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Noncommutative κ-Minkowski ϕ^4 theory: Construction, properties and propagation

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Abstract: Noncommutative (NC) κ-deformation of a spacetime, whose NC coordinates close in a Lie algebra, affects the coalgebra of the Poincaré group and the algebra of physical fields. This leads to a modification of multiplication in the corresponding universal enveloping algebra, thus requiring replacement of the usual pointwise multiplication by a deformed star ($\star$) product. The measure problems in the κ-Minkowski NC spacetime are avoided because the measure function is naturally absorbed within the new $\star_h$-product. That reflects itself in the way we have constructed the deformed NC scalar ϕ^4 action. The action is further modified by including harmonic oscillator term and expanding up to linear order in the κ-deformation parameter a, producing an effective theory on commutative spacetime. Furthermore, we obtain modified equations of motion and conserved currents due to internal symmetry at that order. Next, to compute the tadpole diagram contributions to the scalar field propagation/self-energy, we anticipate that statistics on κ-Minkowski is specifically deformed. Thus our prescription in fact represents *hybrid approach* between standard quantum field theory (QFT) and NCQFT on κ-deformed Minkowski spacetime, producing κ-effective theory. The results are analyzed in the framework of two-point Green's function for low, middle size, and for Planckian energies, respectively. For low energies E, the tadpole diagram dependence on the κ-deformation parameter a completely drops out. At Planckian propagation energies, the tadpole diagram contribution tends to a finite fixed value, which depends only on the Planckian energy and/or the κ-deformation parameter. *Semiclassical/hybrid behavior* of the first order quantum effects *do show up* due to the κ-deformed momentum conservation law. The mass term of the scalar field is shifted and these shifts are dramatically different at different propagation energies. Thus, at Planckian energies we have κ-modified dispersion relations, i.e. our effective theory for the massive scalar field mode *shows genuine effect of birefringence*. We conclude that our results could have physical consequences, in the NC Higgs sector, connection to quantum gravity, etc.

Keywords: kappa-deformed space, noncommutative quantum field theory.

Contents

1. Introduction

Basic areas of interest, quantum gravity and deformation of Poincaré algebra make κ-Minkowski spacetime an important subject of theoretical investigation from both, physical as well as mathematical perspective, respectively.

An argument that favors κ-Minkowski spacetime is the indication that it could arise in the context of quantum gravity coupled to matter fields [1, 2]. These considerations show that after integrating out gravitational topological degrees of freedom of gravity, the effective dynamics of matter fields is described by a noncommutative quantum field theory which has a κ-Poincaré group as its symmetry group [3, 4, 5]. In this context, the κ-Minkowski spacetime can be, and there are some arguments to support this, considered as a flat limit of quantum gravity.

Second, since κ-Minkowski emerges naturally from κ-Poincaré algebra [6, 7, 8, 9, 10], which provides a possible group theoretical framework for describing symmetry lying in the core of the Doubly Special Relativity (DSR) theories [11, 12, 13, 14, 15, 16], it is thus a convenient spacetime candidate for DSR theories too. Although different proposals for DSR theories can be looked upon as different bases [15, 16] for κ-Poincaré algebra, they all have in their core the very same noncommutative structure, encoded within κ-deformed algebra. This situation makes noncommutative field theories on noncommutative κ-Minkowski spacetime an even more interesting subject to study. Various attempts have been undertaken in this direction by many authors [17, 18, 19, 20, 21], including various possibilities for construction and investigation of their properties.

Recently, it was established that if the κ-Poincaré Hopf algebra is supposed to be a plausible model for describing physics in κ-Minkowski spacetime, then it is necessary to accept certain modifications in statistics obeyed by the particles. This means that κ-Minkowski spacetime leads to modification of particle statistics which results in deformed oscillator algebras [22, 23, 24, 25, 26, 27, 28]. Deformation quantization of Poincaré algebra can be performed by means of the twist operator [29, 30, 31, 32, 33] which happens to include dilatation generator, thus belonging to the universal enveloping algebra of the general linear algebra [34, 35, 36, 37, 38]. This twist operator gives rise to a deformed statistics on κ-Minkowski spacetime [23, 35, 39, 40]. However, since the κ-Poincaré Hopf algebra is a quantum symmetry described only approximatively by twisted quantum algebra, there could be problems with identifying charges and current conservation, that is with establishing the Noether theorem. Thus, certain modification towards momentum conservation is necessary to obtain any reasonable physics out of approximatively twisted Hopf algebra prescription of our theory.

Transformation from noncommutative κ-Minkowski to Minkowski spacetime in the case of the free field theory was described in [41], while the star product and interacting fields on κ-Minkowski space were treated in the same approach in [42]. The problem of UV/IR mixing and κ-deformation was discussed in [43]. In correspondence to the above observations, as well as by comparing deformed dispersion relations to corresponding time delay calculations of high energy photons, bounds can be put on the quantum gravity scale [44, 45, 46]. We are continuing along the line where the main aim is to transcribe original NCQFT on κ-Minkowski to a corresponding commutative QFT on standard Minkowski spacetime. With this in mind, we are considering κ-deformation of a Minkowski spacetime whose symmetry has an undeformed Lorentz sector (classical basis [47, 15, 16, 48]) and whose noncommutative coordinates close in κ-deformed Lie algebra and additionally, form a Lie algebra with Lorentz generators [49, 50, 51, 52]. That deformation of the spacetime structure affects the algebra of physical fields, leading to a modification of multiplication in the corresponding universal enveloping algebra, requiring replacement of the usual pointwise multiplication by a deformed star product, i.e. by the new star product $\star_h$, thus reproducing important trace-like property [52]. Instead of being given by classical NC ϕ^4, the action describing a massive and generally complex scalar field in interaction gets modified accordingly. This way the integral measure problems in the κ-Minkowski NC spacetime are avoided since the measure function is naturally absorbed within the new $\star_h$-product. The action is further modified by inclusion of the harmonic type of the interacting term [53, 54] and truncating by expansion only to the first order in the deformation parameter a, producing an effective theory on commutative spacetime. We obtain further modified equations of motion and conserved currents at that order, due to invariance under internal symmetry. Next, to

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compute the tadpole diagram contributions to the scalar field propagation/self-energy, we anticipate that statistics on κ-Minkowski spacetime is specifically deformed. Truncating of the model was necessary to be able to compute any relevant physical quantity, such as self-energy of our complex scalar field ϕ. Above properties are very welcome, however we have to stress that by truncating κ-deformed action to the linear order in deformation parameter a we have lost nonperturbative quantum effects like celebrated UV/IR mixing [43], which, amongst other, connects NC field theories with Holography via UV and IR cutoffs, in a model independent way [55]. Resummation of expanded action could in principle restore nonperturbative character of the model, thus producing UV/IR mixing. Those are general properties of most of NCGFT expanded/resummed in terms of the noncommutative deformation parameter. Holography and UV/IR mixing are in the literature known as possible windows to quantum gravity phenomena.

On the same line of reasoning we should take into account the harmonic oscillator term, despite that it is well known that such a term represents translation invariance breaking [53, 54]. Our approach generally represents *hybrid approach* modeling between standard quantum field theory and NCQFT on κ-Minkowski spacetime involving κ-deformed momentum conservation law [17, 19, 20, 56, 57].

The results are next discussed in the framework of two-point Green's function for low, middle, and for Planck scale energy regimes. We have found semiclassical behavior of the first order quantum effects, and, as a consequence, the mass term of the scalar field is shifted and these shifts are very much different in low and high propagation energy regimes, respectively. Thus the dispersion relations are κ-deformed and we have found genuine birefringence effects, [58, 59], of the massive scalar field mode, at first order in deformation parameter a. This is similar to the fermion field birefringence in truncated Moyal $\star$-product theories [59].

The above describes the main results of this paper, which could be of physical importance, for example for the κ-NC scalar field (Higgs) and its deformed propagation, as well as to quantum gravity.

In the first section, we give some mathematical preliminaries including the Hopf algebra structure of κ-deformed Minkowski spacetime and star products. In the second section, we introduce, the hermitian realization and the star product $\star_h$ corresponding to this realization. The modified κ-deformed scalar field action based on the above notions is introduced next, and the equations of motion are derived. The corresponding currents are conserved. The properties of our new action like the interpretation of κ-deformed action in terms of the deformed theory (with undeformed fields) on ordinary Minkowski spacetime are discussed in the last section. That include the κ-deformed Feynman rules and the field propagation, via computation of two-point Green's function, within the proposed model.

2. Mathematical preliminaries of κ-deformed Minkowski spacetime

2.1 Hopf algebra and star product

We are considering a κ-deformation of a Minkowski spacetime whose symmetry has an undeformed Lorentz sector and whose noncommutative coordinates $\hat{x}_\mu$, $(\mu = 0, 1, ..., n-1)$, close a Lie algebra with the Lorentz generators $M_{\mu\nu},\ (M_{\mu\nu}=-M_{\nu\mu})$,

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)\,, \tag{2.1}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda}\,, \tag{2.2}$$

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda} - i\,(a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda})\,, \tag{2.3}$$

where the deformation parameter a_μ is a constant Lorentz vector and $\eta_{\mu\nu} = diag(-1, 1, \cdot\cdot\cdot, 1)$ defines the metric in this spacetime. The quantity $a^2 = a_\mu a^\mu$ is Lorentz invariant having a dimension of inverse mass squared, $a^2 \equiv \frac{1}{\kappa^2}$. The above algebra has all the Jacobi identities satisfied, thus forming a Lie algebra with the property that in the limit $a_\mu \to 0$, the commutative spacetime with the usual action of the Lorentz algebra is recovered. Throughout the paper we shall work in units $\hbar = c = 1$.

The symmetry of the deformed spacetime (2.1) is assumed to be described by an undeformed Poincaré algebra. Thus, in addition to Lorentz generators $M_{\mu\nu}$, we also introduce momenta p_μ which transform as vectors under the Lorentz algebra,

$$[p_\mu, p_\nu] = 0\,, \tag{2.4}$$

$$[M_{\mu\nu}, p_\lambda] = \eta_{\nu\lambda}\, p_\mu - \eta_{\mu\lambda}\, p_\nu\,. \tag{2.5}$$

For convenience we refer to algebra (2.1)-(2.5) as a deformed special relativity algebra since its different realizations lead to different special relativity models with different physics encoded in deformed dispersion relations resulting from such theories. This algebra, however, does not fix the commutation relation between p_μ and $\hat{x}_\nu$. In fact, there are infinitely many possibilities for the commutation relation between p_μ and $\hat{x}_\nu$, all of which are consistent with the algebra (2.1)-(2.5) in the sense that Jacobi identities are satisfied between all generators of the algebra. In this way, we have an extended algebra, which includes generators $M_{\mu\nu},\ p_\mu$ and $\hat{x}_\lambda$ and has Jacobi identities satisfied for all combinations of the generators. Particularly, the algebra generated by p_μ and $\hat{x}_\nu$ is a deformed Heisenberg-Weyl algebra that can generally be written in the form

$$[p_\mu, \hat{x}_\nu] = -i\Phi_{\mu\nu}(p)\,, \tag{2.6}$$

where $\Phi_{\mu\nu}(p)$ are functions of the generators p_μ that are required to be consistent with the Jacobi identities and satisfy the boundary conditions $\Phi_{\mu\nu}(0) = \eta_{\mu\nu}$, but are otherwise arbitrary. This condition reflects the requirement that deformed NC space, together with the corresponding coordinates, reduces to ordinary commutative space in the limiting case of vanishing deformation parameter, $a_\mu \to 0$.

The momentum $p_\mu = -i\hat{\partial}_\mu$, expressed in terms of deformed derivative $\hat{\partial}_\mu$, can be realized in a natural way [50] by adopting the identification between deformed and undeformed derivatives, $\hat{\partial}_\mu \equiv \partial_\mu$, implying $p_\mu = -i\partial_\mu$. The deformed algebra (2.1)-(2.5) then admits a wide class of realizations

$$\hat{x}_\mu = x^\alpha \Phi_{\alpha\mu}(p)\,, \tag{2.7}$$

$$M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu\,, \tag{2.8}$$

in terms of undeformed Heisenberg algebra

$$[x_\mu, x_\nu] = 0, \qquad [\partial_\mu, \partial_\nu] = 0, \qquad [\partial_\mu, x_\nu] = \eta_{\mu\nu} \tag{2.9}$$

and analytic function $\Phi_{\alpha\mu}(p)$ satisfying the boundary conditions $\Phi_{\alpha\mu}(0) = \eta_{\alpha\mu}$. At this point ∂_μ's represent abstract generators of the undeformed Heisenberg algebra. Later on they shall acquire concrete representation in terms of differential operators, $\partial/\partial x^\mu$. By taking this prescription, the deformed algebra (2.1)-(2.5) is then automatically satisfied, as well as all Jacobi identities among $\hat{x}_\mu$, $M_{\mu\nu}$, and p_μ.

