Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces *

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Abstract

The author shows that the interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces by applying the boundedness of the Hardy-Littlewood maximal operator on $L^{p(x)}$.

As applications the author proves a new Landau-Komogorov type inequality for the second order derivative and a embedding theorem and discusses the equivalent norms in the space $W_0^{1,p(x)}(\Omega)\cap W^{2,p(x)}(\Omega)$.

Keywords: interpolation inequality; Landau-Kolmogorov type inequality; Variable exponent Lebesgue-Sobolev space; maximal function operator;

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1 Introduction

It is well known that upper bounds for L^p norms of the intermediate derivative , $0 < |\beta| < m$ of functions in $W^{m,p}(\Omega)$ are determined by the L^p norms of u and its partial derivatives of order m(see [1]). These estimates for intermediates derivatives play many important roles in the study of partial differential equations and variation problems.

In these years, there emerged a need to study elliptic and parabolic boundary value problems and to obtain boundedness of their solutions in Orlicz spaces and generalize Orlicz spaces. In this context, the interpolations inequalities of these spaces are of particular interest.

Variable exponent Lebesgue spaces $L^{p(\cdot)}$ and Sobolev spaces $W^{m,p(\cdot)}$ are special case of the generalized Orlicz spaces. Many results for variable exponent spaces were obtained, we can refer ([3]-[10]) and references therein.

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Since the spaces $L^{p(\cdot)}$ are not invariant to translations, they have not some undesired properties. For instance, the translation operator is in general not continuous on $L^{p(\cdot)}$. Especially, for every $L^{p(\cdot)}$ with p non-constant there exist $f \in L^{p(\cdot)}$ and a translation τ_h, such that $\tau_h f \notin L^{p(\cdot)}$ (see Theorem 2.10 [10]), and the convolution is in general not continuous. Therefore, Young's inequality for convolutions does not hold in the spaces $L^{p(\cdot)}$. So we will suffer some difficulties for this work. Fortunately, under some conditions one has proved the continuity of the Hardy-Littlewood maximal function [3, 4, 6]. For example, if the bounded exponent $p(x) > 1$ satisfies the following conditions(see [3, 4, 6, 7]):

$$|p(x) - p(y)| \le \frac{C}{\log|x-y|}, \quad x, y \in \mathbb{R}^N \quad |x-y| \le \frac{1}{2}, \tag{1.1}$$

$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)}, \quad x, y \in \mathbb{R}^N \quad |y| \ge |x|, \tag{1.2}$$

where $C > 0$, then maximal operator is bounded on $L^{p(x)}(\mathbb{R}^N)$.

Under the aid of the boundedness of maximal operator and Sobolev integral representation (see[2]), the author obtains some interpolation inequalities in the variable exponent Lebesgue-Sobolev spaces which are similar to those of the classical Sobolev spaces as above. In fact, they are helpful to the study of the differential equations and variational problem with nonstantdard growth conditions.

This article is organized as follows. In section 2, we introduce some definitions and basic properties of the variable exponent Lebesgue-Sobolev spaces $W^{k,p(x)}$ needed in the sequel. In section 3, we prove the our main results. At last, as applications of our results, We discuss a compact Sobolev embedding theorem of the space $W^{k,p(x)}$ and the equivalent norms in the space $W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$. As a consequence we obtain the Landau-Kolmogorov type inequality for the second order derivative on the variable exponent Lebesgue-Sobolev spaces.

2 Preliminaries

Let Ω be a open subset of $\mathbb{R}^N$, let $L_+^\infty(\Omega) = \{p \in L^\infty(\Omega) : \inf_{x\in\Omega} p(x) > 1\}$. For $p \in L_+^\infty(\Omega)$, denote

$$p^- = p^-(\Omega) = \operatorname{ess\,inf}_{x\in\Omega} p(x), \quad p^+ = p^+(\Omega) = \operatorname{ess\,sup}_{x\in\Omega} p(x).$$

On the basic properties of the space $W^{k,p(x)}(\Omega)$, we refer to [8, 9, 10]. Here we display some facts which will be used later.

