Fixed Points Theorems of Ordered Contractive Maps on a Noncommutative Banach Space*

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Abstract The paper introduces the concept of noncommutative Banach spaces and obtains several fixed point theorems for continuous or discontinuous ordered contractive maps in ordered noncommutative Banach spaces. A sufficient condition that the fixed point is unique is given.
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1 Preliminaries

Amann [1] introduced the concepts of ordered topological linear space and ordered Banach space and gave a number of solutions of nonlinear equations in ordered Banach spaces. Based on his work, many authors studied the properties of fixed points of nonlinear equations in ordered Banach spaces [2-6]. Furthermore, [7] introduced some types of ordered contractive maps and obtained some fixed point theorems in ordered Banach spaces. Illumined by [7] and [1], the paper defines the ordered contractive maps and obtains the corresponding theorems of fixed points ordered contractive maps in noncommutative Banach spaces. Let us give the definition of a noncommutative Banach space firstly.

Definition 1.1 Let E be a group. E is called a noncommutative Banach space if the following conditions are satisfied.

1. There exists a metric d on E so that (E, d) is a complete metric space.
2. The d is invariant under the translation operation. That is, $\forall x, y, z \in E$, $d(xz, yz) = d(x, y)$;
3. There exists a binary continuous operation

$$F : \mathbb{R} \times E \longrightarrow E, \quad (\alpha, g) \mapsto g^{\alpha},$$

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which extends the group multiplications in E;

4. The metric d is sub-homogeneous, that is, for $x \in E$, there exists a constant $C_x > 0$ such that for $\alpha \in \mathbb{R}$,

$$d\left(x^{\alpha},\ e\right) \leqslant C_x|\alpha| d\left(x, e\right).$$

It is clear that a Banach space is a noncommutative Banach space. The following is a nontrivial example.

Example 1.1 Suppose that H is a Hilbert space and $U(H)$ is the unitary group of H. As a subset of $L(H)$, $U(H)$ is a complete metric space, where for $S, T \in U(H)$, $d(S, T) = \|S - T\|$. Furthermore, for $T \in U(H)$ and $\alpha \in \mathbb{R}$, set

$$T^{\alpha} = \int_0^{2\pi} e^{i\alpha\theta} dE_{\theta},$$

where E_θ stands for the spectral measure associated with the operator T [8], then $U(H)$ is a noncommutative Banach space.

Proof. It suffices to prove that $U(H)$ possesses properties 3 and 4 of definition 1.1.

Firstly suppose that $d\left(T_n, T\right) \to 0$ where $T_n \in U\left(H\right)$. Since for $k \in \mathbb{N}$,

$$\begin{aligned} d\left(T_n^k, T^k\right) &= \|T_n T_n^{k-1} - T T_n^{k-1} + T T_n^{k-1} - T T^{k-1}\| \\ &\leqslant \|T_n - T\|\|T_n^{k-1}\| + \|T\|\|T_n^{k-1} - T^{k-1}\|, \end{aligned}$$

using induction one can see for an arbitrary polynomial $P\left(x\right)$, $d\left(P\left(T_n\right), P\left(T\right)\right) \to 0$. Since for $\varepsilon > 0$, there exists a polynomial $P_0(x)$ such that $\sup_{x\in[0,2\pi]} |P_0(x) - x^{\alpha}| < \frac{\varepsilon}{3}$, thus

$$d\left(P_0\left(T\right), T^{\alpha}\right) \leqslant \frac{\varepsilon}{3}; \quad d\left(P_0\left(T_n\right), T_n^{\alpha}\right) \leqslant \frac{\varepsilon}{3}.$$

Also for $\frac{\varepsilon}{3} > 0$, $\exists N \in \mathbb{N}$ so that if $n > N$, $\|P_0(T_n) - P_0(T)\| < \frac{\varepsilon}{3}$. Thus when $n > N$,

$$\begin{aligned} d\left(T_n^{\alpha}, T^{\alpha}\right) &\leqslant \|T_n^{\alpha} - P\left(T_n\right) + P\left(T_n\right) + P(T) - P\left(T\right) - T^{\alpha}\| \\ &\leqslant \|T_n^{\alpha} - P(T_n)\| + \|P(T_n) - P(T)\| + \|P(T) - T^{\alpha}\| \\ &\leqslant \varepsilon. \end{aligned}$$

Therefore for $\alpha \in \mathbb{R}$, $d\left(T_n, T\right) \to 0$ implies that $d\left(T_n^{\alpha}, T^{\alpha}\right) \to 0$.