Some comments are in order. Algebra (2.1)-(2.5), as it is defined, is not closed. Closing it by the relations (2.6), we obtain extended algebra containing deformed Heisenberg-Weyl subalgebra. The realization (2.7) indicates that deformed (2.1), (2.4), (2.6) and undeformed (2.9) Heisenberg-Weyl algebras are isomorphic at the level of vector spaces. In the same way one can show that deformed and undeformed extended algebras are isomorphic in the same sence. It is of no importance which concrete formula, (2.7), is used for closing it. That was shown under some general setting in [60], where extended algebra (2.1)-(2.5) has been defined as crossed (smash) product algebra. In Ref [60] authors started from the coproduct (Hopf algebra + module algebra) and then determined crossed commutation relation of extended algebra (2.1)-(2.6). Our route is just opposite: we close the algebra by crossed commutation relation and then accordingly determine the coproduct.

In this paper we shall strictly work with one particular type of realization (as well as with its hermitian variant, see Eq.(3.7) below), namely with the realization [51] of the form

$$\hat{x}_\mu = x_\mu \sqrt{1 + a^2 p^2} - i M_{\mu\nu} a^\nu \,. \tag{2.10}$$

We refer to this type of realization as Maggiore-type of realization of the algebra (2.1), since it leads to phase space noncommutativity analyzed for the first time by Maggiore [61, 62]. This particular kind of phase space noncommutativity is directly related to a generalized uncertainty principle appearing in the contexts of string theory and quantum gravity. It is also considered in [21]. The realization (2.10) belongs to the class of realizations (2.7) and is thus consistent with algebra (2.1)-(2.5). With these particular settings the deformed Heisenberg-Weyl algebra (2.6) receives the following form

$$[p_\mu, \hat{x}_\nu] = -i\eta_{\mu\nu} \left(ap + \sqrt{1 + a^2 p^2} \right) + i a_\mu p_\nu \,. \tag{2.11}$$

As mentioned above, this particular type of phase space noncommutativity leads to uncertainty relations of the form [61, 62, 63]

$$\triangle x_\mu \;\geq\; \frac{\hbar}{\triangle p_\mu} + \alpha G \triangle p_\mu \,, \tag{2.12}$$

(α is a constant and G is gravitational constant) that have been obtained from the study of string collisions at Planckian energies, i.e. so called gravity collapse of string [63], thus manifesting its dynamical origin. The same generalized uncertainty principle emerges from considerations related to quantum gravity [61, 62].

The operator appearing in (2.11) in parenthesis appears to play a very important role in κ-deformed spaces in general. Therefore, we give a special label to it,

$$Z^{-1} = ap + \sqrt{1 + a^2 p^2}\,. \tag{2.13}$$

This operator, among others, has the following properties that define a universal shift operator [51]:

$$[Z^{-1}, \hat{x}_\mu] = -ia_\mu Z^{-1}, \qquad [Z, p_\mu] = 0\,, \tag{2.14}$$

with Z being its inverse, $Z = \frac{1}{Z^{-1}}$. The operator $Z = \frac{1}{Z^{-1}}$ and its properties are described in great detail in Refs. [51, 52].

It can certainly be expected that a deformation of the spacetime structure will affect the algebra of physical fields, leading to a modification of the multiplication in the corresponding universal enveloping algebra. Specifically it means that a spacetime deformation requires one to replace the usual pointwise multiplication by a deformed product or star product, which will finally have its consequences in physics, particularly it will reflect itself in the way in which one should construct the field theoretic action. This modified multiplication, i.e. star product will obviously depend on the particular realization (2.7), which in turn is characterized by an analytic function $\Phi_{\mu\nu}$ of generators p_μ. In this paper it will be understood that noncommutative coordinates $\hat{x}$, appearing in all subsequent course of exposition, will be represented either by the specific realization determined by (2.7) or by its hermitian variant, (3.7) in the next section.

In preparation for our further analysis it is useful to introduce a few notions. First, we denote by $\mathcal{A}$ an algebra constituted of physical fields $\phi(x)$ in commutative coordinates x_μ. Since physical fields $\phi(x)$ are formed out of polynomials in x_μ, the algebra $\mathcal{A}$ is an universal enveloping algebra generated by commuting coordinates x_μ. The algebra $\mathcal{A}$ can also be understood as a module of the deformed Weyl algebra, which is generated by $\hat{x}_\mu$ and ∂_μ, $\mu = 0, 1, ..., n-1$, and allows for infinite series in ∂_μ. In a similar way as for commutative fields $\phi(x)$, the noncommutative fields $\hat{\phi}(\hat{x})$ are built of polynomials in $\hat{x}_\mu$, and thus belong to a universal enveloping algebra $\hat{\mathcal{A}}_\kappa$, generated by noncommutative coordinates $\hat{x}_\mu$. Although the universal enveloping algebras $\mathcal{A}$ and $\hat{\mathcal{A}}_\kappa$ are not isomorphic to each other, there is however a convenient circumstance that for any given realization described by the function $\Phi_{\mu\nu}$ in (2.7), there exists a unique map and even an isomorphism between $\mathcal{A}$ and $\hat{\mathcal{A}}_\kappa$, at the level of vector spaces. Both enveloping algebras, $\mathcal{A}$ and $\hat{\mathcal{A}}_\kappa$, can be shown to have a Hopf algebra structure [6, 8, 64], with the later being obtained from the former by means of a twist deformation [29, 65, 66, 67], satisfying counit and cocycle condition [64]. The full Hopf algebra description of the theory includes algebraic as well as coalgebraic part. However, for theories in commutative spacetime coalgebraic aspects of the symmetry are trivial and basically already contained within the algebraic aspects of the symmetry transformations. It is for this reason that it suffices there to describe symmetries only by specifying their Lie algebra structure. On the contrary, for theories in noncommutative spacetime, the coalgebraic aspects of the symmetry are generally not trivial, so complete characterization of symmetries requires a description in terms of Hopf algebra. The Hopf algebra is a quantum symmetry where the issue of conserved charges and currents is still subject of research, and the Noether theorem generally is still not established. However, to obtain sensible physical picture out of the Hopf algebra prescription of our theory we will need to do certain compromises, most important towards the Noether theorem and momentum conservation. We shall elaborate on this issue in the next subsections. Before doing that, it still remains to introduce the notion of the star product.

2.2 Coalgebra and star product

The star product is introduced in the following way: first let us introduce the unit element $1 \in \mathcal{A}$ and define the action of Poincaré generators ∂_μ and $M_{\mu\nu}$ on 1 as

$$\partial_\mu \triangleright 1 = 0, \qquad M_{\mu\nu} \triangleright 1 = 0\,, \tag{2.15}$$

where $\mathcal{A}$ is understood as a module for the enveloping algebra $\mathcal{U}(\mathfrak{so}(3,1))$ of Poincaré algebra. In other words, $\mathcal{A}$ is considered to be $\mathcal{U}(\mathfrak{so}(3,1))$-module. Suppose we have two associations of the form $\hat{\phi}(\hat{x}) \triangleright 1 = \phi(x)$ and $\hat{\psi}(\hat{x}) \triangleright 1 = \psi(x)$ for two noncommutative functions, $\hat{\phi}(\hat{x})$ and $\hat{\psi}(\hat{x})$. Then the star product is defined by

$$\begin{aligned} \phi(x) \star \psi(x) &= \hat{\phi}(\hat{x})\hat{\psi}(\hat{x}) \triangleright 1 \\ &= \hat{\phi}(\hat{x}) \triangleright (\hat{\psi}(\hat{x}) \triangleright 1) \;=\; \hat{\phi}(\hat{x}) \triangleright \psi(x)\,, \end{aligned} \tag{2.16}$$

where it is understood that $\hat{x}$ is given either by (2.10) or by its hermitian variant, (3.7). In this situation $\mathcal{A}$, considered as a vector space, together with the star product (2.16) constitutes a noncommutative algebra, which we denote by $\mathcal{A}_\kappa$. Unlike the algebra $\mathcal{A}$, the algebra $\mathcal{A}_\kappa$ is isomorphic to the enveloping algebra $\hat{\mathcal{A}}_\kappa$ in NC coordinates. Note that the commutator (2.1) can be written in terms of ordinary coordinates and $\star$-product (2.16) as

$$[x_\mu, x_\nu]_\star = x_\mu \star x_\nu - x_\nu \star x_\mu = i(a_\mu x_\nu - a_\nu x_\mu)\,. \tag{2.17}$$

In the familiar context of theories on commutative spacetime, we describe a symmetry as a transformation of the coordinates that leaves the action of the theory invariant. We keep this notion also in case of noncommutative spacetime. The symmetry underlying κ-deformed Minkowski space, characterized by the commutation relations (2.1), is the deformed Poincaré symmetry which can most conveniently be described in terms of Hopf algebras. As was seen in relations (2.2), (2.4) and (2.5), the algebraic sector of this deformed symmetry is the same as that of the undeformed Poincaré algebra. However, the action of the Poincaré generators on the deformed Minkowski space is modified in such a way, that the whole deformation is contained in the coalgebraic sector. This means that the Leibniz rules, which describe the action of the generators $M_{\mu\nu}$ and p_μ on a product of fields, will no more have the standard form, but instead will be deformed and will depend on the $\Phi_{\mu\nu}$ realization. The Hopf algebra structure describes the properties of the generators of a deformed Poincaré symmetry. Its algebraic sector is determined by the relations (2.2), (2.4) and (2.5). On the other hand, the coalgebraic sector is determined by the coproducts for translation ($p_\mu = -i\partial_\mu$), rotation and boost generators ($M_{\mu\nu}$) [50, 51],

$$\triangle\partial_\mu = \partial_\mu \otimes Z^{-1} + \mathbf{1} \otimes \partial_\mu + ia_\mu(\partial_\lambda Z) \otimes \partial^\lambda - \frac{ia_\mu}{2}\Box\, Z \otimes ia\partial\,, \tag{2.18}$$

$$\begin{aligned}\triangle M_{\mu\nu} &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ &\quad + ia_\mu \left(\partial^\lambda - \frac{ia^\lambda}{2}\Box\right) Z \otimes M_{\lambda\nu} - ia_\nu \left(\partial^\lambda - \frac{ia^\lambda}{2}\Box\right) Z \otimes M_{\lambda\mu}\,, \end{aligned}\tag{2.19}$$

where $\otimes$ denotes, as usual, the tensor product. In the above expressions, Z is the shift operator, determined with (2.13), and which itself has a simple expression for coproduct,

$$\triangle Z = Z \otimes Z\,. \tag{2.20}$$

The operator $\Box$ is a deformed d'Alambertian operator [21, 49, 51],

$$\Box = \frac{2}{a^2}(1 - \sqrt{1 - a^2\partial^2})\,, \tag{2.21}$$

which in the limit $a \to 0$ acquires the standard form, $\Box \to \partial^2$, valid in undeformed Minkowski space.

The Hopf algebra in question also has well defined counits and antipodes. The antipodes for the generators of κ-Poincaré Hopf algebra are given by

$$S(\partial_\mu) = \left(-\partial_\mu + ia_\mu\partial^2 + \frac{1}{2}a_\mu(a\partial)\Box\right) Z\,, \tag{2.22}$$

$$S(M_{\mu\nu}) = -M_{\mu\nu} + ia_\mu \left(\partial_\alpha - \frac{ia_\alpha}{2}\Box\right) M_{\alpha\nu} - ia_\nu \left(\partial_\alpha - \frac{ia_\alpha}{2}\Box\right) M_{\alpha\mu}\,, \tag{2.23}$$

while the counits remain trivial. In the above relations, the operator Z is given by

$$Z \equiv \frac{1}{Z^{-1}} = \frac{1}{-ia\partial + \sqrt{1 - a^2\partial^2}}\,, \tag{2.24}$$

in accordance with (2.13) and $\Box$ is given in (2.21).

