Variational Minimizing Parabolic Orbits for the Restricted 3-Body Problems

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Abstract The Least Action Principle is the simplest and most nature rule in the world.Using variational minimizing methods and some implicit symmetries of the restricted 3-body problems, we prove the existence of the odd symmetric parabolic orbit for the restricted 3-body problems with weak forces.

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1. Introduction

The model for the restricted 3-body problem studied by Sitnikov([15]) and Moser ([8]) is well known:Under Newton's law of attraction,two mass points of equal mass $m_1 = m_2 = \frac{1}{2}$ moving in the plane of their elliptic orbits such that the center of masses is at rest, the third mass point which does not influence the motion of the first two moving on the line perpendicular to the plane containing the first two mass points and going through the center of mass.

Let $z(t)$ be the coordinate of the third mass point, then $z = 0$ corresponds to the center of mass of the first two mass points,$z(t)$ satisfies

$$\ddot{z}(t) = \frac{-z(t)}{(|z(t)|^2 + |r(t)|^2)^{3/2}} \tag{1.1}$$

Where $r(t) = r(t + 2\pi) > 0$ is the distance from the center of mass to one of the first two mass points.

For a small eccentricity $\varepsilon > 0$,the $r(t)$ takes the form (Moser[8]):

$$r(t) = \frac{1}{-}$$

study the homoclinic orbits(parabolic orbits) of (1.1).

Moser[8] used the geometry of the Bernoulli-shift and the symbolic dynamics to study the existence of infinitely many periodic orbits of (1.1).

Mathlouthi [7] used variational methods to give another proof of Moser's result.Souissi [16] used variational minimax methods and approximations to prove the existence of at last one parabolic orbit of the circular restricted 3-body problem for $0 < \alpha \leq 1$:

$$\ddot{z}(t) + \alpha \frac{z(t)}{(|z(t)|^2 + |r|^2)^{\alpha/2+1}} = 0, \tag{1.3}$$

Here we use variational minimizing method to prove

Theorem 1.1 For (1.3) with $0 < \alpha < 2$,there exists one odd parabolic orbit which minimizes the corresponding variational functional.

Remark Comparing our Theorem 1.1 with all the other Theorems,the variational solution in our Theorem 1.1 is the most simple and natural solution,another new point of our Theorem 1.1 is that we found odd symmetrical parabolic solution for all the circular restricted 3-body problem with weak force type potentials ($0 < \alpha < 2$).

2.Variational Minimizing Critical Points

In order to find parabolic orbit of (1.3),firstly,we find mT-periodic solution of (1.3), then let $m \to \infty$ to get the parabolic orbit. Noticing the symmetry of the equation, we can find the odd mT-periodic solution of the restricted 3-body problem:

$$-\ddot{z}(t) = \frac{\alpha z(t)}{|z^2(t) + r^2|^{\alpha/2+1}} \tag{2.1}$$

We define the functional:

$$f(z) = \int_{-mT}^{mT} (\frac{1}{2}|\dot{z}(t)|^2 + \frac{1}{(z^2(t) + r^2)^{\alpha/2}})dt, \tag{2.2}$$

$$z \in H_{mT} = \{z, \dot{z} \in L^2[-mT, mT] | z(-mT) = z(mT), z(-t) = -z(t)\} \tag{2.3}$$

Since $\forall z \in H_{mT}, z(0) = 0$,then we have equivalent norm for $\forall z \in H_{mT}$:

$$\|z\|_{mT} = (\int_{-mT}^{mT} |\dot{z}(t)|^2 dt)^{1/2} \tag{2.4}$$

Lemma 2.1 $f(z)$ is weakly lower semi-continuous(w.l.s.c.) on H_{mT} .

Proof:(1).It is well-known that the norm and its square are w.l.s.c.,
(2).$\forall z_n \subset H_{mT}$,if $z_n \rightharpoonup z$ weakly,then by compact embedding theorem,we have the uniformly convergence:

$$\max_{-mT\leqslant t\leqslant mT} |z_n(t) - z(t)| \to 0, \tag{2.5}$$

so

$$\int_{-mT}^{mT} \frac{1}{(|z_n(t)|^2 + r^2)^{\alpha/2}} dt \to \int_{-mT}^{mT} \frac{1}{(|z(t)|^2 + r^2)^{\alpha/2}} dt \tag{2.6}$$

Hence

$$\underline{\lim}_{n\to\infty} f(z_n) \geq f(z) \tag{2.7}$$

Lemma 2.2 f is coercive on H_{mT}.

Proof: By the definitions of $f(z)$ and the coercivity $(f(z) \to +\infty, \|z\| \to \infty)$.

Lemma2.3(Tonelli,[1]) Let X be a reflexive Banach space,$f : X \to R \cup +\infty$,if $f \neq +\infty$ and is weakly lower semi-continuous and coercive$(f(x) \to +\infty, \|x\| \to \infty)$, then f attains its infimum on X.

