Generic Ising trees

Bergfinnur Durhuus and George M. Napolitano

Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, 2100 Copenhagen Ø, Denmark

Abstract

The Ising model on an infinite generic tree is defined as a thermodynamic limit of finite systems. A detailed description of the corresponding distribution of infinite spin configurations is given. As an application we study the magnetization properties of such systems and prove that they exhibit no spontaneous magnetization. Furthermore, the values of the Hausdorff and spectral dimensions of the underlying trees are calculated and found to be, respectively, $\bar{d}_h = 2$ and $\bar{d}_s = 4/3$.

Contents

1 Introduction

Since its appearance, the Ising model has been considered in various geometrical backgrounds. Most familiar are the regular lattices, where it is well known that in dimension $d = 1$, originally considered by Ising and Lenz [18, 22], there is no phase transition as opposed to dimension $d \geq 2$, where spontaneous magnetization occurs at sufficiently low temperature [24, 25].

The Ising model on a Cayley tree turns out to be exactly solvable [9, 15, 23]. Despite the fact that the free energy, in this case, is an analytic function of the temperature at vanishing magnetic field, the model does have a phase transition and exhibits spontaneous magnetization at a central vertex. One may attribute this unusual behavior to the large size of the boundary of a ball in the tree as compared to its volume.

Studies of the Ising model on non-regular graphs are generally non-tractable from an analytic point of view. For numerical studies see e.g. [4]. See also [6], where the Ising model with external field coupled to the causal dynamical triangulation model is studied via high- and low-temperature expansion techniques. In [2] a grand canonical ensemble of Ising models on random finite trees was considered, motivated by studies in two dimensional quantum gravity [1]. It was argued in [2] that the model does not exhibit spontaneous magnetization at values of the fugacity where the mean size of the trees diverges.

In the present paper we study the Ising model on certain infinite random trees, constructed as "thermodynamic" limits of Ising systems on random finite trees. These are subject to a certain genericity condition for which reason we call them *generic Ising trees*. Using tools developed in [11, 13] we prove for such ensembles that spontaneous magnetization is absent. The basic reason is that the generic infinite tree has a certain one dimensional feature despite the fact that we prove its Hausdorff dimension to be 2. Furthermore, we obtain results on the spectral dimension of the generic Ising trees.

This article is organized as follows. After a brief review of some basic graph theoretic notions that will be used throughout the article and fixing some notation we define, in Section 2, the finite size systems whose infinite size limits are our main object of study. The remainder of Section 2 is devoted to an overview of the main results, including the existence and detailed description of the infinite size limit, the magnetization properties and the determination of the annealed Hausdorff and spectral dimensions of the generic Ising trees.

The next two sections provide detailed proofs and, in some cases, more precise statements of those results. Under the genericity assumption mentioned above we determine, in Section 3, the asymptotic behavior of the partition functions of ensembles of spin systems on finite trees of large size. This allows a construction of the limiting distribution on infinite trees and also leads to a precise description of the limit. In Section 4 we exploit the latter characterization to determine the annealed Hausdorff and spectral dimensions of the generic Ising trees, whereafter we establish absence of magnetization in Section 5.

Finally, some concluding remarks on possible future developments are collected in Section 6.

2 Definition of the models and main results

2.1 Basic definitions

Recall that a *graph* G is specified by its *vertex* set $V(G)$ and its *edge* set $E(G)$. Vertices will be denoted by v or v_i etc. An edge is then an unordered pair (v, v') of different vertices. Both finite and infinite graphs will be considered, i.e. $V(G)$ may be finite or infinite, and all graphs will be assumed to be locally finite, i. e. the number σ_v of edges containing a vertex v, called the *degree* of v, is finite for all $v \in V(G)$. By the *size* of G we shall mean the number of edges in G and denote it by $|G|$, i.e. $|G| = \sharp E(G)$, where $\sharp M$ is used to denote the number of elements in a set M.

A *path* in G is a sequence of different edges $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$ where v_0 and v_k are called the *end vertices.* If $v_0 = v_k$ the path is called a *circuit* originating at v_0. The graph G is called *connected* if any two vertices v and v' of G can be connected by a path, i. e. they are end vertices of a path. The *graph distance* between v and v' is then defined as the minimal number of edges in a path connecting them. A connected graph is called a *tree* if it has no circuits.