Since we are interested in perturbative expansion of the field theoretic action, for later convenience we give $\triangle\partial_\mu$ in form of a series expansion up to second order in the deformation parameter a,

$$\begin{aligned}\triangle\partial_\mu &= \partial_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \partial_\mu - i\partial_\mu \otimes a\partial + ia_\mu\partial_\alpha \otimes \partial^\alpha \\ &\quad - \frac{1}{2}a^2\partial_\mu \otimes \partial^2 - a_\mu(a\partial)\partial_\alpha \otimes \partial^\alpha + \frac{1}{2}a_\mu\partial^2 \otimes a\partial + \mathcal{O}(a^3)\,. \end{aligned}\tag{2.25}$$

Once we have the coproduct (2.18), we can straightforwardly construct a star product between two arbitrary fields f and g of commuting coordinates [49, 50]. For the noncommutative spacetime (2.1), the star product has the following form

$$(f \star g)(x) = \lim_{\substack{u\to x\\ y\to x}} \mathcal{M}\left(e^{x^\mu(\triangle-\triangle_0)\partial_\mu} f(u) \otimes g(y)\right)\,, \tag{2.26}$$

where $\triangle_0\partial_\mu = \partial_\mu \otimes 1 + 1 \otimes \partial_\mu$, $\triangle(\partial_\mu)$ is given in (2.18) and $\mathcal{M}$ is the multiplication map in the undeformed Hopf algebra, namely, $\mathcal{M}(f(x) \otimes g(x)) = f(x)g(x)$ [64]. From

(2.26) we see that star product only depends on the coproduct for translation generators. Its form does not depend on the $\Phi_{\mu\nu}$ realization in (2.7). However, since coproducts depend on the $\Phi_{\mu\nu}$ realization, so does the star product according to (2.26), in an implicit form. At this point we emphasize here once again that in this paper we are doing analysis based on the specific realization (2.10) and its hermitian variant, (3.7) in the next section. The coproducts (2.18) and (2.19) are written for and correspond to this particular type of realization. One can check that the star product (2.26) with the coproduct (2.18) is associative.

3. Modified κ-deformed scalar field action

3.1 Nonhermitian realization of the NC ϕ^4 action

In this subsection, we construct an interacting scalar field theory on noncommutative space-time whose short distance geometry is governed by the κ-deformed symplectic structure (2.1). In particular, we are interested in a field theoretic action describing the dynamics of a massive scalar field, i.e. generalizing the celebrated Grosse-Wulkenhaar action in dimension n,

$$\begin{aligned} S_n[\phi] &= \int d^n x\ \mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi) \\ &= \frac{1}{2}\int d^n x\ (\partial_\mu \phi) \star (\partial^\mu \phi) + \frac{m^2}{2}\int d^n x\ \phi \star \phi + \frac{\xi^2}{2}\int d^n x\ x_\mu \phi \star x^\mu \phi \\ &+ \frac{\lambda}{4!}\int d^n x\ \phi \star \phi \star \phi \star \phi\,. \end{aligned} \tag{3.1}$$

Note that ϕ^4 interaction term is accompanied by an additional harmonic term of the Grosse-Wulkenhaar type, along with the standard kinetic and mass terms. Due to the very definition of the $\star$-product (2.26), all terms in the action (3.1) will be κ-deformed. In the case of nonhermitian realization (2.10) alone, the scalar field ϕ is, up to first order in the deformation parameter a, governed by the action

$$\begin{aligned} S_n[\phi] &= \frac{1}{2}\int d^n x\ \Big[(\partial_\mu \phi)(\partial^\mu \phi) + m^2\phi^2 + \xi^2 x^2 \phi^2 + \frac{\lambda}{12}\phi^4\Big] \\ &+ \frac{1}{2}a_\mu \int d^n x\ \Big\{(n-1)\xi^2 x^\mu \phi^2 \\ &+ (\eta^{\alpha\beta}x^\mu - \eta^{\mu\beta}x^\alpha) \\ &\times \Big[\phi x_\alpha(\partial_\beta\phi) + (m^2 + \xi^2 x^2 + \frac{\lambda}{3}\phi^2)(\partial_\alpha\phi)(\partial_\beta\phi) + (\partial_\alpha\partial_\gamma\phi)(\partial_\beta\partial^\gamma\phi)\Big]\Big\}\,. \end{aligned} \tag{3.2}$$

However, in this case the scalar field cannot be properly defined and it is not clear at all if it is complex or real. Next, besides that various terms from (3.1) attain an explicit x-dependence after the expansion in a, it is easy to note the lack of typical Moyal $\star$-product trace property $\int f \star g = \int f \cdot g$ for star product (2.26). This makes the above nonhermitian models uncontrollable, thus less attractive, since it would be very difficult to find some physical meaning out of them. Generally, to have a scalar field defined in the right way and to be able to say whether it is real or complex, it is necessary to introduce an involution operation $\dagger$ corresponding to adjoint or hermitian conjugation operation $\ddagger$, which in turn requires to consider and work with a hermitian, instead of the nonhermitian realization. The problems just described can thus be properly resolved only after we replace the nonhermitian realization (2.10) with a hermitian one and introduce the notion of an antipode in order to properly define an adjoint or hermitian conjugation operation $\ddagger$. In order to proceed, we introduce scalar product $(\cdot,\cdot)$ on the algebra $\mathcal{A}_\kappa$ as [1]

$$(\psi, \phi)_\kappa = \int d^n x \; \psi^\dagger \star_h \phi\,, \tag{3.3}$$

where $\dagger$ is the involution with respect to this new scalar product, defined in terms of the star product $\star_h$, whose explicit form is going to be given later on. Assuming that functions in the algebra $\mathcal{A}_\kappa$ have the Fourier expansion

$$\phi(x) = \int d^n p \; \tilde{\phi}(p) \; e^{ipx}, \tag{3.4}$$

than the action of the operation $\dagger$ on plane waves is specified as follows

$$\left(e^{ipx}\right)^\dagger = e^{iS(p)x}\,. \tag{3.5}$$

Here $S(p) = -iS(\partial)$ is the antipode (2.22) and $\dagger$ is the involution defined on $\mathcal{A}_\kappa$. Note that reality of the field ϕ can be defined in a more than one way, depending on the conjugation operation demanded: $\phi^\dagger = \phi$ or $\phi^* = \phi$. When we use term real field, we have in mind first case, i.e. the one with $\dagger$. For further details see reference [52].

For the ordinary Moyal case instead of (3.5), we have $(e^{ipx})^\dagger = e^{-ipx}$, as in the commutative case. In order to clarify particle-antiparticle plane waves, one needs to modify the interpretation of (3.5) accordingly. However, this is not trivial for $a \neq 0$. In our approach the algebraic sector of the Poincare algebra, i.e. commutation relations (2.2) are undeformed, thus corresponding Casimirs are also undeformed, and the dispersion relation remains the same, $p^2 = m^2$. From the antipodes, (2.22), for the translation generators of κ-Poincaré Hopf algebra, it follows $S^2(p) = p^2 = m^2$, in agreement with previous conclusion regarding dispersion [52].

[1] To avoid further confusion, we discuss the issue of different notation for different operations and their properties; that is, the usual scalar product $(f,g) = \int d^n x f^* \cdot g$ induces usual hermitian conjugation operation $\dagger$. Here $*$ represents usual complex conjugation; scalar product $(\psi,\phi)_\kappa = \int d^n x\, \psi^\dagger \star_h \phi$ on algebra $\mathcal{A}_\kappa$, where $\dagger$ is corresponding involution (generalized complex conjugation), induces required hermitian conjugation operation $\ddagger$. Note that we denoted by the same symbol two different notions, that is $\dagger$ designates ordinary hermitian conjugation operation $\partial^\dagger = -\partial$ and involution used to define scalar product in $\mathcal{A}_\kappa$. From the context it should be clear which one we are using, namely when $\dagger$ acts on function, it is involution, however when it is applied on the operator it is ordinary hermitian conjugation. It is understood that $\partial^\ddagger = (S(\partial))^\dagger = -\partial$, $M^\ddagger_{\mu\nu} = (S(M_{\mu\nu}))^\dagger$, or compactly written $A^\ddagger = (S(A))^\dagger$, for any generator A of the κ-deformed algebra. It has to be stressed that $\dagger$, in a previous sentence, means involution with respect to the deformed scalar product (3.3). Properties of above operations; generalized trace property and behavior with respect to integration by part, are given in [52].

3.2 Hermitian realization of the NC ϕ^4 action

In order to obtain the physical meaning of the NC ϕ^4 field theory, we have to introduce a complex scalar field ϕ with accompanying notion of hermitian conjugation operation, as explained above and proceed further with the construction of the hermitian theory.

In order to obtain an action that is hermitian, we are necessarily forced to work with a hermitian realization represented by an operator $\hat{x}^h_\mu$, having the property $(\hat{x}^h_\mu)^\dagger = \hat{x}^h_\mu$. This fully hermitian operator $\hat{x}^h_\mu$ can be constructed out of the operator (2.10) as

$$\hat{x}^h_\mu = \frac{1}{2}(\hat{x}_\mu + \hat{x}^\dagger_\mu) = (\hat{x}^h_\mu)^\dagger \,, \tag{3.6}$$

which results in († here means usual hermitian conjugation operation, $x^\dagger_\mu = x_\mu,\ \partial^\dagger_\mu = -\partial_\mu$)

$$\hat{x}^h_\mu = x_\mu\sqrt{1+a^2p^2} - iM_{\mu\nu}a^\nu - i\frac{a^2}{2}\frac{1}{\sqrt{1+a^2p^2}}p_\mu \,. \tag{3.7}$$

The change of realization (2.10) into (3.7) in accordance with (2.16), modifies the form of the star product and we obtain a the new star product denoted as $\star_h$:

$$\begin{aligned} \phi(x)\star_h\psi(x) &= \hat{\phi}(\hat{x}^h)\hat{\psi}(\hat{x}^h)\triangleright 1 \\ &= \hat{\phi}(\hat{x}^h)\triangleright(\hat{\psi}(\hat{x}^h)\triangleright 1) \;=\; \hat{\phi}(\hat{x}^h)\triangleright\psi(x)\,, \end{aligned} \tag{3.8}$$

with $\hat{x}^h$ being given by (3.7). Thus, we are forced to replace the star product (2.26) with a new one of the following form [52]:

$$(f\star_h g)(x) = \lim_{\substack{u\to x\\ y\to x}} \mathcal{M}\left(e^{x^\mu(\triangle-\triangle_0)\partial_\mu}\sqrt[4]{\frac{1-a^2\triangle(\partial^2)}{(1-a^2\ \partial^2\otimes 1)\ (1-a^2\ 1\otimes\partial^2)}}\ f(u)\otimes g(y)\right)\,, \tag{3.9}$$

where it is understood that the coproduct $\triangle(\partial_\mu)$, Eq.(2.18), is a homomorphism, i.e. $\triangle(\partial^2) = \triangle(\partial_\mu)\triangle(\partial^\mu)$. In this way, the nonhermitian version of the star product (2.26) is replaced by the above hermitian one, (3.9).

The star product $\star_h$, (3.9), is associative in the same sense as the star product (2.26). However the star product $\star_h$, contrary to the star product (2.26), has the same property as the usual Moyal-Weyl product [52]:

$$\int d^nx\ \phi^\dagger\star_h\psi = \int d^nx\ \phi^*\cdot\psi\,, \tag{3.10}$$

where the asterisk $*$ denotes usual complex conjugation.

The above results – the new $\star_h$-product (3.9), and the identity (3.10) – embrace a very nice and important property: *the integral measure problems are avoided.* Namely, by usage of the $\star_h$-product, corresponding to the hermitian realization of the κ-Minkowski spacetime, the measure function is naturally absorbed within the new $\star_h$-product (3.9).