Secondly the metric d is pseudo-homogeneous. One can suppose that $T \neq I$. For $\alpha \in \mathbb{R}$,

$$\begin{aligned} d\left(T^{\alpha}, I\right) &= \|\int_0^{2\pi} \left(e^{i\theta\alpha} - 1\right) dE_{\theta}\| \\ &\leqslant \sup_{\theta\in[0,2\pi]} |e^{i\theta\alpha} - 1|. \end{aligned}$$

Since the exponential function is a periodic function, we can consider only the case of $|\alpha| \in [0, 1]$.

$$\sup_{\theta\in[0,2\pi]} |e^{i\theta\pi} - 1| = \begin{cases} 2 < 2\pi|\alpha|, & \text{if } |\alpha| > \frac{1}{2}; \\ |e^{i2\pi\alpha} - 1| \leq 2\pi|\alpha|, & \text{if } 0 \leqslant |\alpha| \leqslant \frac{1}{2}. \end{cases}$$

In all, set $C_T = \frac{2\pi}{d(T,\ I)}$, then for any $\alpha \in \mathbb{R}$, $d(T^\alpha,\ I) \leqslant C_T|\alpha|d(T, I)$.

Similarly one can prove that for $T \in U(H)$, $\lim_{\alpha\to\alpha_0} T^\alpha = T^{\alpha_0}$ and $U(H)$ is a noncommutative Banach space.

2 Ordered contractive maps on the noncommutative Banach Spaces

In this section we will introduce an ordering structure in a noncommutative Banach space, and get basic properties of ordered contractive maps.

Definition 2.1 Suppose that E is a noncommutative Banach space. A set $P \subseteq E$ is called convex if $\forall x,\ y \in P$, $x^p y^q \in P$, where $p, q \in \mathbb{R}^+$ and $p + q = 1$. Furthermore, $P \subseteq E$ is called a cone if P is closed, convex and invariant under exponential operation by element of $[0, \infty)$, and if $P \cap P^{-1} = \{e\}$, where $P^{-1} = \{x^{-1}|x \in P\}$.

It is easy to see that a cone is a semigroup. Each cone can induces a partial ordering in E through the rule $x \lesssim y$ if and only if $y^\beta x^{-\beta} \in E$ for $\beta \in [0,\ 1]$. This ordering is antisymmetry, reflexive and transitive.

Definition 2.2 If there exists a constant $N > 0$ such that for any $e \lesssim x \lesssim y$, $d(x, e) \leqslant Nd(y, e)$, P is called positive, and the constant N is called the positive constant of P.

Let "$\lesssim$" be the partial ordering determined by a cone P. For $u, v \in E$, if one of $u \lesssim v$ and $v \lesssim u$ holds, we say that u and v are comparable and write:

$$\vee(u, v) = \begin{cases} u, & \text{when } v \lesssim u, \\ v, & \text{when } u \lesssim v. \end{cases}$$

Lemma 2.1 *If u and v are comparable, then uv^{-1} and vu^{-1} are comparable, and*

$$e \lesssim \vee\left(uv^{-1}, vu^{-1}\right).$$

Proof. One can suppose that $v \lesssim u$. Then $\forall\alpha \in [0, 1]$,

$$\left(uv^{-1}\right)^\alpha \left(vu^{-1}\right)^{-\alpha} = \left(uv^{-1}\right)^\alpha \left(\left(uv^{-1}\right)^{-1}\right)^{-\alpha} = \left(uv^{-1}\right)^{2\alpha}.$$

Since $2\alpha > 0$ and $uv^{-1} \in P$, $\left(uv^{-1}\right)^{2\alpha} \in P$. Thus the elements uv^{-1} and vu^{-1} are comparable, and $vu^{-1} \lesssim uv^{-1}$. Also, $\forall\alpha \in [0, 1]$, $\left(uv^{-1}\right)^\alpha e^{-\alpha} = \left(uv^{-1}\right)^\alpha \in P$, so $e \lesssim uv^{-1}$, and $e \lesssim \vee\left(uv^{-1}, vu^{-1}\right)$.