Denote by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathcal{U}(\Omega)$ are considered as the same element of $\mathcal{U}(\Omega)$ when they are equal almost everywhere. For $p \in L_{+}^{\infty}(\Omega)$, define the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u \in \mathcal{U}(\Omega) : \int_{\Omega} |u(x)|^{p(x)} \, \mathrm{d}x < \infty\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\},$$

and

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\ D^{\alpha}u| \in L^{p(x)}(\Omega), 1 \leq |\alpha| \leq k \right\}$$

with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha| \leq k} |D^{\alpha}u|_{L^{p(x)}(\Omega)},$$

and let $|u|_{j,p(x),\Omega} = \sum_{|\alpha|=j} |D^{\alpha}u|_{L^{p(x)}(\Omega)}$, here α is multi-index and $|\alpha|$ is the order. Denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Proposition 2.1.([8, 9, 10]) The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left| \int_{\Omega} uv \mathrm{d}x \right| \leq \left(\frac{1}{p^{-}} + \frac{1}{q^{-}} \right) |u|_{p(x)} |v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}.$$

Proposition 2.2. ([8, 9]) Set $\rho_p(u) = \int_{\Omega} |u(x)|^{p(x)} \mathrm{d}x$. For $u, u_k \in L^{p(x)}(\Omega)$, we have

1)for $u \neq 0$, $|u|_{p(x)} = \lambda \iff \rho_p(\frac{u}{\lambda}) = 1$.

2) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho_p(u) < 1\ (= 1; > 1)$.

3) If $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^{-}} \leq \rho_p(u) \leq |u|_{p(x)}^{p^{+}}$.

4) If $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^{+}} \leq \rho_p(u) \leq |u|_{p(x)}^{p^{-}}$.

5)$\lim_{k\to\infty} |u_k|_{p(x)} = 0 \iff \lim_{k\to\infty} \rho_p(u_k) = 0$.

Proposition 2.3. ([8, 9, 10])

i)The space $(W^{k,p(x)}(\Omega), \|\cdot\|_{k,p(x)})$ is a separable, uniform convex Banach space.

ii)In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is, there exists a positive constant c such that

$$|u|_{L^{p(x)}(\Omega)} \leq c\, |\nabla u|_{L^{p(x)}(\Omega)}, \ \forall u \in W_0^{1,p(x)}(\Omega).$$

Hereafter, we always denote by $|E|$ the N-Lebesgue measure of set E. Given a function $f \in L^1_{loc}(\Omega)$, we define the maximal function Mf, by

$$Mf(x) = \sup_{x\in B} \frac{1}{|B|}\int_{B\cap\Omega} |f(y)|\mathrm{d}y$$

where the supermum is taken over all ball center at x. Let $\mathscr{B}(\Omega)$ be the set of $p \in L^\infty_+(\Omega)$ such that M is bounded on $L^{p(x)}(\Omega)$.

3 Main Results

In this section, we give some interpolation inequalities for the variable exponent spaces and their proofs.

Theorem 3.1. If $p \in \mathscr{B}(\mathbb{R}^N)$, for every $u \in W^{n,p(x)}(\mathbb{R}^N)$, $0 \le j < n$ and all $\varepsilon > 0$, there exist finite constants K and K' each depending on N, n, p, but being independent of ε, u. We have

$$\begin{aligned}
|u|_{j,p(x)} &\le K(\varepsilon|u|_{n,p(x)} + \varepsilon^{-\frac{j}{n-j}}|u|_{p(x)}), && (3.1)\\
|u|_{j,p(x)} &\le 2K\,|u|_{n,p(x)}^{\frac{j}{n}}\,|u|_{p(x)}^{\frac{n-j}{n}}, && (3.2)\\
\|u\|_{j,p(x)} &\le K'(\varepsilon\,\|u\|_{n,p(x)} + \varepsilon^{-\frac{j}{n-j}}|u|_{p(x)}), && (3.3)\\
\|u\|_{j,p(x)} &\le 2K'\,\|u\|_{n,p(x)}^{\frac{j}{n}}\,|u|_{p(x)}^{\frac{n-j}{n}}. && (3.4)
\end{aligned}$$

.