Given a connected graph G and $R \geq 0$ and $v \in V(G)$ we denote by $B_R(G, v)$ the *closed ball* of radius R centered at v, i.e. $B_R(G, v)$ is the subgraph of G spanned by the vertices at graph distance $\leq R$ from v.

A *planar graph* is a graph together with an embedding $\phi : V(G) \to \mathbb{R}^2$ and an association to each edge $(v, v') \in E(G)$ of an arc $\psi(v, v')$ in $\mathbb{R}^2$ connecting $\phi(v)$ and $\phi(v')$ such that arcs corresponding to different edges are disjoint except possibly for endpoints. Two planar graphs are considered identical if one can be continuously deformed into the other in $\mathbb{R}^2$.

A *planar tree* is a planar connected graph without circuits. In the following we will often refer to planar trees simply as *trees.*

In this paper we consider planar *rooted* trees, where rooted means that they contain a distinguished oriented edge $e =< r, r' >$, called the *root edge*, and whose initial vertex r is called the *root vertex.* Further, we assume the root r to be of degree 1. We denote by $\mathcal{T}$ the set of such trees, by $\mathcal{T}_N$ the subset of $\mathcal{T}$ of trees of size N and by $\mathcal{T}_\infty$ the subset of infinite trees, such that

$$\mathcal{T} = \left(\bigcup_{N=1}^{\infty} \mathcal{T}_N \right) \cup \mathcal{T}_\infty. \tag{1}$$

The *height* of a finite tree is the maximal distance from the root to one of its vertices.

The set $\mathcal{T}$ is a metric space with the distance between two trees τ and τ' defined by

$$\tilde{d}(\tau, \tau') = \inf \left\{ \frac{1}{R+1} \mid B_R(\tau) = B_R(\tau') \right\}, \tag{2}$$

where $B_R(\tau)$ denotes the ball of radius R centered at the root r, i.e. $B_R(\tau) \equiv B_R(\tau, r)$. See [11] for further details on properties of $\tilde{d}$. In particular, $\tilde{d}$ is an ultrametric, i.e.

$$\tilde{d}(\tau, \tau') \leq \max \left\{ \tilde{d}(\tau, \tau''), \tilde{d}(\tau', \tau'') \right\}$$

for all τ, τ'', $\tau'' \in \mathcal{T}$.

2.2 The models and the thermodynamic limit

The statistical mechanical models considered in this paper are defined in terms of planar trees as follows. Let Λ_N be the set of rooted planar trees of size N decorated with Ising spin configurations,

$$\Lambda_N = \left\{ s : V(\tau) \to \{\pm 1\} \mid \tau \in \mathcal{T}_N \right\}, \tag{3}$$

and set

$$\Lambda = \left(\bigcup_{N=1}^{\infty} \Lambda_N \right) \cup \Lambda_\infty, \tag{4}$$

where Λ_∞ denotes the set of infinite decorated trees. In the following we will often denote by τ_s a generic element of Λ, in particular when stressing the underlying tree structure τ of the spin configuration s. Furthermore, we shall use both s_v and $s(v)$ to denote the value of the spin at vertex v.

The set Λ is a metric space with metric d defined by

$$d(\tau_s, \tau'_{s'}) = \inf \left\{ \frac{1}{R+1} \mid B_R(\tau) = B_R(\tau'),\, s|_{B_R(\tau)} = s'|_{B_R(\tau')} \right\}, \tag{5}$$

as a generalization of (2).

We define a probability measure μ_N on Λ_N by

$$\mu_N(\tau_s) = \frac{1}{Z_N} e^{-H(\tau_s)} \rho(\tau), \quad \tau_s \in \Lambda_N, \tag{6}$$

where the Hamiltonian $H(\tau_s)$, describing the interaction of each spin with its neighbors and with the constant external magnetic field h at inverse temperature β, is given by

$$H(\tau_s) = -\beta \sum_{(v_i, v_j) \in E(\tau)} s_{v_i} s_{v_j} - h \sum_{v_i \in V(\tau) \setminus r} s_{v_i}. \tag{7}$$