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Definition 2.3 Let E be a noncommutative Banach space and P a positive cone of E with the positive constant N. A map $A : E \to E$ is called a β-ordered contractive map if there exists a constant $0 < \beta < 1$ such that for $u, v \in E$, if u and v are comparable, then Au and Av are also comparable, and moreover

$$\vee\left(Av\,(Au)^{-1}, Au\,(Av)^{-1}\right) \lesssim \vee\left(vu^{-1}, uv^{-1}\right)^{\beta}.$$

Here the β is called the constant of the ordered contractive map.

Remark 2.1 *The ordered contractive map need not be continuous.*

Lemma 2.2 *Suppose that for all $n \in \mathbb{N}$, u_n and v_n are comparable. If $v_n \to v_0$, $u_n \to u_0$, then u_0 and v_0 are comparable. That is to say, the ordering structure is compatible with the metric given in E.*

Proof. Since $\forall n \in \mathbb{N}$, one of $u_n \lesssim v_n$ and $v_n \lesssim u_n$ holds, there exist subsequences $\{v_{n_k}\}$ and $\{u_{n_k}\}$ such that for $\forall 0 \leqslant \beta \leqslant 1$, either $u_{n_k}^{\beta} v_{n_k}^{-\beta} \in P$ or $v_{n_k}^{\beta} u_{n_k}^{-\beta} \in P$ holds. Without lose of generality, suppose that $u_{n_k}^{\beta} v_{n_k}^{-\beta} \in P$. Then

$$\begin{aligned} d\left(u_{n_k}^{\beta} v_{n_k}^{-\beta}, u_0^{\beta} v_0^{-\beta}\right) &\leqslant d\left(u_{n_k}^{\beta} v_{n_k}^{-\beta}, u_{n_k}^{\beta} v_0^{-\beta}\right) + d\left(u_{n_k}^{\beta} v_0^{-\beta}, u_0^{\beta} v_0^{-\beta}\right) \\ &= d\left(u_{n_k}^{\beta}, u_0^{\beta}\right) + d\left(v_{n_k}^{-\beta}, v_0^{-\beta}\right). \end{aligned}$$

The last equation holds because the metric is invariant under the translation operation.

Because $v_{n_k} \to v_0$ and $u_{n_k} \to u_0$, we have $v_{n_k}^{-\beta} \to v_0^{-\beta}$ and $u_{n_k}^{-\beta} \to u_0^{-\beta}$. Since the multiplication operation on E is continuous,

$$\lim_{k\to\infty} d\left(u_{n_k}^{\beta} v_{n_k}^{-\beta}, u_0^{\beta} v_0^{-\beta}\right) = 0.$$

Notice the fact that the cone P is closed, $u_0^{\beta} v_0^{-\beta} \in P$. This implies that u_0 and v_0 are comparable.

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Lemma 2.3 *If $x, y \in P$, and $x \lesssim y$, then $\forall 0 < \beta < 1, x^{\beta} \lesssim y^{\beta}$.*

Proof. Since $x \lesssim y$, $\forall \alpha \in [0, 1]$, $y^{\alpha} x^{-\alpha} \in P$. For $0 < \beta < 1$, $\alpha\beta \in [0, 1]$, so $y^{\alpha\beta} x^{-\alpha\beta} \in P$, namely, $x^{\beta} \lesssim y^{\beta}$.

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Lemma 2.4 *If x and y are comparable, then $d\left(\vee\left(xy^{-1}, yx^{-1}\right), e\right) = d\left(x, y\right)$.*

Proof. One can suppose $\vee(x, y) = x$. Then $yx^{-1} \lesssim xy^{-1}$ and $\vee\left(xy^{-1}, yx^{-1}\right) = xy^{-1}$. Since the metric d is invariant under the translation operation,

$$d\left(x, y\right) = d\left(xy^{-1}, yy^{-1}\right) = d\left(\vee\left(xy^{-1}, yx^{-1}\right), e\right).$$

This completes the proof.

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3 Theorems about the Fixed Points

Throughout this section we suppose that E is a noncommutative Banach space which is partially ordered by a positive cone P with the positive constant N, and give several theorems on the fixed points of the ordered contractive maps on E.