Proof. Inequality (3.3) follows from repeated applications of (3.1), in order to obtain (3.2), (3.4) choosing ε in (3.1), (3.3) respectively so that the two terms on the right side are equal. Thus we need only prove (3.1). Let $0 < j < n$ and $u \in W^{n,p(x)}(\mathbb{R}^N)$. Suppose $B_\sigma(x)$ is any ball whose center is $x \in \mathbb{R}^N$ and radius is $\sigma > 0$. By Sobolev's integral representation(see[2]), For any $0 < |\beta| = j < n$ we have

$$\begin{aligned}
|D^\beta u| &\le C\left(\sigma^{-j-N}\int_{B_\sigma(x)}|u|\mathrm{d}y + \sum_{|\alpha|=n}\int_{B_\sigma(x)}\frac{|D^\alpha u|}{|x-y|^{N-n+j}}\mathrm{d}y\right)\\
&\le C\left(\sigma^{-j}\frac{1}{|B_\sigma(x)|}\int_{B_\sigma(x)}|u|\mathrm{d}y + \sum_{|\alpha|=n}\int_{B_\sigma(x)}\frac{|D^\alpha u|}{|x-y|^{N-n+j}}\mathrm{d}y\right)\\
&\le C\left(\sigma^{-j}Mu(x) + \sum_{|\alpha|=n}\int_{B_\sigma(x)}\frac{|D^\alpha u|}{|x-y|^{N-n+j}}\mathrm{d}y\right). \qquad (3.5)
\end{aligned}$$

If $n - j < N$, it is well known that for all $\delta > 0$ and $u \in L^1_{loc}(\mathbb{R}^N)$ holds:

$$\int_{B_\delta}\frac{|u|}{|x-y|^{N-n+j}} \le C\delta^{l-j}Mu(x).$$

Thus $\int_{B_\sigma(x)} \frac{|D^\alpha u|}{|x-y|^{N-n+j}} \mathrm{d}y) \le C\sigma^{n-j} M(D^\alpha u(x))$ therefore,we obtain

$$|D^\beta u| \le C\left(\sigma^{-j} Mu(x) + \sigma^{n-j} \sum_{|\alpha|=n} M(D^\alpha u(x))\right). \qquad (3.6)$$

From (3.5), we can immediately obtain

$$|D^\beta u|^{p(x)} \le C\left((\sigma^{-j} Mu(x))^{p(x)} + \sum_{|\alpha|=n} (\sigma^{n-j} M(D^\alpha u(x)))^{p(x)}\right). \qquad (3.7)$$

C may be different constants which is independent on u and σ in the (3.4)-(3.6). Let $\mu = K_0(\sigma^{-j}|Mu|_{p(x)} + \sum_{|\alpha|=n} \sigma^{n-j}|M(D^\alpha u)|_{p(x)})$, we always suppose that $K_0 \ge 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} \left|\frac{D^\beta u}{\mu}\right|^{p(x)} \mathrm{d}x &\le C\left(\int_{\mathbb{R}^N} \left|\frac{\sigma^{-j} Mu}{\mu}\right|^{p(x)} \mathrm{d}x + \sum_{\alpha=n} \int_{\mathbb{R}^N} \left|\frac{\sigma^{n-j} M(D^\alpha u)}{\mu}\right|^{p(x)} \mathrm{d}x\right)\\ &\le \frac{C}{K_0^{p-}}\left(\int_{\mathbb{R}^N} \left|\frac{Mu}{|Mu|_{p(x)}}\right|^{p(x)} \mathrm{d}x + \sum_{\alpha=n} \int_{\mathbb{R}^N} \left|\frac{M(D^\alpha u)}{|M(D^\alpha u)|_{p(x)}}\right|^{p(x)} \mathrm{d}x\right)\\ &\le \frac{(1+N^n)C}{K_0} \le 1.\end{aligned}$$

