixed point theorems for two multivalued maps in cone metric spaces.

Key words: functional analysis; common fixed point; non normal cone; multivalued mapping; cone metric spaces

1 Introduction

Fixed point theory and common fixed point theory have basic roles in the applications of some branches of mathematics. The theory of common fixed points of multifunctions is a generalization of the theory of fixed point of mappings in a sense. Using the concept of Hausdoff metric, Nadler [1] obtain multivalued version of the Banach contraction principle. Recently, Huang and Zhang [2] introduced the concept of cone metric space, replacing the set of positive real numbers by an ordered Banach space. They obtained some fixed point theorems in cone metric spaces using the normality of the cone, which induces an order in Banach spaces. Later, Wardowski [3] introduced the concept of multivalued contractions in cone metric spaces and, with the condition of normal cones, obtained fixed point theorems for such mappings.

In this paper, we will prove some common fixed points results for multivalued mappings taking closed valued in cone metric spaces without the condition of normal cone. Our result extend and unify various comparable results in the literature ([4][5][6]).

Let E always be real Banach space and P subset of E. P is called cone if and only if:

(i) P is nonempty and closed set, also $P \neq [0]$;

(ii) $a,b \in R, a,b \geq 0, x,y \in P \Rightarrow ax+by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies} \| x \| \leq K \| y \|.$$

The least positive number satisfying above is called the normal constant of P.

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Definition 1.1. Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies:

(i) $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Definition 1.2. Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X, then

(i) $\{x_n\}$ is a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is a natural number N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(ii) $\{x_n\}$ is a convergent sequence if for every $c \in E$ with $0 \ll c$, there is a natural number N and $x \in X$ such that for all $n > N$, $d(x_n, x) \ll c$ for some $x \in X$;

(iii) (X,d) is called a complete cone metric space if every Cauchy sequence is convergent in X;

(iv) A set $B \subseteq X$ is said to be closed if for any sequence $\{x_n\}$ in B converges to x in X, we have $x \in B$.

Let X be a cone metric space. We denote by $P(X)$ the family of all nonempty subsets of X, and by $P_{cl}(X)$ the family of all nonempty closed subsets of X. A point x in X is called a fixed point of a multivalued mapping $T: X \to P_{cl}(X)$ by $x \in T(x)$. The collection of all fixed points of T is denoted by $F(T)$.

Remark 1.3.[5] If $a \leq ha$, for some $a \in P$ and $h \in (0,1)$, then $a = 0$.

2 Main Results

Rus [7] coined the term R-multivalued mapping, and proved the set of the fixed points is nonempty in complete metric space (X,d), for the multivalued mapping $T: X \to P_{cl}(X)$ which is a R-multivalued mapping. Recently, Abbas [5] proved a common fixed point theorem for two R-multivalued mappings in non normal cone metric space.

In this section by introducing a new generalized contractive condition we obtain a common fixed point theorem for two multivalued mappings in a cone metric space without using the condition of a normal cone, which extends a theorem of Abbas [5].

Lemma 2.1. Let (X,d) be a cone metric space with a cone P. If there exist a sequence $\{x_n\}$ in X and a real number $\gamma \in (0,1)$ such that for every $n \in N$,

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}),$$

then $\{x_n\}$ is a Cauchy sequence.

Proof. Let $x_0 \in X$ be an arbitrary but fixed. Note that for all $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq [\gamma^n + \gamma^{n+1} + \cdots + \gamma^{m-1}] d(x_0, x_1) \\ &\leq \frac{\gamma^n}{1-\gamma} d(x_0, x_1), \end{aligned}$$

Now let $0 \ll c$ be given. Choose a symmetric open neighborhood V of 0 such that $c+V \subseteq P$. Also, choose a natural number N such that $\frac{\gamma^n}{1-\gamma} d(x_0, x_1) \ll c$ for all $n > N$, and hence $d(x_n, x_m) \ll c$ for all $n, m > N$. Therefore $\{x_n\}$ is a Cauchy sequence in X.

Theorem 2.2. Let (X, d) be a cone metric space and T_1, $T_1 : X \to P_{cl}(X)$ two multivalued mappings such that for $i, j \in 1,2$ with $i \neq j$ and for each $x, y \in X$, $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ such that

$$d(u_x, u_y) \leq Ad(x, y) + Bd(x, u_x) + Cd(y, u_y) + D[d(x, u_y) + d(y, u_x)],$$

where $A, B, C, D \geq 0$ and $A+B+C+D < 1$. Then $F(T_1) = F(T_2) \neq \varnothing$ and $F(T_1) = F(T_2) \in P_{cl} X$.

Proof. Suppose that x_0 is an arbitrary point of X. For $i, j \in 1,2$ with $i \neq j$, take $x_1 \in T_i(x_0)$. Then there exists $x_2 \in T_j(x_i)$ such that

$$\begin{aligned} d(x_1, x_2) &\leq Ad(x_0, x_1) + Bd(x_0, x_1) + Cd(x_1, x_2) + D[d(x_0, x_2) + d(x_1, x_1)] \\ &\leq (A+B+D) d(x_0, x_1) + (C+D) d(x_1, x_2), \end{aligned}$$

which implies

$$d(x_1, x_2) \leq k d(x_0, x_1),$$

Where $0 < k = \frac{A+B+D}{1-C-D} < 1$. Now for $x_2 \in T_j(x_1)$, there exists $x_3 \in T_i(x_2)$ such that $d(x_2, x_3) \leq k d(x_1, x_2)$. Continuing this process we obtain a sequence x_n in X with $x_{2n-1} \in T_i(x_{2n-2})$, $x_{2n} \in T_j(x_{2n-1})$ such that $d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n)$. From Lemma 2.1 we get $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