Theorem 3.1 *Suppose that the β-ordered contractive map $A : E \to E$ is continuous. If there exists an element $x_0 \in E$ such that x_0 and Ax_0 are comparable, then the sequence $A^n x_0$ converges to some fixed point x^* of A. Moreover, there is a number C_{x_0} depending on the choice of x_0, so that*

$$d(x_0, x^*) \leqslant \left(\frac{C_{x_0} \cdot N \cdot \beta}{1-\beta} + 1\right) d(x_0, Ax_0).$$

Proof. Consider the sequence

$$x_1 = Ax_0,\ x_2 = Ax_1 ..., x_{n+1} = Ax_n, ...$$

Since x_0 and $x_1 = Ax_0$ are comparable and the map A is a β-ordered contractive map, x_1 and $x_2 = Ax_1$ are comparable, and hence, x_n and $x_{n+1} = Ax_n$ are comparable. Since

$$\vee\left(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}\right) = \vee\left(Ax_{n-1}(Ax_n)^{-1}, Ax_n(Ax_{n-1})^{-1}\right) \lesssim \vee\left(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1}\right)^{\beta},$$

using lemma 2.3,

$$\begin{aligned}
\vee\left(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}\right) &\lesssim \vee\left(x_{n-1}x_n^{-1}, x_n x_{n-1}^{-1}\right)^{\beta} \\
&\lesssim \vee\left(x_{n-2}x_{n-1}^{-1}, x_{n-1} x_{n-2}^{-1}\right)^{\beta^2} \\
&\lesssim \dots \\
&\lesssim \vee\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right)^{\beta^n}.
\end{aligned}$$

Thus,

$$d\left(\vee\left(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}\right), e\right) \leqslant Nd\left(\vee\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right)^{\beta^n}, e\right). \tag{1}$$

Because the metric d is sub-homogeneous, there exists a constant C_{x_0}, which depends on the choice of x_0, so that

$$d\left(\vee\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right)^{\beta^n}, e\right) \leqslant C_{x_0} \cdot \beta^n d\left(\vee\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right), e\right). \tag{2}$$

Notice that the metric d is invariant under the translation operation,

$$d\left(\vee\left(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}\right), e\right) = d(x_n, x_{n+1}). \tag{3}$$

In the same way,

$$d\left(\vee\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right), e\right) = d(x_0, x_1). \tag{4}$$

So the inequality (2) turns into

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &= d\left(\vee\left(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}\right), e\right) \\ &\leqslant N \cdot d\left(\vee\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right)^{\beta^n}, e\right) \\ &\leqslant N \cdot C_{x_0} \cdot \beta^n d\left(\vee\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right), e\right) \\ &= N \cdot C_{x_0} \cdot \beta^n d(x_0,\ x_1). \end{aligned}$$

Thus the sequence $\{x_n\}$ is a Cauchy sequence since $\beta \in (0,\ 1)$. Suppose that $x_n \to x^*$, then

$$Ax^* = A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x^*,$$

which implies that x^* is a fixed point of A. Moreover

$$\begin{aligned} d\left(x^*, x_0\right) &\leqslant d\left(x_0, x_1\right) + d\left(x_1, x_2\right) + \cdots + d\left(x_n, x_{n+1}\right) + \cdots \\ &= \sum_{n=1}^{\infty} d\left(x_n, x_{n+1}\right) + d\left(x_0, x_1\right) \\ &\leqslant \sum_{n=1}^{\infty} C_{x_0} \cdot N \cdot \beta^n d\left(x_0, x_1\right) + d\left(x_0, x_1\right) \\ &= \left(\frac{C_{x_0} \cdot N \cdot \beta}{1-\beta} + 1\right) d\left(x_0, Ax_0\right). \end{aligned}$$

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Corollary 3.2 *Conditions and assumptions are the same as in theorem 3.1. Let* $\widetilde{x}$ *be another fixed point of* A*. If* $\widetilde{x}$ *and* x^* *are comparable, then* $\widetilde{x} = x^*$*.*

Proof. Since $\widetilde{x}$ and x^* are comparable, one can suppose that $\widetilde{x} \lesssim x^*$. Using the definition of contractive map,

$$\vee\left(A\widetilde{x}(Ax^*)^{-1}, Ax^*\left(A\widetilde{x}\right)^{-1}\right) \lesssim \vee\left(x^*\widetilde{x}^{-1}, \widetilde{x}x^{*-1}\right)^{\beta},$$

namely,

$$\vee\left(x^*\widetilde{x}^{-1}, \widetilde{x}x^{*-1}\right) \lesssim \vee\left(x^*\widetilde{x}^{-1}, \widetilde{x}x^{*-1}\right)^{\beta}.$$