Hence

$$|D^\beta u|_{p(x)} \le K_0(\sigma^{-j}|Mu|_{p(x)} + \sigma^{n-j} \sum_{\alpha=n} |M(D^\alpha u)|_{p(x)}), \qquad (3.8)$$

Since $p \in \mathscr{B}(\mathbb{R}^N)$, $|Mu|_{p(x)} \le C_1|u|_{p(x)}$, $|M(D^\alpha u)|_{p(x)} \le C_1|D^\alpha u|_{p(x)}$, whence (3.8) implies

$$|D^\beta u|_{p(x)} \le K(\sigma^{-j}|u|_{p(x)} + \sigma^{n-j} \sum_{\alpha=n} |(D^\alpha u)|_{p(x)}). \qquad (3.9)$$

If $n - j > N$, then

$$\begin{aligned}|D^\beta u| &\le C(\sigma^{-j-N} \int_{B_\sigma(x)} |u| \mathrm{d}y + \sum_{\alpha=n} \sigma^{n-j-N} \int_{B_\sigma(x)} |D^\alpha u| \mathrm{d}y)\\ &\le C(\sigma^{-j} Mu + \sum_{\alpha=n} \sigma^{n-j} M(D^\alpha u)) \qquad (3.10)\end{aligned}$$

Similarly, we get

$$|D^\beta u|_{p(x)} \le K(\sigma^{-j}|u|_{p(x)} + \sigma^{n-j} \sum_{\alpha=n} |(D^\alpha u)|_{p(x)}). \qquad (3.11)$$

It is easy to see that the constant$K$ in (3.9) and (3.11) only depends on N, n, p, but is independent of σ and u. (3.1) follows by setting $\varepsilon = \sigma^{n-j}$.

In [4], They have pointed out that it is not clear whether every exponent $p \in \mathscr{B}(\Omega)$ can be extended to an exponent function in $\mathscr{B}(\mathbb{R}^N)$. But we have

Theorem 3.2. Given an open set $\Omega \subset \mathbb{R}^N$ which has uniform cone property (see [1, 2, 4]), $p \in L^\infty_+(\Omega)$ such that (1.1) and (1.2) hold on the the Ω, for every $u \in W^{n,p(x)}(\Omega)$, $0 \le k < n$ and all $\varepsilon > 0$, there exist finite constants K depending on N, n, p, Ω but being independent of ε. We have

$$\begin{aligned} \|u\|_{j,p(x)} &\le K(\varepsilon \|u\|_{n,p(x)} + \varepsilon^{-\frac{j}{n-j}} |u|_{p(x)}), && (3.12)\\ \|u\|_{j,p(x)} &\le 2K \|u\|^{\frac{j}{n}}_{n,p(x)} |u|^{\frac{n-j}{n}}_{p(x)}. && (3.13) \end{aligned}$$

Proof. Thanks to lemma 4.3 in [4], there exist a function $\tilde{p} \in L^\infty_+(\mathbb{R}^N)$ such that $\tilde{p}(x)$ satisfies (1.1)and (1.2), $\tilde{p}|_\Omega = p$ and $\tilde{p}^- = p^-, \tilde{p}^+ = p^+$, thus $\tilde{p} \in \mathscr{B}(\mathbb{R}^N)$. Using Calderón extension theorem for variable Sobolev spaces due to D. Cruz-Uribe et.al.(see Theorem 4.5 [4]), there exists an extension operator

$$E : W^{n,p(x)}(\Omega) \to W^{n,p(x)}(\mathbb{R}^N),$$

such that $Eu(x) = u(x), a.e. x \in \Omega$, and

$$|Eu|_{\tilde{p}(x),\mathbb{R}^N} \le C(p, n, \Omega)|u|_{p(x),\Omega}, \|Eu\|_{n,\tilde{p}(x),\mathbb{R}^N} \le C(p, n, \Omega)\|u\|_{n,p(x),\Omega}.$$