Therefore, we replace the action (3.1) by

$$
\begin{aligned}
S_n[\phi] &= \int d^n x\ \mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi) \\
&= \int d^n x\ (\partial_\mu\phi)^\dagger \star_h (\partial^\mu\phi) + m^2 \int d^n x\ \phi^\dagger \star_h \phi + \xi^2 \int d^n x\ x_\mu \phi^\dagger \star_h x^\mu \phi \\
&+ \frac{\lambda}{4}\int d^n x\ \frac{1}{2}(\phi^\dagger \star_h \phi^\dagger \star_h \phi \star_h \phi + \phi^\dagger \star_h \phi \star_h \phi^\dagger \star_h \phi)\,.
\end{aligned} \tag{3.11}
$$

In the above action, the interaction ϕ^4 term should in fact incorporate six terms corresponding to all possible permutations of fields ϕ and $\phi^\dagger$. However, due to integral property (3.10) of the star product (3.9), these six different permutations can be reduced to only two mutually nonequivalent terms, $\phi^\dagger \star_h \phi^\dagger \star_h \phi \star_h \phi$ and $\phi^\dagger \star_h \phi \star_h \phi^\dagger \star_h \phi$.

When expanded up to first order in the deformation parameter a, the action (3.11) after rearrangements, including integration by parts, receives the following form

$$
\begin{aligned}
S_n[\phi] &= \int d^n x\ \Big[(\partial_\mu\phi^*)(\partial^\mu\phi) + (m^2 + \xi^2 x^2)\phi^*\phi + \frac{\lambda}{4}(\phi^*\phi)^2\Big] \\
&+ i\frac{\lambda}{4}\int d^n x \Big[a_\mu\ x^\mu\Big(\phi^{*2}(\partial_\nu\phi)\partial^\nu\phi - \phi^2(\partial_\nu\phi^*)\partial^\nu\phi^*\Big) \\
&+ a_\nu\ x^\mu\Big(\phi^2(\partial_\mu\phi^*)\partial^\nu\phi^* - \phi^{*2}(\partial_\mu\phi)\partial^\nu\phi\Big) \\
&+ \frac{1}{2}a_\nu x^\mu\ \phi^*\phi\Big((\partial_\mu\phi^*)\partial^\nu\phi - (\partial_\mu\phi)\partial^\nu\phi^*\Big)\Big] + \mathcal{O}(a^2)\,.
\end{aligned} \tag{3.12}
$$

Note that the oscillator term proportional to ξ^2 attain no correction in the deformation parameter a. These two features of our model separate completely in above action.

At the end of this subsection note that the Hopf algebra, yielding (3.11), is a twisted symmetry algebra, where existence/conservation of charges and currents are still subject of research. However the action (3.12), obtained by expansion of (3.11) up to first order in the deformation parameter a, is invariant under the internal symmetry transformations:

$$
\begin{pmatrix} \phi \\ \phi^* \end{pmatrix} \to \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix} \begin{pmatrix} \phi \\ \phi^* \end{pmatrix}, \tag{3.13}
$$

thus the corresponding Noether current should be conserved.

3.3 Equations of motion and Noether currents of internal symmetry

We proceed further by evaluating the equations of motion for the fields ϕ and ϕ^*:

$$
\begin{aligned}
\partial_\mu\partial^\mu\phi - (m^2 + \xi^2 x^2)\phi &= \frac{\lambda}{6}\Big[\phi^*\phi^2 + ia_\mu x^\mu\Big(\phi^2\partial_\nu\partial^\nu\phi^* + \phi^*(\partial_\nu\phi)\partial^\nu\phi + \phi(\partial_\nu\phi)\partial^\nu\phi^*\Big) \\
&- ia^\mu x^\nu\Big(\phi^*(\partial_\nu\phi)\partial_\mu\phi + \phi(\partial_\nu\phi)\partial_\mu\phi^* + \phi^2\partial_\mu\partial_\nu\phi^* + \phi(\partial_\mu\phi)\partial_\nu\phi^*\Big) \\
&+ \frac{i}{4}(1-n)a^\mu\phi^*\phi\partial_\mu\phi + \frac{i}{2}(1-n)a^\mu\phi^2\partial_\mu\phi^*\Big]\,,
\end{aligned} \tag{3.14}
$$

where $\phi = 0$ is the trivial solution of the above equation, as it should be. The equation of motion for ϕ^* can be obtained from (3.14).

Next, we present Noether currents derived from the Lagrangian densities (3.12):

$$\begin{aligned} j^\mu(x) &= i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\phi - i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\phi^*\,, \qquad (3.15)\\ \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} &= \frac{1}{2}\partial^\mu\phi^* + \frac{i\lambda}{4}\Big[\phi^{*2}(2a_\nu x^\nu\partial^\mu - a^\nu x^\mu\partial_\nu - a^\mu x^\nu\partial_\nu)\phi + \frac{1}{2}\phi^*\phi(a^\mu x^\nu\partial_\nu - a^\nu x^\mu\partial_\nu)\phi^*\Big]\,,\\ \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)} &= \frac{1}{2}\partial^\mu\phi - \frac{i\lambda}{4}\Big[\phi^{2}(2a_\nu x^\nu\partial^\mu - a^\nu x^\mu\partial_\nu - a^\mu x^\nu\partial_\nu)\phi^* + \frac{1}{2}\phi^*\phi(a^\mu x^\nu\partial_\nu - a^\nu x^\mu\partial_\nu)\phi\Big]\,. \end{aligned}$$

The above current (3.15), is conserved; that is

$$\partial_\mu j^\mu(x) = 0\,, \qquad (3.16)$$

as it should be due to the invariance of the Lagrangian (3.12) under the internal symmetry transformations (3.13). This is showed by straightforward computations from (3.11)-(3.16).

4. Quantum properties of the model

4.1 Feynman rules

Even though the S matrix LSZ formalism, including Wick theorem, is not quite clearly defined on κ-Minkowski noncommutative spacetime, we continue bona fide towards the research of the quantum properties of the model defined by the action (3.12). To do that, we first derive relevant Feynman rules and than compute the tadpole diagram contributions to the 2-point Green's function of our model.

Due to the κ-deformation of our theory, the statistics of particles is twisted, so that we are generally no more dealing with pure bosons. We are in fact dealing with *something* whose statistics is governed by the statistics flip operator [68, 69, 70, 71] and the quasitriangular structure (universal R-matrix) on the corresponding quantum group [64, 72, 73, 74]. It would be interesting to investigate these mutual relations more thoroughly, but at the surface level, we can argue that it is possible to pick up the basic characteristics of the twisted statistics by using the nonabelian momentum addition law [18, 19, 20, 41, 42, 56, 57, 75, 76]. It can be seen that the accordingly induced deformation of the δ-function (arising from the implementation of the nonabelian momentum addition/subtraction rule) yields the usual δ-function multiplied by a certain statistical factor, which could have its origin in κ-modified statistics. When we speak about deformed statistics, we have in mind less rigid notion of statistics as applied to the symmetry properties of the states, where multiparticle change of 4-momenta may change the state's symmetry properties.

To obtain the Feynman rules in momentum space, we are suggesting to use the following line of reasoning, which we shall further on call *hybrid approach*.

(A) We use the methods of standard QFT and treat the modifications in action (3.12) as a perturbation. In doing this, we obtain propagators and Feynman rules for vertices.

(B) We know that the statistics of particles is twisted and that it has to be implemented into the formalism. Thus we require that ordinary addition/subtraction rule induces addition rule for twisted statistics on κ-Minkowski spacetime. This in momentum space means

$$\sum_i k_i^\mu \;\to\; \sum_{\oplus i} k_i^\mu \quad \& \quad \sum_i k_i^\mu - \sum_j p_j^\mu \;\to\; \sum_{\oplus i} k_i^\mu \ominus \sum_{\oplus j} p_j^\mu\,, \qquad (4.1)$$

where induced *direct* addition/subtraction rules are going to be defined in the Subsection 4.1.2, for simplest cases of two to four momenta. Note that the associativity for direct summ $\oplus$ is satisfied due to the associativity of the star product (3.9). We proceed in two steps:
(1) Following above arguments we implement the induced conservation law within the delta functions in the Feynman rule, and
(2) whenever needed, we use the modified/deformed conservation law along the course of evaluation of the Feynman diagrams.

4.1.1 Feynman rules (A): standard momentum addition law

From now on we continue to work with Euclidean metric. Note that in transition from Minkowski to Euclidean signature we are using transition rules: $a^M = (a_0^M, a_i^M) \longrightarrow a^E = (a_i^E, a_n^E)$, where $a_n^E = ia_0^M$, and similary for any n-vector. Thus scalar product is defined as $a^E k^E = a_\mu^E k_\mu^E = a_i^E k_i^E + a_n^E k_n^E = -a_0^M k_0^M + a_i^M k_i^M = a^M k^M$. In subsequent consideration we drop the M and E superscripts, but it is understood that we work with Euclidean (E) quantities. As the quadratic part of the action (3.12) in κ-space is modified by the harmonic oscillator term, the propagator in momentum space is also going to be modified,

$$\begin{aligned} \tilde{\Gamma}^\xi(k_1,k_2) &= -i\left(\frac{\delta^2 S[\tilde{\phi}]}{\delta\tilde{\phi}^*(k_1)\delta\tilde{\phi}(k_2)}\right)_{\tilde{\phi}=\tilde{\phi}^*=0} \\ &= -i(2\pi)^n\left[(k_{1\mu}k_{2\mu}+m^2)+\xi^2\partial^2_{k_1}\right]\delta^{(n)}(k_1-k_2)\,, \end{aligned} \tag{4.2}$$

with all fields ϕ, ϕ^* having the same nonzero mass. The free propagator is than symbolically written as

$$\tilde{G}^\xi(k_1,k_2) = \frac{(2\pi)^n}{\tilde{\Gamma}^\xi(k_1,k_2)} = i\bigg((k_{1\mu}k_{2\mu}+m^2+\xi^2\partial^2_{k_1})\,\delta^{(n)}(k_1-k_2)\bigg)^{-1}. \tag{4.3}$$

Since the presence of the harmonic oscillator term in the action (3.12) breaks the translation invariance, from above it is clear that translation invariance breaking produces a modification of the scalar field mass. Parameter ξ^2, of dimension $dim[\xi] = length^{-2}$, represents the magnitude of translation invariance breaking.

If one includes a harmonic oscillator term, the propagator is given by the so-called Mehler kernel which depends on different incoming and outgoing momenta since the propagator does not respect momentum conservation [77]. So for full computation, one needs to take into account the Mehler kernel from the beginning. However, in the spirit of our *hybrid approach* and under the assumption of small perturbation due to the harmonic term, we approximate the full propagator by

$$G^\xi \equiv G^\xi(k_1,k_2) \simeq \frac{i}{k_1^2+m^2}\left(1-\frac{\xi^2}{(k_1^2+m^2)}\partial^2_{k_1}\right)\delta^{(n)}(k_1-k_2)\,. \tag{4.4}$$

This is going to be used in the further computation of the 2-point Green's function. We believe that the approximative expression (4.4) is good enough to help us to indicate the

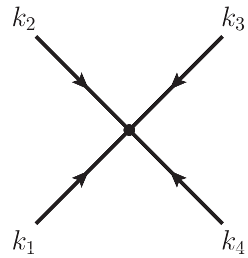

Figure 1: Scalar 4-field vertex

influence of the ξ^2 term on the one-loop quantum corrections. This is going to be presented in Subsection 4.2. Of course the full computation of quantum corrections including the Mehler kernel is out of scope for this paper, but it is certainly going to be performed in the future.

If a is of the order of the Planck length, ξ despite being small, carries contributions to Green's functions that are still larger than the terms linear in a.