Since $\widetilde{x} \lesssim x^*$, $\vee\left(\widetilde{x}x^{*-1}, x^*\widetilde{x}^{-1}\right) = x^*\widetilde{x}^{-1}$, then $x^*\widetilde{x}^{-1} \lesssim \left(x^*\widetilde{x}^{-1}\right)^{\beta}$, $\left(x^*\widetilde{x}^{-1}\right)^{\beta-1} \in P$. Notice that $1-\beta \in [0,1]$, $\left(x^*\widetilde{x}^{-1}\right)^{1-\beta} \in P$. By $\left(x^*\widetilde{x}^{-1}\right)^{\beta-1} \in P$ and $\left(x^*\widetilde{x}^{-1}\right)^{1-\beta} \in P$, $x^*\widetilde{x}^{-1} = e$, and $x^* = \widetilde{x}$.

■

Theorem 3.3 *Suppose that* $A : E \to E$ *is a* β*-ordered contractive map. If there exists an element* $x_0 \in E$ *so that* $\forall n$, x_0 *and* $A^n x_0$ *are comparable, then* A *has some fixed point, and the sequence* $\{A^n x_0\}$ *converges to one fixed point* x^* *of* A*. Moreover, there exists a constant* C_{x_0} *such that*

$$d\left(x^*, x_0\right) \leqslant \left(\frac{C_{x_0} \cdot N \cdot \beta}{1-\beta} + 1\right) d\left(x_0, Ax_0\right).$$

Proof. Similar to the proof of theorem 3.1, the sequence $\{x_n = A^n x_0\}$ is a Cauchy sequence. By the completeness of E, let $x_n \to x^* \in E$. Now we prove that x^* is a fixed point of A.

For all m, n, suppose that $m > n$, using the given condition, x_0 and x_{m-n} are comparable. Then Ax_n and Ax_{m-n} are comparable, and so are $x_n = A^n x_0$ and $x_m = A^n x_{m-n}$. Let $m \to \infty$, using lemma 2.2, $\forall n, x_n$ and x^* are comparable, therefore Ax_n and Ax^* are comparable, and so

$$e \lesssim \vee\left(Ax_n\left(Ax^*\right)^{-1}, Ax^*\left(Ax_n\right)^{-1}\right) \lesssim \vee\left(x_n x^{*-1}, x^* x_n^{-1}\right)^{\beta}.$$

Since P is a positive cone,

$$d\left(\vee\left(Ax_n\left(Ax^*\right)^{-1}, Ax^*\left(Ax_n\right)^{-1}\right), e\right) \leqslant C_{x_0} \cdot N \cdot \beta d\left(\vee\left(x_n x^{*-1}, x^* x_n^{-1}\right), e\right),$$

that is

$$d\left(x_{n+1}, Ax^*\right) = d\left(Ax_n, Ax^*\right) \leqslant C_{x_0} \cdot N \cdot \beta d\left(x_n, x^*\right) \to 0.$$

Therefore $x^* = Ax^*$ and x^* is a fixed point of A. At last, similar to the proof of Theorem 3.1, we can get the estimation of $d\left(x^*, x_0\right)$ and we omit it here. ■

Theorem 3.4 *Let $A : E \to E$ be a continuous map and satisfy the following condition:*

(C1) If u and v are comparable, then Au and Av are comparable. Also, if u and Au are comparable, and v and Av are comparable, then there exists a $\lambda \in \left(0, \frac{1}{2}\right)$ so that for $\forall \beta \in [0, 1]$,

$$\vee\left(Av\left(Au\right)^{-1}, Au\left(Av\right)^{-1}\right)^{\beta} \lesssim \vee\left(Au \circ u^{-1}, u \circ \left(Au\right)^{-1}\right)^{\lambda\beta} \circ \vee\left(Av \circ v^{-1}, v\left(Av\right)^{-1}\right)^{\lambda\beta}.$$

If there exists an element $x_0 \in E$, such that x_0 and Ax_0 are comparable, then the sequence $\{A^n x_0\}$ converges to a fixed point x^* of A. Moreover, there exists a constant C_{x_0} such that

$$d\left(x_0, x^*\right) \leqslant \left(1 + \frac{C_{x_0} \cdot N \cdot \lambda}{1 - 2\lambda}\right) d\left(x_0, Ax_0\right).$$