From theorem 3.1,we have

$$\begin{aligned} \|u\|_{j,p(x)} &\le \|Eu\|_{j,\tilde{p}(x),\mathbb{R}^N}\\ &\le C(\varepsilon \|Eu\|_{n,\tilde{p}(x),\mathbb{R}^N} + \varepsilon^{-\frac{j}{n-j}} |Eu|_{\tilde{p}(x),\mathbb{R}^N})\\ &\le K(\varepsilon \|u\|_{n,p(x)} + \varepsilon^{-\frac{j}{n-j}} |u|_{p(x)}), \end{aligned}$$

and

$$\begin{aligned} \|u\|_{j,p(x)} &\le \|Eu\|_{j,\tilde{p}(x),\mathbb{R}^N}\\ &\le 2C \|Eu\|^{\frac{j}{n}}_{n,\tilde{p}(x),\mathbb{R}^N} |Eu|^{\frac{n-j}{n}}_{\tilde{p}(x),\mathbb{R}^N}\\ &\le 2K \|u\|^{\frac{j}{n}}_{n,p(x)} |u|^{\frac{n-j}{n}}_{p(x)}. \end{aligned}$$

Theorem 3.3. If $\Omega \subset \mathbb{R}^N$ is an open set, $p(x)$ as in theorem 3.2, then for all $\varepsilon > 0$ the inequalities (3.1)-(3.4) hold for any $u \in W^{n,p(x)}_0(\Omega)$.

Proof. Applying lemma 4.3 in [4] again, we have there exist a function $\tilde{p} \in L^\infty_+(\mathbb{R}^N)$ such that $\tilde{p}(x)$ satisfies (1.1)and (1.2), $\tilde{p}|_\Omega = p$ and

$\tilde{p}^- = p^-, \tilde{p}^+ = p^+$, thus $\tilde{p} \in \mathscr{B}(\mathbb{R}^N)$. Let $\tilde{u}$ denote the zero extension of u to $\mathbb{R}^N \setminus \Omega$. As same to proof of lemma 3.27 in [1], one can show the mapping $u \mapsto \tilde{u}$ maps $W_0^{n,p(x)}(\Omega)$ isometrically into $W^{n,p(x)}(\mathbb{R}^N)$. We can immediately get the conclusion.

4 Applications

4.1 Sobolev embedding theorems for the space $W^{k,p(x)}$

In this subsection, firstly, we give a Sobolev embedding theorem for variable exponent spaces which generalize Lemma 13 of chapter 4 in [2].

Theorem 4.1. Let $n \in \mathbb{Z}_+, m \in \mathbb{Z}_+ \cup \{0\}, m < n, p, q \in L_+^\infty(\Omega)$ and let $\Omega \subset \mathbb{R}^N$ be an open set.

1.If the embedding

$$W^{n,p(x)}(\Omega) \hookrightarrow W^{m,q(x)}(\Omega) \tag{4.1}$$

is compact, then $\forall \varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that $\forall f \in W^{n,p(x)}(\Omega)$

$$\|f\|_{m,q(x)} \le C(\varepsilon)|f|_{p(x)} + \varepsilon\|f\|_{n,p(x)}. \tag{4.2}$$

2. If $\varepsilon > 0$ (4.2) holds and the embedding $W^{n,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact, then embedding (4.1) is also compact.

Proof. 1. Suppose that inequality (4.2) does not hold for all $\varepsilon > 0$, i.e. there exist $\varepsilon_0 > 0$ and functions $f_k \in W^{n,p(x)}(\Omega), k \in \mathbb{N}$, such that $\|f\|_{n,p(x)} = 1$ and

$$\|f_k\|_{m,q(x)} \le k|f_k|_{p(x)} + \varepsilon_0\|f_k\|_{n,p(x)}. \tag{4.3}$$

Since $\|f_k\|_{n,p(x)} = 1$, by (4.1) it follows that $\|f_k\|_{m,p(x)} \le M$, where M is independent of k. Consequently, by (4.3) we have $|f_k|_{p(x)} < \frac{M}{k}$. Thus $\lim\limits_{k\to\infty} |f_k|_{p(x)} = 0$. Employing (4.3) again we have

$$\liminf_{k\to\infty}\|f_k\|_{m,p(x)} \ge \varepsilon_0. \tag{4.4}$$

Since embedding (4.1) is compact, there exists a subsequence f_{k_j} converging to a function f in $W^{m,q(x)}(\Omega)$. Since $f_{k_j} \to 0$ in $L^{p(x)}(\Omega)$. Thus $f = 0$ a.e. in Ω. This contradicts the inequality (4.4).