let $0 \ll c$ be given and $0 < \delta < \min\{\frac{1-C-D}{2(A+B+D)}, \frac{1-C-D}{2(B+D+1)}\}$. Choose a natural number N_1 such that $d(x_m, x^*) \ll \delta c$ for all $m \geq N_1$ is given. If $x_{2n} \in T_j(x_{2n-1})$, there exists $u_n \in T_i(x^*)$ such that

$$d(x_{2n}, u_n) \leq Ad(x_{2n-1}, x^*) + Bd(x_{2n-1}, x_{2n}) + Cd(x^*, u_n) + D[d(x_{2n-1}, u_n) + d(x^*, x_{2n})],$$

which further gives

$$d(x^*, u_n) \leq Ad(x_{2n-1}, x^*) + Bd(x_{2n-1}, x_{2n}) + Cd(x^*, u_n) + D[d(x_{2n-1}, u_n) + d(x^*, x_{2n})] + d(x_{2n}, x^*)$$

and so

$$\begin{aligned} (1-C) d(x^*, u_n) &\leq Ad(x_{2n-1}, x^*) + Bd(x_{2n-1}, x_{2n}) + Dd(x_{2n-1}, u_n) + (D+1) d(x_{2n}, x^*) \\ &\leq Ad(x_{2n-1}, x^*) + Bd(x_{2n-1}, x^*) + Bd(x^*, x_{2n}) + Dd(x_{2n-1}, x^*) + Dd(x^*, u_n) \\ &\quad + (D+1) d(x_{2n}, x^*). \end{aligned}$$

Thus

$$
\begin{aligned}
d(x^*,u_n) &\le \frac{A+B+D}{1-C-D} d(x_{2n-1},x^*) + \frac{B+D+1}{1-C-D} d(x_{2n},x^*) \\
&\ll \frac{A+B+D}{1-C-D}\cdot\frac{1-C-D}{2(A+B+D)} c + \frac{B+D+1}{1-C-D}\cdot\frac{1-C-D}{2(B+D+1)} c \\
&= c,
\end{aligned}
$$

which shows that $u_n \to x^*$ as $n \to \infty$. Since $T_i(x^*)$ is closed, $x^* \in F(T_i)$ and so $F(T_i) \ne \varnothing$.

Let $x^* \in X$ be a fixed point of T_1. Then, by hypothesis, there exists $x \in T_2(x^*)$ such that

$$
\begin{aligned}
d(x^*,x) &\le Ad(x^*,x^*) + Bd(x^*,x^*) + Cd(x^*,x) + D[d(x^*,x) + d(x^*,x^*)] \\
&= (C+D) d(x^*,x),
\end{aligned}
$$

which by using Remark 1.3, implies that $d(x^*,x)=0$. And so $x^* = x$. Thus, $x^* \in T_2(x^*)$. And so $F(T_1) \subseteq F(T_2)$. Similarly, $F(T_2) \subseteq F(T_1)$. Therefore $F(T_1) = F(T_2) \ne \varnothing$.

Now we prove that $F(T_i)$ is closed. Let $\{x_n\}$ be a sequence in $F(T_j) = F(T_i)$ such that $x_n \to x$ as $n \to \infty$. let $0 \ll c$ be given and $0 < \delta < \frac{1-C-D}{2(A+1)}$. Choose a natural number N_2 such that $d(x_m, x^*) \ll \delta c$ for all $m \ge N_2$ is given. For $i \ne j$, since $x_n \in T_i(x)$, there exists $v_n \in T_j(x)$ such that

$$
d(x_n,v_n) \le Ad(x_n,x) + Bd(x_n,x_n) + Cd(x,v_n) + D[d(x_n,v_n) + d(x,x_n)],
$$

which further gives

$$
(1-D)d(x_n,v_n) \le (A+D)d(x_n,x) + Cd(x,v_n).
$$

And so

$$
d(x_n,v_n) \le \frac{A+D}{1-D} d(x_n,x) + \frac{C}{1-D} d(x,v_n).
$$

From

$$
\begin{aligned}
d(x,v_n) &\le d(x,x_n) + d(x_n,v_n) \\
&\le \frac{A+D}{1-D} d(x_n,x) + \frac{C}{1-D} d(x,v_n) + d(x_n,v_n) \\
&= \frac{A+1}{1-D} d(x_n,x) + \frac{C}{1-D} d(x,v_n),
\end{aligned}
$$

we get

$$
d(x,v_n) \le [\frac{1-D}{1-D-C} \times \frac{1+A}{1-D}] d(x_n,x) = \frac{A+1}{1-D-C} d(x_n,x) \ll c.
$$

Thus $d(x,v_n) \to 0$ as $n \to \infty$. Since $v_n \in T_j(x)$ for $n \in \mathbb{N}$ and $T_j(x)$ is closed, $x \in T_j(x)$. Hence $x \in F(T_i) = F(T_j)$. The proof is completed.

Remark 2.3.

(i) Lemma 2.1 have the same result of [4, Lemma 3.1] without the condition of a normal cone.

(ii) In Theorem 2.2, if $D=0$, we can easily obtain the result of [5, Theorem 2].

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锥度量空间中两个集值映射的公共不动点定理

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摘要：在度量空间中引入了一种新的广义压缩条件，在不要求正规锥的前提下，利用这种条件得到了满足广义压缩条件的集值映射的公共不动点的存在性．这些结果推广了一些锥度量空间中关于两个集值映射的最常见的公共不动点定理.

关键词：泛函分析；公共不动点；非正规锥；集值映射；锥度量空间

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