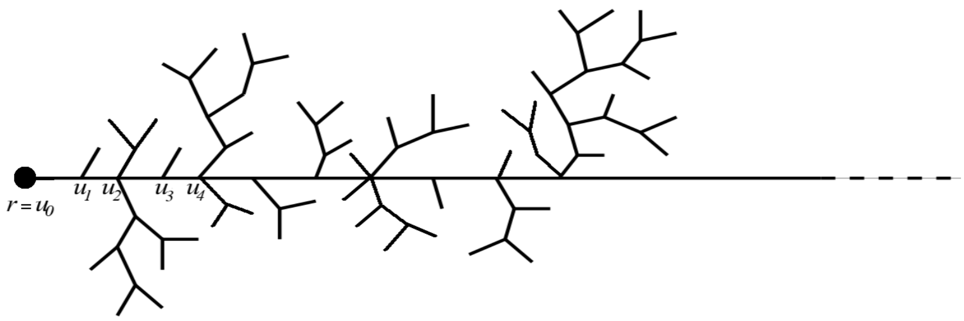

Figure 1: Example of an infinite tree, consisting of a spine and left and right branches.

The weight function $\rho(\tau)$ is defined in terms of the *branching weights* $p_{\sigma_v - 1}$ associated to vertices $v \in V(\tau) \setminus r$, and is given by

$$\rho(\tau) = \prod_{v \in V(\tau) \setminus r} p_{\sigma_v - 1}. \tag{8}$$

Here $(p_n)_{n \geq 0}$ is a sequence of non-negative numbers such that $p_0 \neq 0$ and $p_n \neq 0$ for some $n \geq 2$ (otherwise only linear chains would contribute). We will further assume the branching weights to satisfy a *genericity condition* explained below in (25), and which defines the *generic Ising tree ensembles* considered in this paper (see also [13]). Finally, the partition function Z_N in (6) is given by

$$Z_N(\beta, h) = \sum_{\tau \in \mathcal{T}_N} \sum_{s \in S_\tau} e^{-H(\tau_s)} \rho(\tau), \tag{9}$$

where $S_\tau = \{\pm 1\}^{V(\tau)}$.

Our first result (see Sec. 3) is to establish the existence of the thermodynamic limit of this model, in the sense that we prove the existence of a limiting probability measure $\mu = \mu^{(\beta,h)} = \lim_{N \to \infty} \mu_N$ defined on the set of trees of infinite size decorated with spin configurations. Here, the limit should be understood in the weak sense, that is

$$\int_\Lambda f(\tau_s)\, d\mu_N(\tau_s) \xrightarrow{N \to \infty} \int_\Lambda f(\tau_s)\, d\mu(\tau_s)$$

for all bounded continuous functions f on Λ. In particular, we find that the measure μ is concentrated on the set of infinite trees with a single infinite path, the *spine*, starting at the root r, and with finite trees attached to the spine vertices, the *branches*, see Fig. 1.

As will be shown, the limiting distribution μ can be expressed in explicit terms in such a way that a number of its characteristics, such as the Hausdorff dimension, the spectral dimension, as well as the magnetization properties of the spins, can be analyzed in some detail. For the reader's convenience we now give a brief account of those results.

2.3 Magnetization properties

As a first result we show that the generic Ising tree exhibits no single site spontaneous magnetization at the root r or at any other spine vertex, i. e.

$$\lim_{h\to 0} \mu^{(\beta,h)}(\{\tau_s \mid s(v) = +1\}) = \frac{1}{2},$$

for any vertex v on the spine and all $\beta \in \mathbb{R}$. Details of this result can be found in Theorem 5.2.

The fact that the measure μ is supported on trees with a single spine gives rise to an analogy with the one-dimensional Ising model. In fact, we show that the spin distribution on the spine equals that of the Ising model on the half-line at the same temperature but in a modified external magnetic field. As a consequence, we find that also the mean magnetization of the spine vanishes for $h \to 0$.