Lemma 2.4(Palais's Symmetry Principle[9])Let G be a finite or compact group,σ be an orthogonal representation of G, let H be a real Hilbert space,$f : H \to R$ and satisfies

$$f(\sigma \cdot x) = f(x), \forall \sigma \in G, \forall x \in H. \tag{2.8}$$

Let

$$F \triangleq \{x \in H | \sigma \cdot x = x, \forall \sigma \in G\}. \tag{2.9}$$

Then the critical point of f in F is also a critical point of f in H.

Lemma 2.5 (1).$f(z)$ attains the infimum on H_{mT},the minimizer $\tilde{z}_{\alpha,m}(t)$ is an odd mT-periodic solution.
(2).Furthermore ,when $m \to \infty$, $\tilde{z}_{\alpha,m}(t) \to \tilde{z}_\alpha(t)$ and $\tilde{z}_\alpha(t)$ has the properties:
(i).

$$|\tilde{z}_\alpha(t)| \to \infty, |t| \to \infty, \tag{2.10}$$

(ii).

$$|\dot{\tilde{z}}_\alpha(t)| \to 0, |t| \to \infty. \tag{2.11}$$

So $\tilde{z}_\alpha(t)$ is a parabolic solution.
The proof of (1) is obvious by Lemmas 2.1-2.4.

In the following,we will give the proof of (2) of Lemma 2.5, in order to do this, we give a key Lemma 2.6 ,in its proof,we use the properties of the odd test loop and the variational functional, the proof about the estimates for infsup values in [16] is totally different from here.

Lemma 2.6 There exist constants $c > 0$ and $0 < \theta < 1$ independent of m such that the variational minimizing value for $f(z)$ on H_{mT} satisfies

$$a_m \leq cm^\theta \tag{2.12}$$

Proof: We choose special odd $mT-$periodic function defined by

$$g_\beta(t) = t^\beta, t \in [-mT, 0], \tag{2.13}$$

where $\frac{1}{2} < \beta = \frac{p}{q} < \frac{1}{\alpha}$,$p, q$ are odd numbers and $(p, q) = 1$.
Then

$$\begin{aligned} &f(g_\beta(t)) = \int_0^{mT} \beta^2 t^{2(\beta-1)} dt + 2\int_0^{mT} \frac{dt}{(t^{2\beta}+r^2)^{\alpha/2}} \\ &\leq \frac{\beta^2}{2\beta-1}(mT)^{2\beta-1} + \frac{2}{1-\alpha\beta}(mT)^{1-\alpha\beta} \leq cm^\theta, \end{aligned} \tag{2.14}$$

where

$$\theta = max(2\beta - 1, 1 - \alpha\beta) \tag{2.15}$$

$$c = \frac{\beta^2}{2\beta - 1}T^{2\beta-1} + \frac{2}{1 - \alpha\beta}T^{1-\alpha\beta} > 0, \tag{2.16}$$

When $0 < \alpha < 2$,we have $\frac{1}{\alpha} > \frac{1}{2}$,we can choose $\frac{1}{2} < \beta < \frac{1}{\alpha}$,then $2\beta - 1 > 0, 1 - \alpha\beta > 0$,hence $\theta > 0$;when $\beta < 1, 2\beta - 1 < 1$,then $0 < \theta < 1$.

Similar to infsup-critical points in the paper [16],here for our minimizer ,we have

Lemma 2.7 Let $\tilde{z}_{\alpha,m}$ be critical points corresponding to the minimizing critical values $a_m = \min\limits_{H_{mT}} f(z)$,then

$$\|\tilde{z}_{\alpha,m}\|_\infty \to +\infty, when \quad m \to +\infty \tag{2.17}$$

Proof: By the definition of $f_m(\tilde{z}_{\alpha,m})$ and Lemma 2.6,we have

$$cm^\theta \geq f_m(\tilde{z}_{\alpha,m}) \geq \int_{-mT}^{mT} \frac{dt}{(\|\tilde{z}_{\alpha,m}\|_\infty^2 + r^2)^{\alpha/2}} = \frac{2mT}{(\|\tilde{z}_{\alpha,m}\|_\infty^2 + r^2)^{\alpha/2}} \tag{2.18}$$

Hence

$$\|\tilde{z}_{\alpha,m}\|_\infty^2 \geq (\frac{2T}{c})^{\alpha/2} \cdot m^{\frac{2(1-\theta)}{\alpha}} - r^2 \to +\infty, as \quad m \to +\infty. \tag{2.19}$$

Lemma 2.8 $\{\dot{\tilde{z}}_{\alpha,m}\}$ is uniformly bounded.