2. Let $M > 0$ and $\mathcal{S} = \{f \in W^{n,p(x)} : \|f\|_{n,p(x)} \le M\}$, Since the embedding $W^{n,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact, there exists a sequence $f_k \in \mathcal{S}, k \in \mathbb{N}$, which is a cauchy consequence in $L^{p(x)}(\Omega)$. Furthermore from (4.2) for any $\varepsilon > 0$,

$$\|f_k - f_j\|_{m,q(x)} \le C(\varepsilon)|f_k - f_j|_{p(x)} + \varepsilon\|f_k - f_j\|_{n,p(x)}. \tag{4.5}$$

From (4.5), we have $\lim\limits_{k,j\infty} \|f_k - f_j\|_{m,q(x)} \leq M\varepsilon$. Since ε is arbitrary, thus the sequence f_k is a cauchy sequence in $W^{m,q(x)}(\Omega)$. Therefore there exists a function $f \in W^{m,q(x)}(\Omega)$ such that $f_k \to f$ in $W^{m,q(x)}(\Omega)$ as $k \to \infty$, which completes the proof.

Remark 4.2. It is well know that if $p(x) \equiv p_0$ a constant then theorem 4.1 is valid. But in [2], it is not true that it doesn't require $1 < p_0 < \infty$ which make the space $W^{k,p_0}(\Omega)$ is a reflexive space in Lemma 13 of chapter 4.

Corollary 4.3. Let $n, m \in \mathbb{N}, m < n, \Omega$ be a bounded domain with Lipchitz boundary and $1 < q \leq p < \frac{N}{n}$. Supposed that $p(x)$ satisfies $p(x) < \frac{Np(x)}{N-np(x)} - \epsilon, a.e. x \in \Omega$ for some $\epsilon > 0$ such that (1.1) and (1.2) hold. Then the embedding $W^{n,p(x)}(\Omega) \hookrightarrow W^{m,q(x)}(\Omega)$ is compact.

Proof. Ω be a bounded domain with Lipchitz boundary, then Ω has uniform cone property from [1]. Since $q(x) \leq p(x), m < n$, the embedding $W^{m,p(x)}(\Omega) \hookrightarrow W^{m,q(x)}(\Omega)$ is continuous by theorem[ ] in [9]. Thus the inequality (4.2) holds by theorem 3.2. $p(x) < \frac{Np(x)}{N-np(x)} - \epsilon$ follows the embedding $W^{n,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact from theorem[ ] in[**?**]. From theorem 4.1, we can get this conclusion.

4.2 Equivalent norms in the space $W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$

In this section, we will discuss an application of our results to the quivalent norms in the space $W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$. At first, we set a definition.

Definition 4.4. Assume spaces X, Y are Banach spaces, we define the norm on the space $X \cap Y$ is $\|u\|_{X\cap Y} = \|u\|_X + \|u\|_Y$.

In this section, we always assume Ω and $p(x)$ satisfy the conditions of theorem 3.3. From definition 4.3, we can know that for any $u \in W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$, $\|u\| = \|u\|_{W_0^{1,p(x)}(\Omega)\cap W^{2,p(x)}(\Omega)} = \|u\|_{1,p(x)} + \|u\|_{2,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum\limits_{|\alpha|=2} |D^\alpha u|_{p(x)}$.

Theorem 4.5. In the space $W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$, the norm $\|\cdot\|$ and $|\triangle \cdot|_{p(x)}$ are equivalent norms.