A different and perhaps more relevant result concerns the the total mean magnetization, which may be stated as follows. First, let us define the mean magnetization in the ball of radius R around the root by

$$M_R(\beta, h) = \langle |B_R(\tau)| \rangle_{\beta,h}^{-1} \left\langle \sum_{v \in B_R(\tau)} s_v \right\rangle_{\beta,h} \tag{10}$$

and the mean magnetization on the full infinite tree as

$$M(\beta, h) = \limsup_{R\to\infty} M_R(\beta, h). \tag{11}$$

For the generic Ising tree, we prove in Theorem 5.4 that this quantity satisfies

$$\lim_{h\to 0} M(\beta, h) = 0, \quad \beta \in \mathbb{R}.$$

2.4 Hausdorff dimension

Given an infinite connected graph G, if the limit

$$d_h = \lim_{R\to\infty} \frac{\ln |B_R(G, v)|}{\ln R} \tag{12}$$

exists, we call d_h the *Hausdorff dimension* of G. It is easily seen that the existence of the limit as well as its value do not depend on the vertex v.

For an ensemble of infinite graphs $\mathcal{G}_\infty$ with a probability measure ν, we define the *annealed* Hausdorff dimension by

$$\bar{d}_h = \lim_{R\to\infty} \frac{\ln \langle\, |B_R(G)|\, \rangle_\nu}{\ln R}, \tag{13}$$

provided the limit exists, where $< \cdot >_\nu$ denotes the expectation value w.r.t. ν.

We show in Theorem 4.1 that the annealed Hausdorff dimension of a generic Ising tree can be evaluated and equals that of generic random trees as introduced in [13], i.e.

$$\bar{d}_h = 2\,.$$

2.5 Spectral dimension

A *walk* on a graph G is a sequence $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ of (not necessarily different) edges in G. We shall denote such a walk by $\omega : v_0 \to v_k$ and call v_0 the *origin* and v_k the *end* of the walk. Moreover, the number k of edges in ω will be denoted by $|\omega|$. To each such walk ω we associate a weight

$$\pi_G(\omega) = \prod_{i=0}^{|\omega|-1} \sigma_{\omega(i)}^{-1}$$

where $\omega(i)$ is the i'th vertex in ω. Denoting by $\Pi_n(G, v_0)$ the set of walks of length n originating at vertex v_0 we have

$$\sum_{\omega \in \Pi_n(G,v_0)} \pi_G(\omega) = 1\,.$$

i.e. π_G defines a probability distribution on $\Pi_n(G, v_0)$. We call π_G the *simple random walk* on G.

For an infinite connected graph G and $v \in V(G)$ we denote by $\pi_t(G, v)$ the *return probability* of the simple random walk to v at time t, that is

$$\pi_t(G, v) = \sum_{\substack{\omega : v \to v \\ |\omega| = t}} \pi_G(\omega)\,.$$

One can in a standard manner relate this quantity to the discrete heat kernel on G, but we shall not need this interpretation in the following. If the limit

$$d_s = -2 \lim_{t \to \infty} \frac{\ln \pi_t(G, v)}{\ln t} \tag{14}$$

exists, we call d_s the *spectral dimension* of G. Again in this case, the existence and value of the limit are independent of v.

If G is the hyper-cubic lattice $\mathbb{Z}^d$ it is clear that $d_h = d$ and by Fourier analysis it is straight-forward to see that also $d_s = d$. However, examples of graphs with $d_h \neq d_s$ are abundant, see e.g. [12].

The *annealed* spectral dimension of an ensemble $(\mathcal{G}_\infty, \nu)$ of rooted infinite graphs is defined as

$$\bar{d}_s = -2 \lim_{t \to \infty} \frac{\ln \langle\, \pi_t(G, r)\, \rangle_\nu}{\ln t} \tag{15}$$

provided the limit exists.

We show in Theorem 4.6 that the annealed spectral dimension of a generic Ising tree is

$$\bar{d}_s = \frac{4}{3}. \tag{16}$$

The values of the Hausdorff dimension and the spectral dimension of generic Ising trees are thus found to coincide with those of generic random trees [13]. This indicates that the geometric structure of the underlying trees is not significantly influenced by the coupling to the Ising model as long as the model is generic.

3 Ensembles of infinite trees

In this section we establish the existence of the measure $\mu^{(\beta,h)}$ on the set of infinite trees for values of β, h that will be specified below. Our starting point is the Ising model on finite but large trees. We first consider the dependence of its partition function on the size of trees.