The vertex function, which in momentum space is given by

$$\tilde{\Gamma}(k_1, k_2, k_3, k_4; a) = i\frac{\delta^4 S[\tilde{\phi}]}{\delta\tilde{\phi}(k_1)\delta\tilde{\phi}(k_2)\delta\tilde{\phi}^*(k_3)\delta\tilde{\phi}^*(k_4)}\,, \tag{4.5}$$

is modified too. It is illustrated in Fig. 1 and amounts to the following expression:

$$\begin{aligned}
\tilde{\Gamma}(k_1, k_2, k_3, k_4; a) \,=\, & i(2\pi)^n\frac{\lambda}{2}\,a_\nu\Bigg[\frac{a_\nu}{a^2} + \frac{1}{4}\bigg(k_{4\mu}k_{3\nu} + k_{3\mu}k_{4\nu} - 2\delta_{\mu\nu}k_{4\rho}k_{3\rho} \\
& +\frac{1}{2}(k_{2\mu}k_{4\nu} - k_{4\mu}k_{2\nu} + k_{2\mu}k_{3\nu} - k_{3\mu}k_{2\nu})\bigg)\partial^{k_1}_\mu \\
& +\frac{1}{4}\bigg(k_{4\mu}k_{3\nu} + k_{3\mu}k_{4\nu} - 2\delta_{\mu\nu}k_{4\rho}k_{3\rho} \\
& +\frac{1}{2}(k_{1\mu}k_{4\nu} - k_{4\mu}k_{1\nu} + k_{1\mu}k_{3\nu} - k_{3\mu}k_{1\nu})\bigg)\partial^{k_2}_\mu \\
& +\frac{1}{4}\bigg(k_{1\mu}k_{2\nu} + k_{2\mu}k_{1\nu} - 2\delta_{\mu\nu}k_{1\rho}k_{2\rho}\bigg)(\partial^{k_3}_\mu + \partial^{k_4}_\mu)\Bigg]\delta^{(n)}(k_1 + k_2 - k_3 - k_4),
\end{aligned} \tag{4.6}$$

where we denote $\partial^k_\mu = \frac{\partial}{\partial k_\mu}$, and all four momenta k_i are flowing into the vertex. Both couplings, ξ and λ have to be dimensionally regularized.

4.1.2 Feynman rules (B): κ-deformed momentum addition law

Next, we discuss the notion which anticipates induced momentum conservation law on κ-space, within our *hybrid approach*. Namely, the δ-function in (4.6) comes from the contraction of fields, where the momentum conservation should be obeyed in accordance with the κ-deformed momentum addition rule [18, 19, 41]. We have two cases for summation/subtraction of 4-vectors k^μ with respect to the physical situation of four particles

and/or quantum fields propagating in space with respect to an interaction point:
(I) all particle momenta flowing into the vertex, as given in Fig. 1

$$k_{1\mu} + k_{2\mu} + k_{3\mu} + k_{4\mu} = 0 \quad \to \quad k_{1\mu} \oplus k_{2\mu} \oplus k_{3\mu} \oplus k_{4\mu} = 0\,, \tag{4.7}$$

(II) the process of scattering "2 particle $\to$ 2 particle", where we have

$$(k_{1\mu} + k_{2\mu}) - (k_{3\mu} + k_{4\mu}) = 0 \quad \to \quad (k_{1\mu} \oplus k_{2\mu}) \ominus (k_{3\mu} \oplus k_{4\mu}) = 0\,. \tag{4.8}$$

Having obtained the Feynman rules (4.4) and (4.6), we have completed the first stage of our program, that is to deduce the free propagation and interaction properties of the model by using the standard quantization.

At this point we turn to the second part, which includes the effective description of the statistics of particles described by the model. As already indicated before, the statistics of particles is twisted in κ-space [22, 23, 35, 40], with deformation being encoded within the nonabelian momentum addition rule. It is known that the rule for addition of momenta is governed by the coproduct structure of the Hopf algebra in question. In our case, the relevant Hopf algebra is the κ-Poincaré algebra and the corresponding coalgebra structure is given by (2.18), (2.19), and (2.22), (2.23). In particular, the coproduct (2.18) for translation generators determines the required momentum addition rule, which in momentum space and up to the first order in deformation a, from (2.22) and (2.25), yields:

$$\begin{aligned} S(p_\mu) &= -iS(\partial_\mu) = -p_\mu - a_\mu p^2 + (ap)p_\mu + \mathcal{O}(a^2)\,, \qquad (4.9)\\ (p_\mu \oplus k_\mu) &= (p+k)_\mu + (ak)p_\mu - a_\mu(pk) + \mathcal{O}(a^2)\,. \qquad (4.10)\end{aligned}$$

Here we have nonabelian momentum addition rule (4.10), while $S(p)$ is the antipode with the property $p^\mu \oplus S(p^\mu) = 0$, that in fact represents the very definition of the antipode. Namely, since commutativity in momentum space is not satisfied, i.e. $k \oplus p \neq p \oplus k$, certain *ordering* has to be implemented. However, instead of implementation of possibly *complicated unknown ordering*, we shall proceed in the most simple way by taking into account all possible type of contributions; for example $k \oplus p \oplus q$, $p \oplus k \oplus q$, etc. Combining (4.9) and (4.10) we obtain the following momentum subtraction rule:

$$p_\mu \ominus k_\mu \equiv (p \oplus S(k))_\mu = (p-k)_\mu(1-ak) + a_\mu(pk - k^2) + \mathcal{O}(a^2)\,. \tag{4.11}$$

This enables us to rewrite the energy-momentum conservation which is supposed to be satisfied at each vertex. Thus, if two external momenta k_1 and k_2 flow into the vertex and the other two external momenta k_3 and k_4 flow out of the vertex, then, writing in components, we have the induced momentum conservation law (4.8), which corresponds to our physical situation while computing 4-field tadpole diagram in the next subsection.

In order to obtain expressions for the δ-functions in Feynman rules, we are starting with

$$\delta^{(n)}(p \ominus k) = \sum_i \left| \det\left(\frac{\partial(p\ominus k)_\mu}{\partial p_\nu}\right)_{p=q_i}\right|^{-1} \delta^{(n)}(p - q_i)\,, \tag{4.12}$$

where we have to sum up over all zeros q_i for the expression in the argument of δ-function. Since there is only one zero, $q_i = k$, with the help of subtraction rule (4.11), we find the following first order contribution to the above $\delta^{(n)}$-function

$$\delta^{(n)}(p \ominus k) = \frac{\delta^{(n)}(p-k)}{(1-ap)^{n-1}} = (1+(n-1)ap + \mathcal{O}(a^2))\delta^{(n)}(p-k)\,. \tag{4.13}$$

It was shown in [52] that the star product $\star_h$ (3.9) breaks translation invariance (in the sense of definition introduced in [18]). However, this feature does not show up until computations are extended to the second order in the deformation parameter a. The important point is that the translation invariance is thus intact within the first order deformation. Since we are carrying out our investigation in exactly this order, we are allowed to invoke the energy momentum conservation albeit in a modified form, dictated by the modified coproduct structure and by the oscillator term. Relations between Hopf algebra symmetries and conservation laws is important subject of investigation. This is the issue of generalizing the Noether theorem, thus the whole subject is still appealing [78].

With the idea of implementing the new physical features that have just been described, we modify the Feynman rule (4.6). Taking into account all possible contributions, with the help of (4.10) and (4.11), and choosing the following replacement of δ-function from (4.6) [17, 19, 20, 79],

$$\delta^{(n)}(k_1+k_2-k_3-k_4) \to \delta^{(n)}((k_1\oplus k_2)\ominus(k_3\oplus k_4)) + \delta^{(n)}((k_1\oplus k_2)\ominus(k_4\oplus k_3))\,, \tag{4.14}$$

we obtain the *hybrid* Feynman rule which obeys κ-deformed momentum addition/subtraction rule via sum of δ-functions (4.14):

$$\begin{aligned}
\tilde{\Gamma}(k_1,k_2,k_3,k_4;a) =\; & i(2\pi)^n\frac{\lambda}{2}\,a_\nu\Bigg[\frac{a_\nu}{a^2} + \frac{1}{4}\bigg(k_{4\mu}k_{3\nu} + k_{3\mu}k_{4\nu} - 2\delta_{\mu\nu}k_{4\rho}k_{3\rho} \\
& +\frac{1}{2}(k_{2\mu}k_{4\nu} - k_{4\mu}k_{2\nu} + k_{2\mu}k_{3\nu} - k_{3\mu}k_{2\nu})\bigg)\partial^{k_1}_\mu \\
& +\frac{1}{4}\bigg(k_{4\mu}k_{3\nu} + k_{3\mu}k_{4\nu} - 2\delta_{\mu\nu}k_{4\rho}k_{3\rho} \\
& +\frac{1}{2}(k_{1\mu}k_{4\nu} - k_{4\mu}k_{1\nu} + k_{1\mu}k_{3\nu} - k_{3\mu}k_{1\nu})\bigg)\partial^{k_2}_\mu \\
& +\frac{1}{4}\bigg(k_{1\mu}k_{2\nu} + k_{2\mu}k_{1\nu} - 2\delta_{\mu\nu}k_{1\rho}k_{2\rho}\bigg)(\partial^{k_3}_\mu + \partial^{k_4}_\mu)\Bigg] \\
& \times\Big[\delta^{(n)}((k_1\oplus k_2)\ominus(k_3\oplus k_4)) + \delta^{(n)}((k_1\oplus k_2)\ominus(k_4\oplus k_3))\Big]\,.
\end{aligned} \tag{4.15}$$

In the above, sum of δ-functions represents all mutually different physical situations.

The δ-functions in (4.15), should in principle come from the contraction of fields quite naturally, if the noncommutative version of the LSZ formalism is applied to our model. Since such formalism is not developed so far, we choose to follow a kind of approach that combines the standard quantum field theory consideration (used when treating terms in the Lagrangian linear in a as small perturbations) with the peculiarities resulting from the statistics properties of particles in κ-space. The later part is realized through embedding a nonabelian momentum-energy conservation within the 4-point vertex function. That approach may seem as a *hybrid construction* raised in trying to move our understanding one step forward towards a complete quantum theory on noncommutative spaces in general. In this sense, such an approach can serve as an intermediate step bridging the gap between the standard quantum field theory and a complete field theory on κ-space in as much the similar way as for example the semiclassical theory of radiation can be considered as a cross-over towards the quantum theory of radiation. In following the described path we have to rely in part on intuition, especially when peculiarities of κ-space statistics have to be taken into account.

The Feynman rule (4.15) now appears to be consistent with the energy-momentum conservation that respects κ-deformed momentum addition rule. In order to obtain the complete expression for the δ-functions appearing in (4.15), we are proceeding in two steps. First, with the help of (4.10) and (4.13), up to linear order in a, and with $j, l = 3, 4$; $j \neq l$, we have:

$$
\begin{aligned}
\delta^{(n)}((k_1 \oplus k_2) \ominus (k_j \oplus k_l)) &= \frac{\delta^n((k_1 \oplus k_2) - (k_j \oplus k_l))}{\Big(1 - a(k_1 \oplus k_2)\Big)^{n-1}} \\
&= \frac{\delta^{(n)}((k_1 \oplus k_2) - (k_j \oplus k_l))}{\Bigg[1 - \Big(a(k_1 + k_2) + (ak_1)(ak_2) - a^2(k_1 k_2) + \mathcal{O}(a^3)\Big)\Bigg]^{n-1}} \\
&= (1 + (n-1)a(k_1 + k_2) + \mathcal{O}(a^2))\delta^{(n)}((k_1 \oplus k_2) - (k_j \oplus k_l)) \,.
\end{aligned}
\tag{4.16}
$$

The second step is to compute delta functions from (4.16) in the same way as we computed the one in (4.13):

$$
\delta^{(n)}((k_1 \oplus k_2) - (k_j \oplus k_l)) = \sum_i \left| \det \left(\frac{\partial((k_1 \oplus k_2) - (k_j \oplus k_l)_\mu}{\partial k_{1\nu}} \right)_{k_1 = q_i} \right|^{-1} \delta^{(n)}(k_1 - q_i) \,,
\tag{4.17}
$$

where we have to sum up over all zeros q_i for the expression in the argument of δ-function.