Proof. Set $\beta = \frac{\lambda}{1-\lambda}$, then $1 + \frac{N\beta}{1-\beta} = 1 + \frac{\lambda \cdot N}{1-2\lambda}$. Consider the sequence:

$$x_1 = Ax_0, x_2 = Ax_1 = A^2 x_0, \cdots, x_{n+1} = Ax_n, \cdots.$$

Since x_0 and Ax_0 are comparable, for $n \in \mathbb{N}$, x_n and Ax_n are comparable, and

$$\begin{aligned} e &\lesssim \vee\left(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}\right) \\ &= \vee\left(Ax_{n-1}\left(Ax_n\right)^{-1}, Ax_n\left(Ax_{n-1}^{-1}\right)\right) \\ &\lesssim \vee\left(Ax_{n-1} x_{n-1}^{-1}, x_{n-1}\left(Ax_{n-1}\right)^{-1}\right)^{\lambda} \vee\left(Ax_n x_n^{-1}, x_n\left(Ax_n\right)^{-1}\right)^{\lambda} \\ &= \vee\left(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1}\right)^{\lambda} \vee\left(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}\right)^{\lambda}. \end{aligned}$$

Therefore

$$e \lesssim \vee\left(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}\right) \lesssim \vee\left(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1}\right)^{\lambda} \vee\left(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}\right)^{\lambda},$$

and $\forall \beta \in [0,1]$,

$$\vee\left(x_n x_{n+1}^{-1}, x_{n+1} x_n^{-1}\right)^{\beta} \lesssim \vee\left(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1}\right)^{\beta\lambda} \vee\left(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}\right)^{\beta\lambda}.$$

That is,

$$\vee\left(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1}\right)^{\beta\lambda} \vee\left(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}\right)^{-(1-\lambda)\beta} \in P,$$

and so

$$\vee\left(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}\right)^{(1-\lambda)} \lesssim \vee\left(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1}\right)^{\lambda}.$$

Since $0 < \lambda < \frac{1}{2}, 0 < \frac{\lambda}{1-\lambda} < 1$, using lemma 2.3

$$\begin{aligned} e &\lesssim \vee\left(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}\right) \\ &\lesssim \vee\left(x_n x_{n-1}^{-1}, x_{n-1} x_n^{-1}\right)^{\frac{\lambda}{1-\lambda}} \\ &\lesssim \cdots \\ &\lesssim \vee\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right)^{\left(\frac{\lambda}{1-\lambda}\right)^n}. \end{aligned}$$

Using inequality (2) in the proof of theorem 3.1 there exists a constant C_{x_0} such that,

$$d\left(\vee\left(x_{n+1} x_n^{-1}, x_n x_{n+1}^{-1}\right), e\right) \leqslant C_{x_0} \cdot N \cdot \left(\frac{\lambda}{1-\lambda}\right)^n d\left(x_0 x_1^{-1}, x_1 x_0^{-1}\right).$$

Since the metric d is invariant under the translation operation,

$$d\left(x_n, x_{n+1}\right) \leqslant C_{x_0} \cdot N \cdot \left(\frac{\lambda}{1-\lambda}\right)^n d\left(x_0, x_1\right).$$

This implies that $\{x_n\}$ is a Cauchy sequence. By the completeness of E, let $x_n \to x^* \in E$, then

$$Ax^* = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus A has a fixed point in E, and the sequence $\{A^n x\}$ converges to a fixed point of A. ■

Theorem 3.5 *Let $A : E \to E$ be a map satisfying the condition (C1) of theorem 3.4. If there exists an element $x_0 \in E$ so that $\forall n \in \mathbb{N}$, x_0 and $A^n x_0$ are comparable, then A has a fixed point in E, and the sequence $\{A^n x_0\}$ converges to a fixed point x^* of A. Moreover, there exists a constant C_{x_0} such that*

$$d\left(x_0, x^*\right) \leqslant \left(1 + \frac{N \cdot C_{x_0} \cdot \lambda}{1 - 2\lambda}\right) d\left(x_0, Ax_0\right).$$