3.1 Asymptotic behavior of partition functions

Let the branching weights $(p_n)_{n\geq 0}$ be given as above and consider the generating functions

$$\varphi(z) = \sum_{n=0}^{\infty} p_n z^n, \tag{17}$$

which we assume to have radius of convergence $\xi > 0$, and

$$Z(\beta, h, g) = \sum_{N=0}^{\infty} Z_N(\beta, h) g^N, \tag{18}$$

where Z_N is given by (9).

Decomposing the set S_τ into the two disjoint sets

$$S_\tau^{\pm} = \left\{ s \in S_\tau \;\middle|\; s(r) = \pm 1 \right\}, \tag{19}$$

gives rise to the decompositions

$$\Lambda_N = \Lambda_{N+} \cup \Lambda_{N-}$$

and

$$\Lambda = \Lambda_+ \cup \Lambda_-.$$

Correspondingly, we get

$$Z(\beta, h, g) = Z_+(\beta, h, g) + Z_-(\beta, h, g),$$

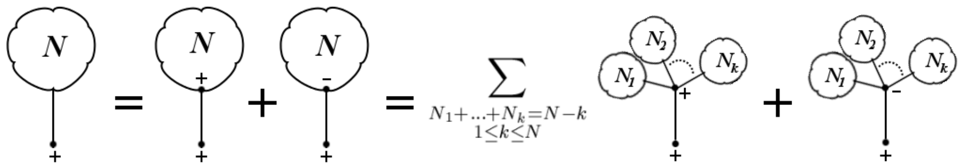

Figure 2: Decomposition of a tree of size $N+1$ with $s(r)=+1$. The tree is decomposed according to the spin and the degree of the root's neighbor.

where the generating functions $Z_\pm(\beta,h,g)$ are given by

$$Z_\pm(\beta,h,g)=\sum_{N=0}^{\infty} Z_{N\pm}(\beta,h)g^N, \tag{20}$$

and $Z_{N\pm}$ are defined by restricting the second sum in (9) to $S_\tau^\pm$.

Decomposing the tree as in Fig.2, it is easy to see that the functions $Z_\pm(g)$ are determined by the system of equations

$$\begin{cases} Z_+ = g(a\,\varphi(Z_+)+a^{-1}\,\varphi(Z_-)) \\ Z_- = g(b\,\varphi(Z_+)+b^{-1}\,\varphi(Z_-)) \end{cases} \tag{21}$$

where

$$a=e^{\beta+h}, \quad b=e^{-\beta+h}. \tag{22}$$

Let us define $F:\{|z|<\xi\}^2\times\mathbb{C}\to\mathbb{C}^2$ by

$$F(Z_+,Z_-,g)=\mathcal{Z}-g\,\Phi(Z_+,Z_-), \tag{23}$$

where,

$$\mathcal{Z}\equiv\begin{pmatrix}Z_+\\Z_-\end{pmatrix}, \qquad \Phi(Z_+,Z_-)\equiv\begin{pmatrix}a\,\varphi(Z_+)+a^{-1}\,\varphi(Z_-)\\ b\,\varphi(Z_+)+b^{-1}\,\varphi(Z_-)\end{pmatrix}. \tag{24}$$

With the assumption $\xi>0$, we have

$$\frac{\partial F}{\partial\mathcal{Z}}=\mathbb{1}-g\,\frac{\partial\Phi}{\partial\mathcal{Z}}=\mathbb{1}-g\begin{pmatrix}a\,\varphi'(Z_+) & a^{-1}\,\varphi'(Z_-)\\ b\,\varphi'(Z_+) & b^{-1}\,\varphi'(Z_-)\end{pmatrix},$$

and in particular, $F(0,0,0)=0$ and $\frac{\partial F}{\partial\mathcal{Z}}(0,0,0)=\mathbb{1}$. The holomorphic implicit function theorem (see e.g. [16], Appendix B.5 and refs. therein) implies that the fixpoint equation (21) has a unique holomorphic solution $Z_\pm(g)$ in a neighborhood of $g=0$. Let g_0 be the radius of convergence of the Taylor series of $Z_+(g)$. Since the Taylor coefficients of Z_+ are non-negative, $g=g_0$ is a singularity of $Z_+(g)$ by Pringsheim's Theorem ([16] Thm.IV.6). Setting