To proceed, we shall choose specific momenta $k_2 = k_3 = \ell$ we need for tadpole diagram evaluation. Since there are no zeros for delta function $\delta^{(n)}((k_1 \oplus \ell) - (\ell \oplus k_4))$, the only contribution comes from second combination $\delta^{(n)}((k_1 \oplus \ell) - (k_4 \oplus \ell))$. In order to perform that computation, we start with (4.10) and orient the vector a in the direction of time, $a = (0, ..., 0, ia_0)$. Then due to covariance, the obtained result will also be valid for an arbitrary orientation of a. Hence

$$\det\left(\frac{\partial((k_1\oplus\ell)-(k_4\oplus\ell))_\mu}{\partial k_{1\nu}}\right)_{k_1=k_4} = \begin{vmatrix} 1 & -ia_0\ell_1 & \cdots & -ia_0\ell_{n-2} & -ia_0\ell_{n-1} \\ 0 & 1+a\ell & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1+a\ell & 0 \\ 0 & 0 & \cdots & 0 & 1+a\ell \end{vmatrix}$$
$$= (1+a\ell)^{n-1}. \tag{4.18}$$

Since there is only one zero, $q_i = k_4$, we find

$$\delta^{(n)}((k_1\oplus\ell)-(k_4\oplus\ell)) = \frac{\delta^{(n)}(k_1-k_4)}{(1+a\ell)^{n-1}} = \Big(1-(n-1)a\ell\Big)\delta^{(n)}(k_1-k_4)+\mathcal{O}(a^2)\,. \tag{4.19}$$

which gives final expression for the δ-function in (4.15) via (4.16), up to the first order in a

$$\begin{aligned}\delta^{(n)}((k_1\oplus\ell)\ominus(k_4\oplus\ell)) &= \Big(1+(n-1)a(k_1+\ell)\Big)\frac{\delta^{(n)}(k_1-k_4)}{(1+a\ell)^{n-1}}+\mathcal{O}(a^2)\\ &= (1+(n-1)ak_1)\delta^{(n)}(k_1-k_4)+\mathcal{O}(a^2)\,.\end{aligned} \tag{4.20}$$

In the above expression the ℓ dependences drop out as we expected, thus showing the consistency of the *hybrid* Feynman rule derivation. The remaining factor $(1+(n-1)ak_1)$ in (4.20) is due to the κ-space twisted particle statistics of our *hybrid approach* modeling.

4.2 Massive scalar field propagation

4.2.1 Tadpole diagram: standard momentum conservation

In order to compute the tadpole diagram from Fig. 2, using dimension regularization technique, we have to introduce in the action (3.12) *new* dimensionfull regularization masses denoted by μ for the coupling λ, and by μ' for ξ^2, respectively. In accordance with Quantum Field Theory [80], the regularization of the ϕ^4 model requires that they are given in the following form:

$$\lambda_{new} = \lambda_{old}(\mu^2)^{\frac{n}{2}-2} \quad\to\quad (\mu^2)^{2-\frac{n}{2}}\lambda\,,\quad \lambda=\lambda_{new}\,, \tag{4.21}$$
$$\xi_{new} = \xi_{old}(\mu'^2)^{\frac{n}{2}-4} \quad\to\quad (\mu'^2)^{4-\frac{n}{2}}\xi\,,\quad \xi=\xi_{new}\,. \tag{4.22}$$

Here μ' is defined in a way to retain the same dimension of constant $dim[\xi^2]=dim[mass^4]$, for $n=4$, under the loop integral contribution from ξ-term in (3.12).

In this paper, we shall further restrict our computation only to the first order contribution of the two-point function $\Pi_2^{a,\xi}$ in our $\lambda\phi^4$ model (3.12) corresponding to the tadpole diagram from Fig. 2:

$$\Pi_2^{a,\xi} = \Pi_2^{0,0}+\Pi_2^{a\neq0,0}+\Pi_2^{0,\xi\neq0}+\Pi_2^{a\neq0,\xi\neq0} = \int\frac{d^nk_2}{(2\pi)^n}\frac{d^nk_3}{(2\pi)^n}\,\tilde\Gamma(k_1,k_2,k_3,k_4;a,\mu)G^\xi(k_2,k_3;\mu')\,. \tag{4.23}$$

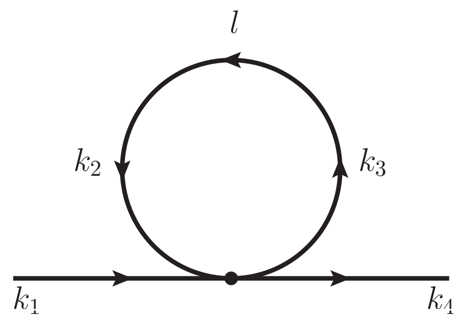

Figure 2: Scalar 4-field tadpole

In general, the one-loop integral (4.23) produces four different contributions, where for $G^\xi(k_2, k_3; \mu')$ we are using the expanded expression (4.4) further on. In computing the first two terms from (4.23) we assume momentum conservation, $k_1 + k_2 = k_3 + k_4$ and $k_2 = k_3 = \ell$. Thus we have

$$\Pi_2^{a,0} = \int \frac{d^n\ell}{(2\pi)^n}\, \tilde{\Gamma}(k_1, \ell, \ell, k_4; a, \mu)\, \frac{i}{\ell^2 + m^2}\,. \tag{4.24}$$

First we need $\tilde{\Gamma}$ from Feynman rule (4.6), in accordance with notations on Fig. 1; that is for incoming momenta k_1 and outgoing momenta k_4 we have to replace $k_3 \to -k_3 = -\ell$ and $k_4 \to -k_4$. Thus, we have

$$\begin{aligned}
\tilde{\Gamma}(k_1, \ell, \ell, k_4^{out}; a, \mu) &= i(2\pi)^n \mu^{4-n}\, \frac{\lambda}{2} \bigg\{ 1 + a_\nu \frac{1}{8} \bigg[\Big(k_{4\mu} k_{1\nu} - k_{1\mu} k_{4\nu} \Big) \partial_\mu^\ell \\
&- 2\Big(\ell_\mu k_{4\nu} - 3\ell_\nu k_{4\mu} + 8\delta_{\mu\nu} k_{4\rho} \ell_\rho \Big) \partial_\mu^{k_1} - 2\Big(\ell_\mu k_{1\nu} + k_{1\mu} \ell_\nu - 2\delta_{\mu\nu} k_{1\rho} \ell_\rho \Big) \partial_\mu^{k_4} \\
&+ \Big(\ell_\mu (2k_4 - k_1)_\nu + \ell_\nu (2k_4 - 3k_1)_\mu - 4\delta_{\mu\nu} (k_4 - k_1)^\rho \ell_\rho \Big) \partial_\mu^\ell \bigg] \bigg\} \delta^{(n)}(k_1 - k_4)\,.
\end{aligned} \tag{4.25}$$

In order to obtain $\tilde{\Gamma}(k_1, \ell, \ell, k_4^{in}; a, \mu)$ we just have to replace $k_4 \to -k_4$ in (4.25).

As a next step, we compute $\Pi_2^{a,\xi}$ straightforward with help of the notion for integral (4.24) as an effective action describing the given one-loop quantum process. So, employing integration by parts in (4.24) and using dimensional regularization scheme, we obtain

$$\Pi_2^{0,0} = -\frac{\lambda}{2}\, I_0\,, \tag{4.26}$$

$$\Pi_2^{a\neq 0,0} = -\frac{\lambda}{2} \bigg\{ \frac{3}{8}(aK) I_0 - \frac{1}{4}(aK) \Big(\delta_{\mu\nu} - \frac{a_\mu K_\nu}{(aK)} \Big) I_{2,\mu\nu} \bigg\}\,, \tag{4.27}$$

where $K = 2k_4 - k_1$. The oscillator contribution from ξ-term we obtain by rude approximation of (4.23). That is, from

$$\Pi_2^{0,\xi\neq 0} = -i \int \frac{d^n\ell}{(2\pi)^n}\, \tilde{\Gamma}(k_1, \ell, \ell, k_4; a, \mu)\, \frac{\xi^2}{(\ell^2 + m^2)^2} \partial_\ell^2 \delta^{(n)}(\ell)\,, \tag{4.28}$$

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integrating by part, we found

$$\Pi_2^{0,\xi\neq 0} = -\frac{\lambda}{2}\xi^2\frac{(8n)(\mu'^2)^{4-\frac{n}{2}}}{m^6}\,. \tag{4.29}$$

In above equations the presence of $(2\pi)^n\delta^{(n)}(k_1-k_4)$ is understood, although not being explicitly shown. For $n=4-\epsilon$, we have the well known integrals

$$\begin{aligned} I_0 &= (\mu^2)^{2-\frac{n}{2}}\int\frac{d^n\ell}{(2\pi)^n}\frac{1}{\ell^2+m^2} = \frac{m^2}{(4\pi)^2}\left[\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}}\Gamma(-1+\frac{\epsilon}{2})\right]_{\epsilon\to 0} \\ &= \frac{-1}{8\pi^2}m^2\left[\frac{1}{\epsilon}+\frac{\psi(2)}{2}+\log\sqrt{\frac{4\pi\mu^2}{m^2}}+...\right], \end{aligned} \tag{4.30}$$

$$I_{2,\mu\nu} = (\mu^2)^{2-\frac{n}{2}}\int\frac{d^n\ell}{(2\pi)^n}\frac{\ell_\mu\ell_\nu}{(\ell^2+m^2)^2} = \frac{1}{2}\delta_{\mu\nu}I_0\,,\quad \delta_{\mu\nu}\delta_{\mu\nu}=n\,, \tag{4.31}$$

with a simple pole at $\epsilon=0$. Thus the expression (4.30) is divergent in the UV cut-off.

The non-vanishing contributions come from commutative cases, that is from (4.26), and from harmonic oscillator term (4.29) which is finite for finite scalar field mass. Integrals (4.30) and (4.31) for $n=4$ give $\Pi_2^{a\neq 0,0}=0$, producing in the case $\xi=0$, very well known commutative $\lambda\phi^4$ theory result (4.26). All contributions proportional to a, coming from κ-Minkowski NC ϕ^4 theory cancel out, as one would naively expect by inspecting vertex (4.25).

Clearly, the one loop computation has to be modified by anticipating the momentum conservation on κ-space. To illustrate that something nonstandard appears in our model (4.6), we start with the general one-loop integral (4.23). It should be noted that one cannot integrate over k_3 using the first delta $\delta^{(n)}(k_2-k_3)$ – from the propagator – and replace k_3 by k_2 in the above expression as it stands, because of the derivative with respect to k_2. So, as a first step of computation we are using a simple trick

$$\begin{aligned} \partial_\mu^{k_1}\delta^{(n)}(k_1+k_2-k_3-k_4) &= \partial_\mu^{k_2}\delta^{(n)}(k_1+k_2-k_3-k_4)\,, \\ \partial_\mu^{k_1}\delta^{(n)}(k_1+k_2-k_3-k_4) &= -\partial_\mu^{k_3}\delta^{(n)}(k_1+k_2-k_3-k_4)\,,\ \text{etc.}, \end{aligned} \tag{4.32}$$

and than we rewrite (4.6) and (4.4) as follows:

$$\begin{aligned} &\tilde\Gamma(k_1,k_2,k_3,k_4;a)G^\xi(k_2,k_3) = \\ &\Bigg\{i(2\pi)^n\frac{\lambda}{4}\Bigg[1+a_\nu\Bigg(2(-k_{1\mu}k_{2\nu}-k_{2\mu}k_{1\nu}+k_{3\mu}k_{4\nu}+k_{4\mu}k_{3\nu}+2\delta_{\mu\nu}(k_{1\rho}k_{2\rho}-k_{3\rho}k_{4\rho})) \\ &+\frac{1}{2}(k_{1\mu}k_{3\nu}-k_{3\mu}k_{1\nu}+k_{2\mu}k_{3\nu}-k_{3\mu}k_{2\nu}+k_{1\mu}k_{4\nu}-k_{4\mu}k_{1\nu}+k_{2\mu}k_{4\nu}-k_{4\mu}k_{2\nu})\Bigg)\partial_\mu^{k_1}\Bigg] \\ &\times\delta^{(n)}(k_1+k_2-k_3-k_4)\Bigg\}\left[\frac{i}{k_2^2+m^2}\left(1-\frac{\xi^2}{(k_2^2+m^2)}\partial_{k_3}^2\right)\delta^{(n)}(k_2-k_3)\right]. \end{aligned} \tag{4.33}$$

For the ξ independent part, we obtain after performing the integration over k_3 in (4.33) the following expression:

$$\Pi_2^{a\neq 0,0} = -\frac{\lambda}{8(2\pi)^n}((ak_1)k_4-(ak_4)k_1)_\mu\left[\partial_\mu^{k_1}\delta^{(n)}(k_1-k_4)\right]\int\frac{d^nk_2}{k_2^2+m^2}\,, \tag{4.34}$$

where k_1 and k_4 are the external momenta. This expression is quadratically divergent in the UV cut-off representing a quantum loop modification of the free action (3.12), which can be nonzero because of the momentum conservation violation at the vertex. The next term we obtain after the partial integration,

$$\Pi_2^{0,\xi\neq 0} = -\frac{\lambda\xi^2}{4(2\pi)^n}\Big[\partial_{k_1}^2 \delta^{(n)}(k_1-k_4)\Big]\int \frac{d^n k_2}{(k_2^2+m^2)^2}\,. \tag{4.35}$$