Proof. Choose $u \in C_0^\infty(\Omega)$, we will demonstrate that $\sum\limits_{|\alpha|=2} |D^\alpha u|_{p(x)} \leq C|\triangle u|_{p(x)}$. From [11], we know $\frac{\partial^2 u}{\partial x_i x_j} = -R_i R_j(\triangle u)$, where $R_i u$ is called Rieze transform of u, namely, $R_i u = \lim\limits_{\varepsilon\to 0} C_N \int_{|x-y|>\varepsilon} \frac{x_i - y_i}{|x-y|^{N+1}} u(y)dy$, where

$C_N = \frac{\Gamma(\frac{N+1}{2})}{\pi^{\frac{N+1}{2}}}$. Let $k(x,y) = C_N \frac{x_i - y_i}{|x-y|^{N+1}}$, it is easy to prove that $k(x,y)$ is a standard kernel of definition 4.3 in [5]. Indeed, there exist $\delta = 1 > 0, A = (N+2)NC_N > 0,$, we have $|k(x,y)| \le A|x-y|^{-N}$ and $|\frac{\partial k(x,y)}{\partial x}| \le A|x-y|^{-N-1}$, $|\frac{\partial k(x,y)}{\partial y}| \le A|x-y|^{-N-1}$. We can check that $k(x,y)$ satisfies condition(a) and (b) of proposition 4.3 in [5]. Since $N(x,z) = k(x, x-z) = C_N \frac{z_i}{|z|^{N+1}}$. Obviously, $\int_{|z|=1} N(x,z)\mathrm{d}S = \int_{|z|=1} C_N \frac{z_i}{|z|^{N+1}}\mathrm{d}S = 0$, and let $\sigma = 2 > 1, \int_{|z|=1} |N(x,z)|^\sigma \mathrm{d}S = \int_{|z|=1} C_N \frac{z_i^2}{|z|^{2(N+1)}}\mathrm{d}S = \frac{2\pi^{\frac{1}{2}}\Gamma(\frac{N+1}{2})}{N\Gamma(\frac{N}{2})}$ is bounded uniformly with respect to x. From corollary 4.12 in [5], we know the operators $R_j (j = 1, \cdots, N)$ are uniformly bounded in $L^{p(x)}(\mathbb{R}^N)$. Therefore, for $u \in C_0^\infty(\Omega) \subset L^{p(x)}(\mathbb{R}^N)$, we can get $\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{p(x)} = |-R_i R_j(\triangle u)|_{p(x)} \le C_1|\triangle u|_{p(x)}$, for $i, j = 1, 2, \cdots, N$, moreover,

$$|\triangle u|_{p(x)} \le \sum_{|\alpha|=2} |D^\alpha u|_{p(x)} \le C_2|\triangle u|_{p(x)}. \tag{4.6}$$

From theorem 3.3, we know $|\nabla u|_{p(x)} \le K|u|_{p(x)}^{\frac{1}{2}}|u|_{2,p(x)}^{\frac{1}{2}}$. Applying proposition 2.3. and Cauchy inequality with ε, we have

$$|\nabla u|_{p(x)} \le C_3|\triangle u|_{p(x)} \text{ and } |u| \le C_4|\triangle u|_{p(x)}. \tag{4.7}$$

From (4.6) and (4.7) we can claim that

$$|\triangle u|_{p(x)} \le \|u\|_{W_0^{1,p(x)}(\Omega)\cap W^{2,p(x)}(\Omega)} \le C_5|\triangle u|_{p(x)}. \tag{4.8}$$

$C_0^\infty(\Omega)$ is dense in $W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$, hence the inequality (4.8) holds for any $u \in W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$, which completes this proof.

From the proof above and theorem 3.1, It is easy to obtain the following Landau-Kolmogorov type inequality.

Theorem 4.6. If $p \in \mathscr{B}(\mathbb{R}^N)$ for any $f \in C_0^2(\mathbb{R}^N)$, then one can have

$$|\nabla f|_{p(x)} \le C|f|_{p(x)}^{\frac{1}{2}}|\triangle f|_{p(x)}^{\frac{1}{2}}$$

where C is independent on the function f.

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