Proof. Similar to the proof of theorem 3.3, $\{x_n = A^n x_0\}$ is a Cauchy sequence.Let $x_n \to x^* \in E$. Now we prove that x^* is a fixed point of A. As we have proved in theorem 3.2, for all n, x_n and x^* are comparable. Using (C1), for all n, x_{n-1} and $x_n = Ax_{n-1}$ are comparable. Let $n \to \infty$, using lemma 2.2, x^* and Ax^* are comparable. Hence,

$$e \lesssim \vee\left(Ax_n (Ax^*)^{-1}, Ax^* (Ax_n)^{-1}\right) \lesssim \vee\left(Ax_n x_n^{-1}, x_n (Ax_n)^{-1}\right)^{\lambda} \circ \vee\left(Ax^* x^{*-1}, x^* (Ax^*)^{-1}\right)^{\lambda}.$$

Let $n \to \infty$, we obtain

$$e \lesssim \vee\left(x^* (Ax^*)^{-1}, Ax^* (x^*)^{-1}\right) \lesssim \vee\left(Ax^* x^{*-1}, x^* (Ax^*)^{-1}\right)^{\lambda} \circ \vee\left(Ax^* x^{*-1}, x^* (Ax^*)^{-1}\right)^{\lambda},$$

that is

$$e \lesssim \vee\left(x^* (Ax^*)^{-1}, Ax^* (x^*)^{-1}\right) \lesssim \vee\left(Ax^* x^{*-1}, x^* (Ax^*)^{-1}\right)^{2\lambda}.$$

Thus, $\vee\left(Ax^* x^{*-1}, x^* (Ax^*)^{-1}\right)^{2\lambda-1} \in P$. Since $2\lambda-1 < 0$ and $\vee\left((Ax)^* x^{*-1}, x^* (Ax^*)^{-1}\right) \in P$, $x^* (Ax^*)^{-1} = Ax^* (x^*)^{-1} = e$, namely, $Ax^* = x^*$. Therefore x^* is a fixed point of A. ■

Remark 3.1 *In Theorems* 3.3, 3.4 *and* 3.5, *the estimations of* $d(x_0, x^*)$ *are the same as that in theorem 3.1. This is because* $\{x_n\}$ *is the Cauchy sequence which makes* $d(x_n, x^*) \leqslant C_{x_0} \cdot N \cdot \beta^n d(x_1, x_0)$ *hold. In theorems 3.4 and 3.5,* $\beta = \frac{\lambda}{1-\lambda}$.

Theorem 3.6 *Suppse that* $u_0, v_0 \in E$ *with* $u_0 \lesssim v_0$, *and* $[u_0, v_0] = \{u \in E | u_0 \lesssim u \lesssim v_0\}$ *is a ordered interval in* E. *If* $A : [u_0, v_0] \to [u_0, v_0]$ *is a* β*-ordered contractive map, then* A *has a unique fixed point. Moreover, for all* $x \in [u_0, v_0]$, *the sequence* $\{A^n x\}$ *converges to the only fixed point of* A.

Proof. Define the sequences:

$$u_1 = Au_0, u_2 = Au_1, \cdots, u_{n+1} = Au_n, \cdots,$$

$$v_1 = Av_0, v_2 = Av_1, \cdots, v_{n+1} = Av_n, \cdots,$$

then $\{u_n\}, \{v_n\} \subset [u_0, v_0]$. Since $u_0 \lesssim v_0$ and A is a β-ordered contractive map, for all n, u_n and v_n are comparable, and

$$\begin{aligned} e &\lesssim \vee\left(u_n v_n^{-1}, v_n u_n^{-1}\right) \\ &= \vee\left(Au_{n-1} (Av_{n-1})^{-1}, Av_{n-1} (Au_{n-1})^{-1}\right) \\ &\lesssim \vee\left(u_{n-1} v_{n-1}^{-1}, v_{n-1} u_{n-1}^{-1}\right)^{\beta} \\ &\lesssim \cdots \\ &\lesssim \vee\left(u_0 v_0^{-1}, v_0 u_0^{-1}\right)^{\beta^n}. \end{aligned}$$

Because P is positive, there exists a constant $C_{u_0 v_0^{-1}}$ and a positive integer N such that

$$d(u_n, v_n) = d\left(u_n v_n^{-1}, e\right) = d\left(v_n u_n^{-1}, e\right) \leqslant C_{u_0 v_0^{-1}} \cdot N \cdot \beta^n d(u_0, v_0).$$