$$Z_+(g_0)=\lim_{g\nearrow g_0} Z_+(g)$$

we have that $Z_+(g_0) < +\infty$. In fact, if $\xi = \infty$ this follows from (21), since $\varphi(Z_+)$ increases faster than linearly at $+\infty$, assuming that $p_n > 0$ for some $n \geq 2$. If $\xi < +\infty$ we must have $Z_\pm(g_0) \leq \xi$, because otherwise there would exist $0 < g_1 < g_0$ such that $Z_+(g_1) = \xi$ and $Z_-(g_1) \leq \xi$ (or vice versa), contradicting (21) (the LHS would be analytic at g_1 and the RHS not). In particular, we also have $g_0 < +\infty$ and that g_0 equals the radius of convergence for the Taylor series of $Z_-(g)$ by (21).

The *genericity assumption* mentioned above states that

$$Z_\pm(g_0) < \xi\,, \tag{25}$$

which we shall henceforth assume is valid.

Remark 3.1. *It should be noted that, in the absence of an external magnetic field, i. e. for $h = 0$, one has $Z_+(\beta, 0, g) = Z_-(\beta, 0, g) \equiv \bar{Z}(\beta, g)$ and the system (21) determining $Z_\pm$ reduces to the single equation $\bar{Z} = 2g \cosh \beta\, \varphi(\bar{Z})$. On the other hand, this equation characterizes the random tree models considered in [13] except for a rescaling of the coupling constant g by the factor $2\cosh\beta$. It follows that the condition* (25) *can be considered as a generalization of the genericity condition introduced in [13]. For this reason, the results on the Hausdorff dimension and the spectral dimension established in this paper follow from [13] in case $h = 0$.*

Under the assumption (25), the implicit function theorem gives

$$\det\left(\mathbb{1} - g_0\, \Phi_0'\right) = 0, \tag{26}$$

where

$$\Phi_0' = \Phi'(Z_+^0, Z_-^0) = \begin{pmatrix} a\,\varphi'(Z_+^0) & a^{-1}\,\varphi'(Z_-^0) \\ b\,\varphi'(Z_+^0) & b^{-1}\,\varphi'(Z_-^0) \end{pmatrix}, \tag{27}$$

with $Z_\pm^0 = Z_\pm(g_0)$. Expanding (21) around $Z_\pm^0$ we get

$$\Delta\mathcal{Z} = \Delta g\, \Phi_0 + g_0\, \Phi_0'\, \Delta\mathcal{Z} + \frac{g_0}{2}\, \Phi_0''\, \Delta\mathcal{Z}^2 + O(\Delta\mathcal{Z}^3, \Delta g\, \Delta\mathcal{Z})\,, \tag{28}$$

where

$$\Delta\mathcal{Z}^n = \begin{pmatrix} (\Delta Z_+)^n \\ (\Delta Z_-)^n \end{pmatrix} = \begin{pmatrix} (Z_+ - Z_+^0)^n \\ (Z_- - Z_-^0)^n \end{pmatrix}, \qquad \Delta g = g - g_0, \tag{29}$$

$$\Phi_0'' = \begin{pmatrix} a\,\varphi''(Z_+^0) & a^{-1}\,\varphi''(Z_-^0) \\ b\,\varphi''(Z_+^0) & b^{-1}\,\varphi''(Z_-^0) \end{pmatrix}. \tag{30}$$

By (26), we have

$$\begin{pmatrix} c_1 & c_2 \end{pmatrix} \left(\mathbb{1} - g_0\, \Phi_0'\right) = 0, \tag{31}$$

where

$$c_1 = g_0\, b\, \varphi'(Z_+^0), \quad c_2 = 1 - g_0\, a\, \varphi'(Z_+^0). \tag{32}$$

Hence, multiplying (28) on the left by $c = (c_1\ c_2)$ gives

$$\Delta g\, c\, \Phi_0 + \frac{g_0}{2}\, c\, \Phi_0''\, \Delta \mathcal{Z}^2 + O(\Delta \mathcal{Z}^3, \Delta g\, \Delta \mathcal{Z}) = 0. \tag{33}$$