It is logarithmically divergent in $n=4$ dimensions. The last term depends on both deformation parameters, a and ξ, it reads

$$\begin{aligned}\Pi_2^{a\neq 0,\xi\neq 0} = &-\frac{\lambda\xi^2}{8(2\pi)^n}a_\nu\bigg[\Big(\partial_{k_1}^2\partial_\mu^{k_1}\delta^{(n)}(k_1-k_4)\Big)(k_{1\mu}k_{4\nu}-k_{4\nu}k_{1\mu})\\ &-4\Big(\partial_\sigma^{k_1}\partial_\mu^{k_1}\delta^{(n)}(k_1-k_4)\Big)\Big(2(\delta_{\mu\sigma}k_{4\nu}+\delta_{\nu\sigma}k_{4\mu}-2\delta_{\mu\nu}k_{4\sigma})\\ &\qquad+\frac{1}{2}(\delta_{\nu\sigma}k_{1\mu}-\delta_{\mu\sigma}k_{1\nu})\Big)\bigg]\int\frac{d^n k_2}{(k_2^2+m^2)^2}\,,\end{aligned} \tag{4.36}$$

which is also logarithmically divergent in $n=4$ dimensions. The one-loop corrections (4.35)-(4.36) have to be included in the action as counterterms. Their structure is different from the tree-level terms, and therefore the tree-level action is not stable under 1-loop quantum corrections. For this reason, we have to question the approximations we have employed: namely expansion of the action up to first order in the deformation parameter a and expansion of the propagator up to first order in ξ. The results (4.26) and (4.27) (as well as the result (4.37) that will follow shortly) are obtained directly from (4.24). We assume that (4.24) replaces the expression (4.23) for $\xi=0$. As for (4.29) (and (4.38)), it is a rude estimate of what ξ term should produce. It is obtained by making approximations in (4.23), by approximating/adjusting the form of the propagator (4.4) and by using κ-deformed addition/subtraction of momenta instead of commutative one. The difference between (4.29) and (4.38) is that unlike the former expression, the later one is obtained by making use of κ-deformed addition/subtraction rule for momenta. Due to the results (4.34)-(4.36) we obviously stumbled across the momentum nonconservation. Such results seem to favor our *hybrid* approach.

4.2.2 Tadpole diagram: κ-deformed momentum conservation

In the following computation of the tadpole diagram from Fig. 2, we fully implement the *hybrid approach*. That is the notion that standard momentum conservation is not satisfied, i.e. we use induced momentum conservation on κ-space represented within delta functions in (4.14). However, in accordance with our *hybrid approach*, at the end undeformed momentum conservation has to be applied. General one-loop integral (4.23) can be roughly reduced to two terms, (4.24) and (4.28), respectively.

In the next step we are applying integration by parts, which is possible due to the notion that remaining integral in (4.23) is an effective action of the given one-loop quantum process. This of course plays an essential role in our *hybrid approach*. Performing the computation of all terms in the tadpole one-loop integral (4.24) with Feynman rule (4.15) and delta function (4.20) and for an arbitrary number of dimensions n, we obtain the following first order result:

$$\Pi_2^{0,0} + \Pi_2^{a\neq 0,0} = -\frac{\lambda}{2}\left[(1+(n-1)ak_1) + \frac{1}{2}(1-\frac{n}{4})(ak_1 - 2ak_4)\right] I_0 \quad (4.37)$$

$$\Pi_2^{0,\xi\neq 0} + \Pi_2^{a\neq 0,\xi\neq 0} = -\frac{\lambda}{2}\xi^2 \frac{8n(\mu'^2)^{4-\frac{n}{2}}}{m^6}(1+(n-1)ak_1)\,. \quad (4.38)$$

First terms in (4.37) and (4.38) for $n = 4$ corresponds to results (4.26) and (4.29) from previous subsection. From above formulas it is clear that there exist the non-vanishing contributions even for number of dimensions $n = 4$. They are arising via the κ-deformed momentum conservation rule, entering through the deformed δ-function (4.20) in the *hybrid* Feynman rule (4.15), and from the harmonic oscillator term in the action (3.12) via the modified propagator (4.4).

For $n = 4 - \epsilon$, we obtain modified expression for the tadpole in Fig. 2 in the limit $\epsilon \to 0$, where $1/\epsilon$ divergence is explicitly isolated. For conserved external momentum in accordance with (4.20), i.e. for $k_1 = k_4 \equiv k$, from (4.30), (4.37) and (4.38) we finally have,

$$\Pi_2^{a,\xi} = \frac{\lambda m^2}{32\pi^2}(1+3ak)\left[\frac{2}{\epsilon} + \psi(2) + \log\frac{4\pi\mu^2}{m^2} - \frac{9}{4}\frac{ak}{1+3ak} - (4-\epsilon)128\pi^2\frac{\xi^2\mu^{4-\epsilon}}{m^8}\right], \quad (4.39)$$

where for simplicity we have used $\mu' = \mu$, and in ξ-term we retain the explicit ϵ-dependence in order to keep dimensional and/or limiting procedure under control. The above finite parts represent modifications of the scalar field self-energy and depend explicitly on the regularization parameter, the mass of the scalar field, the magnitude of the translation invariance breaking, and it contains a correction ak due to the dependence on energy $|k|$, where actual scalar field self-energy modification occurs. Dependence of (4.39) on κ-deformation parameter a enters explicitly, as we expected. Note that there is no need to do renormalization at the point $(1 + 3ak) \to 0$.

The result (4.39) is next discussed in the framework of Green's functions. Generally we know that by summing all the 1PI contributions, for full free propagator modification (4.3), we get the following expression for *two-point connected* Green's function

$$G^{a,\xi}_{(c,2)}(k_1,k_4) = \left[\frac{i}{k_1^2+m_1^2} + \frac{i}{k_1^2+m_1^2}\Pi_2^{a,\xi}\frac{i}{k_1^2+m_1^2} + ...\right], \quad (4.40)$$

where, symbolically, $m_1^2 = m^2 + \xi^2\partial^2_{k_1}\delta^{(n)}(k_1 - k_4)$ represents redefined mass. As an illustration we resume the above series in the limit $\xi \to 0$:

$$G^{a,\xi}_{(c,2)}(k_1,k_4) \overset{\xi\to 0}{\longrightarrow} (2\pi)^n\delta^{(n)}(k_1-k_4)\left[\frac{i}{k_1^2+m^2-\Pi_2^{a,0}}\right]. \quad (4.41)$$

The genuine $1/\epsilon$ divergence in $\Pi_2^{a,\xi}$, can only be removed by introducing the following counter term δm^2:

$$\delta m^2\tilde{\phi}^*(k)\tilde{\phi}(k) = \frac{\lambda m^2}{32\pi^2}\left[2(1+3ak)\frac{1}{\epsilon} + f\left(\frac{4-\epsilon}{2},\frac{\mu^2}{m^2},ak,\frac{\xi^2}{m^4}\right)\right]\tilde{\phi}^*(k)\tilde{\phi}(k)\,, \quad (4.42)$$

with f as an arbitrary dimensionless function, which is fixed by the normalization conditions. Adding above counterterm contribution to the previous expression (4.40), results during the renormalization procedure in the shift $m^2 \to m^2 + \delta m^2$ in (4.40)/(4.41), thus leading to

$$\begin{aligned}\tilde{G}^{a,\xi}_{(c,2)}(k_1,k_4) &= \Big[G^{a,\xi}_{(c,2)}(k_1,k_4) + G^{a,\xi}_{(c,2)}(k_1,k_4)(-\delta m^2)G^{a,\xi}_{(c,2)}(k_1,k_4) + ...\Big] \\ &\stackrel{\xi\to 0}{\longrightarrow} (2\pi)^n \delta^{(n)}(k_1-k_4)\left[\frac{i}{k_1^2+m^2+\delta m^2 - \Pi_2^{a,0}}\right],\end{aligned} \tag{4.43}$$

where $\tilde{G}^{a,\xi}_{(c,2)}$ denotes Green's function with the contribution from counter term incorporated.

However, since $\Pi_2^{a,\xi}$ was computed for expanded free propagator (4.4), it is consistent to compute *two-point connected* Green's function under the same approximation. After the resummation of (4.40), with usage of (4.4), we obtain

$$G^{a,\xi}_{(c,2)}(k_1,k_4) = \frac{G^\xi}{1-G^\xi\,\Pi_2^{a,\xi}}\,. \tag{4.44}$$

To identify proper counter term for above expression we resume series in (4.43), but with replacement (4.40) $\to$ (4.44), and the full free propagator $i/(k_1^2+m_1^2)$ replaced with expanded one, (4.4), giving:

$$\tilde{G}^{a,\xi}_{(c,2)}(k_1,k_4) = \frac{G^\xi}{1+G^\xi(\delta m^2 - \Pi_2^{a,\xi})}\,. \tag{4.45}$$

In (4.45) δm^2 is generic quantity. This is due to the fact that expression (4.39) contains the finite parts too. The requirement $\delta m^2 = \Pi_2^{a,\xi}$ removes infinity. Thus we have

$$\tilde{G}^{a,\xi}_{(c,2)}(k_1,k_4) = \frac{G^\xi}{1+G^\xi\frac{\lambda m^2}{32\pi^2}\left[f-(1+3ak)\left(\psi(2)+\log\frac{4\pi\mu^2}{m^2}-\frac{9}{4}\frac{ak}{1+3ak}-512\pi^2\frac{\xi^2\mu^4}{m^8}\right)\right]}\,. \tag{4.46}$$

Precise extraction and removal of genuine UV divergence is performed next via (4.42) in the context of the analysis of $\tilde{G}^{a,\xi}_{(c,2)}(k_1,k_4)$ for different energy regimes; that is from low energy to extremely high -Planck scale- energy propagation.

There is a very interesting property of the expression (4.39) at extreme energies. Namely, there exists a term $(1+3ak)$ which for $(1+3ak \to 0)$ tends to zero linearly. For low energies and/or small κ-deformation a, i.e. for $ak=0$, (equivalent to $a=0$), which is far away from the point $(1+3ak=0)$, this is not the case.

Low energy limit
Using finite combination $(\delta m^2 - \Pi_2^{0,0})$ for low energy $(ak=0)$, and at order λ

$$\delta m^2 - \Pi_2^{0,0} = \frac{\lambda m^2}{32\pi^2}\left[f-\psi(2)-\log\frac{4\pi\mu^2}{m^2}\right], \tag{4.47}$$

from (4.39) and (4.46), we get

$$\tilde{G}^{0,\xi}_{(c,2)}(k_1,k_4) = \frac{G^\xi}{1 + G^\xi \frac{\lambda m^2}{32\pi^2}\left(f - \psi(2) - \log\frac{4\pi\mu^2}{m^2} + 512\pi^2 \frac{\xi^2\mu^4}{m^8}\right)}\,, \tag{4.48}$$

while in the case $\xi = 0$

$$\tilde{G}^{0,0}_{(c,2)}(k_1,k_4) = \frac{(2\pi)^4\delta^{(4)}(k_1-k_4)}{k_1^2 + m^2\left(1 + \frac{\lambda}{32\pi^2}\left[f - \psi(2) - \log\frac{4\pi\mu^2}{m^2}\right]\right)}\,. \tag{4.49}$$

This expression has a pole in Minkowski space, and we can define the renormalization condition by requiring that the inverse propagator at the physical mass is $k_1^2 + m^2_{phys/low}$. This choice, in the case $\xi^2 = 0$, determines uniquely the sum of the residual terms in (4.47), which is in accordance with commutative ϕ^4 theory result [80].

Planckian energy limit
At the limiting point $(1+3ak \to 0)$, which corresponds to extreme energy propagation $|k|$, where the components of κ-deformation parameter a_μ are extremely small, of Planck length order, the divergence in (4.39) gets removed *under the reasonable assumption that* $(1+3ak)$ *tends to zero with the same speed as* ϵ *does.* That is, in the Planckian propagation energy limit

$$\frac{(1+3ak) \to 0}{\epsilon \to 0} \longrightarrow \mathcal{O}(\epsilon^0)\,, \tag{4.50}$$

the $1/\epsilon$ and ak terms from (4.39) contributes.