Proof: The Proof is easy (ref.[16]) by the conservation of the energy along the solution and existing $t_m \in [-mT, mT]$ such that

$$\dot{\tilde{z}}_{\alpha,m}(t_m) = 0.$$

Now we prove **Theorem 1.1:**

The proof is standard as [16].
By $\tilde{z}_{\alpha,m}(0) = 0$ and Lemma 2.8,we have

(i).$\{\tilde{z}_{\alpha,m}\}$ is uniformly bounded on any compact set of R.

(ii).$\{\tilde{z}_{\alpha,m}\}$ is uniformly equi-continuous on R.

By Ascoli-Arzelà Theorem,we know $\{\tilde{z}_{\alpha,m}\}$ has a sub-sequence converging uniformly to a limit $\tilde{z}_\alpha(t)$ on any compact set of R,and $\tilde{z}_\alpha(t)$ is a solution of (2.1) and $\tilde{z}_\alpha(t) \in C^2(R, R)$.By the energy conservation law,we have

$$E = \frac{1}{2}|\dot{\tilde{z}}_\alpha|^2 - \frac{1}{(|\tilde{z}_\alpha|^2 + r^2)^{\alpha/2}} = 0 \tag{2.20}$$

Then

$$\frac{1}{2}|\dot{\tilde{z}}_\alpha|^2 = \frac{1}{(|\tilde{z}_\alpha|^2 + r^2)^{\alpha/2}} > 0, \tag{2.21}$$

Now we claim

(i).

$$|\tilde{z}_\alpha(t)| \to +\infty, as \quad |t| \to +\infty, \tag{2.22}$$

in fact,if $\exists \beta > 0$ s.t

$$|\tilde{z}_\alpha| < \beta, \forall t \in R, \tag{2.23}$$

By (2.21),$\exists \gamma > 0$ s.t

$$|\dot{\tilde{z}}_\alpha| > \gamma, \forall t \in R, \tag{2.24}$$

Then we have

$$|\tilde{z}_\alpha(t)| = |\tilde{z}_\alpha(t) - \tilde{z}(0)| = |\int_0^t \dot{\tilde{z}}_\alpha dt| > \gamma|t| \to +\infty, as \quad |t| \to +\infty, \tag{2.25}$$

This is a contradiction.
Now by (2.21) we have

(ii).

$$|\tilde{z}_\alpha(t)| \to 0, as \quad |t| \to +\infty. \tag{2.26}$$

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References

[1] Ambrosetti A. and Coti Zelati V.,Periodic solutions of singular Lagrangian systems,Birkhäuser,Basel,1993.

[2] Bolotin S.V.,Existence of homoclinic motions,Vestnik Moskov Univ. ser.I Mat. Mekh. 6(1983),98-103.

[3] Chang K.C.,Infinite dimensional Morse thory and multiple solution problems,Progress in Nonlinear Diff. Equ. and their Appl.,Vol.6,Birkhäser,1993.

[4] Long Y.,Index theory for symplectic paths with applications,Birkhäuser,2002.

[5] McGehee R.,Parabolic orbits of the restricted three-body problem,Academic Press,New York and London,1973.

[6] Mawhin J. and Willem M.,Critical point theory and Hamiltonian systems,Springer,1989.

[7] Mathlouthi S.,Periodic orbits of the restricted three-body problem,Trans.AMS 350(1998),2265-2276.

[8] Moser J.,Stable and random motions in dynamical systems,Ann.Math.Studies 77,Princeton Univ. Press,1973.

[9] Palais R.,The principle of symmetric criticality,CMP 69(1979),19-30.

[10] Poincaré H.,Les Méthodes Nouvelles de la Mécanique Céleste,Gauthier-Villars,Paris,1899.

[11] Rabinnowtz P. H.,On the existence of periodic solutions for a class of symmetric Hamiltonian systems,Nonlinear Anal. 11(1987),595-611.

[12] Rabinnowtz P. H.,Homoclinic orbits for a class of Hamiltonian systems,Proc. Roy.Soc.Edinburgh Sect.A 114(1990),33-38.

[13] Sere E.,Existence of infinitely many homoclinics Hamiltonian systems,Math.Z. 209(1992),27-42.

[14] Serra E. and Terracini S.,Collisionless periodic solutions to some 3-body problems,Arch. Rational. Mech. Anal. 120(1992),305-325.

[15] Sitninkov K.,Existence of oscillating motion for the three-body problem,J. Dokl. Akad. Nauk USSR 133(1960),303-306.

[16] Souissi C.,Existence of parabolic orbits for the restricted three-body problem, Annals of University of Craiova,Math. Comp. Sci. Ser. 31(2004),85-93.

[17] Tanaka K.,Homoclinic orbits for a singular second order Hamiltonian system,Ann. Inst. H. Poincaré Anal. Non Linéaire 7(1990),427-438.