Since again $u_0 \lesssim u_1$, for all n, u_n and u_{n+1} are comparable and

$$\begin{aligned} e &\lesssim \vee\left(u_n u_{n+1}^{-1}, u_{n+1} u_n^{-1}\right) \\ &= \vee\left(Au_{n-1}\left(Au_n\right)^{-1}, Au_n\left(Au_{n-1}^{-1}\right)\right) \\ &\lesssim \vee\left(u_{n-1} u_n^{-1}, u_n u_{n-1}^{-1}\right)^{\beta} \\ &\lesssim \cdots \\ &\lesssim \vee\left(u_0 u_1^{-1}, u_1 u_0^{-1}\right)^{\beta^n}. \end{aligned}$$

Thus,

$$d\left(u_n, u_{n+1}\right) \leqslant \beta^n N d\left(u_0, u_1\right).$$

Notice that $\beta < 1$, $\{u_n\}$ is a Cauchy sequence with a limit point $u^* \in [u_0,\ v_0]$. Similarly $\{v_n\}$ is a Cauchy sequence with a limit point $v^* \in [u_0, v_0]$. Then

$$d\left(u^*, v^*\right) = \lim_{n\to\infty} d\left(u_n, v_n\right) \leqslant \lim_{n\to\infty} C_{u_0 v_0^{-1}} \cdot N \cdot \beta^n d\left(u_0,\ v_0\right) = 0.$$

This implies that $u^* = v^*$.

Now we prove that u^* is a fixed point of A. For all $m > n$, since u_0 and u_{m-n} are comparable, $A^n u_0 = u_n$ and $A^n u_{m-n} = u_m$ are comparable. Let $m \to \infty$, then u_n and u^* are comparable, and Au_n and Au^* are also comparable,

$$e \lesssim \vee\left(Au_n\left(Au^*\right)^{-1}, Au^*\left(Au_n\right)^{-1}\right) \lesssim \vee\left(u_n u^{*-1}, u^* u_n^{-1}\right)^{\beta}.$$

So,

$$\lim_{n\to\infty} d\left(u_{n+1}, Au^*\right) = \lim_{n\to\infty} d\left(Au_n, Au^*\right) \leqslant \lim_{n\to\infty} C_{u_n u^{*-1}} \cdot N \cdot \beta d\left(u_n, u^*\right) = 0$$

Thus $Au^* = u^*$, u^* is a fixed point of A.

For all $x \in [u_0, v_0]$, since x and u_0 are comparable, $A^n x$ and $A^n u_0$ are comparable, and

$$e \lesssim \vee\left(A^n x\left(A^n u_0\right)^{-1}, A^n u_0\left(A^n x\right)^{-1}\right) \lesssim \left[\vee\left(u_0 x^{-1}, x u_0^{-1}\right)\right]^{\beta^n} \to e$$

Therefore $A^n x \to u^*$.

Now we prove that the fixed point of A is unique. Suppose that v is another fixed point of A in $[u_0, v_0]$, then

$$d(u^*, v) \quad \leq \quad d(u^*, A^n u_0) + d(A^n u_0, A^n v).$$

Notice that $u_0 \lesssim v$, we have

$$\begin{aligned} e &\lesssim \vee\left(A^n u_0\left(A^n v\right)^{-1}, A^n v\left(A^n u_0\right)^{-1}\right) \\ &= \vee\left(\left(AA^{n-1}u_0\right)\left(AA^{n-1}v\right)^{-1}, \left(AA^{n-1}v\right)\left(AA^{n-1}u_0\right)^{-1}\right) \\ &\lesssim \vee\left(A^{n-1}u_0\left(A^{n-1}v\right)^{-1}, A^{n-1}v\left(A^{n-1}u_0\right)^{-1}\right)^{\beta} \\ &\lesssim \cdots \\ &\lesssim \vee\left(u_0 v^{-1}, v u_0^{-1}\right)^{\beta^n}. \end{aligned}$$

Thus

$$\begin{aligned} d\left(A^n u_0, A^n v\right) &= d\left(\vee A^n u_0 \left(A^n v\right)^{-1}, A^n v \left(A^n u_0\right)^{-1}\right) \\ &\leq Nd\left(\vee \left(u_0 v^{-1}, v u_0^{-1}\right)^{\beta^n}, e\right) \\ &\leq NC_{u_0 v^{-1}} \beta^n d\left(u^*, v\right). \end{aligned}$$

Since u^* and v are fixed points, $d\left(A^n u_0, A^n v\right) \to d\left(u^*, v\right)$, we have $d\left(u^*, v\right) = 0$. The uniqueness is proved.

■

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