Assuming that our κ-noncommutativity is spatial $a_\mu = (\vec{a}, 0)$, and using the momentum along the third axis $k_\mu = (0,0,E,iE)$, i.e. for $ak = Ea_3$, Eq. (4.39) in the Planckian propagation energy limit (4.50) gives

$$\Pi_2^{a,0}\bigg|_{(3Ea_3+1\to 0)} \longrightarrow \frac{\lambda m^2}{32\pi^2}\left[2 - \frac{9}{4}ak\right]_{(3Ea_3+1\to 0)}\,, \tag{4.51}$$

producing the following modified Green's function (4.41):

$$\Pi_2^{a,0}\bigg|_{(3Ea_3+1\to 0)} \to \frac{\lambda m^2}{32\pi^2}\,\frac{11}{4} \;\Rightarrow\; \tilde{G}^{a,0}_{(c,2)}(k_1,k_4)\bigg|_{(3Ea_3+1\to 0)} \simeq \frac{i(2\pi)^4\delta^{(4)}(k_1-k_4)}{k_1^2 + m^2\left(1 - \frac{\lambda}{32\pi^2}\frac{11}{4}\right)}\,. \tag{4.52}$$

At exact zero-point $3Ea_3 + 1 = 0$ however, from (4.39) we obtain different result

$$\Pi_2^{a,0}\bigg|_{(3Ea_3+1=0)} = \frac{\lambda m^2}{32\pi^2}\,\frac{3}{4} \quad\Rightarrow\quad \tilde{G}^{a,0}_{(c,2)}(k_1,k_4)\bigg|_{(3Ea_3+1=0)} \simeq \frac{i(2\pi)^4\delta^{(4)}(k_1-k_4)}{k_1^2 + m^2\left(1 - \frac{\lambda}{32\pi^2}\frac{3}{4}\right)}\,. \tag{4.53}$$

For time-space noncommutativity with the time component of κ-deformation parameter ia_0 being also of Planck length order and vanishing space component, i.e. for $ak = -Ea_0$, Eq. (4.39) in the Planckian energy limit (4.50) produces result,

$$\Pi_2^{a,0}\,|_{(3Ea_0-1\to 0)} \longrightarrow \frac{\lambda m^2}{32\pi^2}\left[2-\frac{9}{4}ak\right]_{(3Ea_0-1\to 0)}, \tag{4.54}$$

equivalent to (4.51), leading to final results identical with (4.52) and (4.53).

The existence of linear type of limit $(1+3ak\to 0)$ which removes the genuine UV divergence $1/\epsilon$ is a new, previously unknown feature of NC κ-Minkowski ϕ^4 theory at linear order in a. Mass shift receive fixed value (4.51), independent on function f, but this does depend on energy $|k|$ and κ-deformation parameter a. Above expressions gives κ-deformed dispersion relations, thus, (4.51), (4.52), (4.53) and (4.54) represents birefringence [58, 59] of the massive scalar field mode. Namely, all finite terms in (4.46)-(4.49), do not contribute in the limit $(1+3ak)\to 0$, and the possibility of their internal cancellation is diminished. This way massive scalar field mode birefringence arise as genuine effect. The above inverse propagator determines the physical mass $m^2_{phys/Planck}$ at respected Planck scale energies.

5. Discussion and conclusion

Let us discuss the final outcome of the above constructed ϕ^4 scalar field theory on the NC κ-deformed spacetime and describe the main results of this work.

First, we point out the most important new results:

(i) The integral measure problems on κ-Minkowski spacetime are avoided by the introduction of the new $\star_h$-product (3.9), into which the measure function is naturally absorbed due to the hermitian realization (3.6).

(ii) The trace-like property (the integral identity (3.10)) is valid for κ-deformed spaces, but only if one is dealing with the hermitian realization (3.6), as one should because only hermitian realizations have physical meaning. Due to the integral identity (3.10), the only deformation with respect to the standard scalar field action comes from the interaction term in (3.11).

(iii) The action (3.12) includes an harmonic type of the interaction term and is expanded up to first order in the deformation parameter a, producing an effective theory on commutative spacetime. Despite κ-deformation mixture with ξ^2 term in (3.11) via $\star_h$ product, at first order in κ-deformation these two features of our model separate completely in (3.12). The action (3.12) further produces modified equations of motion (3.14) and conserved deformed currents (3.15) due to the internal symmetry satisfied at that order. The above properties are very welcome.

(iv) Truncated κ-deformed action (3.12) does not possess celebrated quantum effect of UV/IR mixing [43]. Lack of UV/IR mixing is general feature of most of NCGFT expanded in terms of noncommutative deformation parameter. However, resummation of expanded action could in principle restore nonperturbative character of the model, thus restoring presence of UV/IR mixing in quantum loop computations. The UV/IR mixing connects NCGFT with Holography via UV and IR cutoffs, in a model independent way [55]. Both,

the UV/IR mixing and Holography, represent important windows to quantum gravity phenomena.
(v) Next, we discuss the result for the tadpole diagram contribution to the propagation and/or self-energy, of our scalar field ϕ for arbitrary number of dimensions n, depicted in Fig. 2, as a function of κ-deformed momentum conservation law which originates from deformed statistics on κ-Minkowski spacetime. Thus our approach represents kind of *hybrid approach modeling* between standard QFT and NCQFT on κ-Minkowski spacetime, involving κ-deformed δ-functions in the Feynman rules.
(vi) When we worked with the standard conservation of momenta (standard addition of momenta) and undeformed δ-function, the contributions to the tadpole diagram of the first order in a are zero. The deformed δ-function can be written in terms of leading term plus corrections in a. Since the Feynman rule (4.6) has already terms linear in a, we have to retain only zeroth order term in the modified δ-function, because otherwise we get terms of quadratic and higher orders in a (and this is not what we are interested in). The only term where we need to take into account δ-function correction linear in a is the leading order term in Feynman rule (4.6). However, this term is vanishing due to the integration over the loop momentum. Analyzing (4.34)-(4.36), we recognize that something nonstandard appears in the model for the tadpole integral at first order in λ and κ-deformation a.
(vii) It appears that in the computation of the tadpole diagram integrals for arbitrary number of dimensions n, for $n = 4$ all contributions linear in a canceled each other automatically. For dimensions $n \neq 4$, the same linear in a contributions become nonzero, regardless of the need for using any momentum conservation rule, even undeformed one. However the harmonic oscillator term from the action (3.12) modifies the mass term in the free propagator (4.3), thus producing the additional contribution (4.29) from the tadpole in Fig. 2. Propagation of the scalar field ϕ for dimension $n = 4$ also receive a modification from κ-deformation at linear order in deformation parameter a, and it does receive contribution due to oscillator term in the action for any number of dimensions.
(viii) In the final computation of the tadpole diagram from Fig. 2 we fully implement the notion of our *hybrid approach modeling*, that is that standard momentum conservation is not explicitly satisfied, i.e. we have to use the momentum conservation on κ-space given in (4.8), while at the end of computation undeformed momentum conservation has to be applied. We have found a non-vanishing contributions even for number of dimensions $n = 4$. They are arising from the harmonic oscillator term in the action (3.12) via modified propagator (4.4), and via κ-deformed momentum conservation rule entering through the deformed δ-function in *hybrid* Feynman rule (4.15). We have found fully modified expression for tadpole in Fig. 2 in the limit $\epsilon \to 0$, where the genuine $1/\epsilon$ (UV) divergence is explicitly isolated. For conserved external momenta, i.e. for $k_1 = k_4 \equiv k$, we obtain two-point function (4.39), where the finite parts represent the modification of the scalar field self-energy $\Pi_2^{a,\xi}$ and depend explicitly on the regularization parameter μ^2, the mass of the scalar field m^2, and the magnitude of translation invariance breaking ξ^2. Most important is that (4.39) contains the finite correction ak due to the deformed statistics on κ-Minkowski spacetime, thus, via ak term we obtain an explicit dependence on the scale of propagating energy involved $|k| = E$, and the κ-deformation parameter a, as we expected.

(ix) Two-point function (4.39) is next applied in the framework of *two-point connected* Green's function for three energy regimes, that is for low energies, for Planck scale energies, and for middle size energies respectively. For low energy scale and/or small κ-deformation a, i.e. for $ak \simeq 0$, which is far away from the point $(1 + 3ak = 0)$, κ-deformation dependence of two-point Green's function completely drops out (4.48). Genuine UV divergence in (4.39) as well as spurious δ-function term in (4.3) have been removed by subtracting counterterm δm^2 (4.42), from previous contribution (4.40), or through shifting m^2 into $(m^2 + \delta m^2)$ in (4.41). In this case mass shift (4.48) could increase or decrease m^2 depending on values of function f and/or parameter ξ^2.
(x) For energies within limits $\frac{-1}{3a_3} << E << 0$, the full expression (4.39), with mass the counterterm (4.42) has to be used in determining the Green's function (4.43). In that particular case the harmonic oscillator term in (4.39) will also give a non-negligible contribution. It's coupling ξ^2 could be in principle determined via higher order contributions to the Green's function, which is certainly an issue to be addressed in future work.
(xi) At Planckian energy scale, due to the existence of linear type of limits $(1 + 3ak) \to 0$, we have new situation and, distinguish two cases. They both are new, previously unknown, features of linear order in a NC κ-Minkowski ϕ^4 theory. In the first case we have limit (4.50) which produces self-energy and/or modified Green's function (4.52).
(xii) Second case, that is exact zero-point $1 + 3Ea_3 = 0$ case, represents in fact genuine type of zero-point which exactly removes genuine UV divergence $1/\epsilon$, giving self-energy and/or modified Green's function (4.53). In both cases the mass term is shifted in the same direction, (*the same sign!*), but for different amount, $+11/4$ versus $+3/4$ respectively. Or more precisely we can say that mass shift during the limiting process $(3Ea_3 + 1 \to 0)$ drops from the value proportional to $+11/4$ to exact value proportional to $+3/4$.
(xiii) The results (4.52) and (4.53) are the same for two different choices of κ-noncommutativity, (with appropriate choice of referent system for momentum k_μ), i.e. for $a_\mu = (0, 0, a_3, 0)$ and $a_\mu = (0, 0, 0, ia_0)$ respectively, since (4.51) and (4.54) are equivalent.
(xiv) At Planckian propagation energy scale $E \simeq \frac{-1}{3a_3}$ contribution of tadpole in Fig. 2 tends to finite fixed value, between (4.52) and (4.53), respectively. That, due to effects of κ-Minkowski statistics, depends only on Planckian propagation energy and/or κ-deformation parameter a. This way (4.51), (4.52), (4.53) and (4.54) represents κ-deformed dispersion relations, producing birefringence, [58, 59], of the massive scalar field mode, which arise as genuine effect at the first order in κ-deformation parameter a. It is similar to the chiral fermion field birefringence in truncated Moyal $\star$-product theories [59].
(xv) Considering full renormalization, besides δm^2 counterterm, the other divergent parts have to be added as counterterms to free Lagrangian (3.12) as well:

$$\int d^4x \, (\mathcal{L} + \mathcal{L}_{ct}) = S[\phi_B, m_B, \lambda_B, \xi_B, a_B] \,,$$

where index B denotes bare quantities. That would include analisys of $4-$*point* one-loop contributions, counterterms $(\mu^2)^{2-\frac{n}{2}}\delta\lambda$ and $(\mu^2)^{4-\frac{n}{2}}\delta\xi$, as well as 2-loop expansion for the 2-point Green's function with insertion of counterterms in multi-loop diagrams. Certainly, the full analysis of the renormalization group equations is also under the same schedule.

However, the full renormalization of our action (3.12) is anyhow beyond the scope of this paper and it is planned for our next project.

Regarding the effects of statistics according to the described arguments, important is to repeat that within first order in κ-deformation a, *statistics effects on κ-Minkowski in our hybrid model do arrises as semiclassical/hybrid behavior of first order quantum effects, thus showing birefringence of the massive scalar field mode.* We believe that this property of such a constructed model, is of importance for further possible research towards quantum gravity. At higher orders in a the matter would become growingly interesting and complicated.

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