This equation, together with (28), gives

$$(\Delta Z_\pm)^2 = -K_\pm\, \Delta g + o(\Delta g), \tag{34}$$

where the constants $K_\pm$ (depending only on β and h) are given by

$$K_+ = \alpha^2 K_- \tag{35}$$

with

$$\alpha \equiv \frac{g_0\, a^{-1}\, \varphi'(Z_-^0)}{1 - g_0\, a\, \varphi'(Z_+^0)} = \frac{1 - g_0\, b^{-1}\, \varphi'(Z_-^0)}{g_0\, b\, \varphi'(Z_+^0)}, \tag{36}$$

where the identity follows from (26), and

$$K_- \equiv \frac{2}{g_0} \frac{\alpha\, a\, \varphi(Z_+^0) + b^{-1}\, \varphi(Z_-^0)}{\alpha^3\, a\, \varphi''(Z_+^0) + b^{-1}\, \varphi''(Z_-^0)}\,. \tag{37}$$

This proves that $Z_\pm(g)$ has a square root branch point at $g = g_0$ in the disc $\{g \mid |g| \le g_0\}$.

Remark 3.2. *The transpose of the matrix $g_0 \Phi_0'$ has positive entries and eigenvalues 1 and λ, with*

$$\lambda = \det g_0 \Phi'(Z_+^0, Z_-^0) = g_0^2 (ab^{-1} - a^{-1} b) \varphi'(Z_+^0) \varphi'(Z_-^0). \tag{38}$$

In particular, we have $\lambda < 1$ by construction and $\lambda > -1$ since

$$1 + \lambda = g_0 (a \varphi'(Z_+^0) + b^{-1} \varphi'(Z_-^0)) > 0. \tag{39}$$

Hence 1 is the Perron-Frobenius eigenvalue of the transpose of $g_0 \Phi_0'$ (cf. [16] and refs. therein) and we have $c_1, c_2 > 0$ and accordingly $\alpha > 0$.

Making further use of the implicit function theorem we next show that $Z_\pm(g)$ have extensions to a so-called Δ-domain (cf. [16]), as described by the following proposition.

Proposition 3.3. *Suppose the greatest common divisor of $\{n \mid p_n > 0\}$ is 1. Then the functions $Z_\pm(g)$ can be analytically extended to a domain*

$$D_{\epsilon,\vartheta} = \{z \mid |z| < g_0 + \epsilon,\ z \neq g_0,\ |\arg(z - g_0)| > \vartheta\} \tag{40}$$

and (34) holds in $D_{\epsilon,\vartheta}$, for some $\epsilon > 0$ and $0 \le \vartheta < \frac{\pi}{2}$.

Proof. From $\det(\mathbb{1} - g\Phi'(Z_+, Z_-))|_{g=g_0} = 0$ and $\det(\mathbb{1} - g\Phi'(Z_+, Z_-))|_{g=0} = 1$, we have

$$\det(\mathbb{1} - g\Phi'(Z_+, Z_-)) > 0, \qquad 0 \le g < g_0. \tag{41}$$

Hence

$$|\det(\mathbb{1} - g\Phi'(Z_+, Z_-))| \ge \det\big(\mathbb{1} - |g|\Phi'(Z_+(|g|), Z_-(|g|))\big) > 0 \tag{42}$$

for $|g| < g_0$, where we have used that φ and $Z_\pm$ have positive Taylor coefficients. Moreover, in the limiting case $|g| = g_0$ we get that $\det(\mathbb{1} - g\Phi'(Z_+, Z_-)) = 0$ if and only if

$$g\varphi'(Z_\pm(g)) = g_0\varphi'(Z_\pm(g_0)). \tag{43}$$

In particular, $|\varphi'(Z_\pm(g))| = \varphi'(Z_\pm(g_0))$ which implies

$$|Z_\pm(g)| = Z_\pm(g_0). \tag{44}$$

By the definition of $Z_{N\pm}(\beta, h)$ we have that $Z_{N\pm}(\beta, h) > 0$ for all N of the form

$$N = 1 + n_1 + n_2 + \cdots + n_s, \tag{45}$$

where n_i are such that $p_{n_i} > 0$, $i = 1, \ldots, s$. Hence, eq. (44) implies

$$g^N = e^{i\theta} g_0^N \tag{46}$$

for some fixed $\theta \in \mathbb{R}$ and all such N. By the assumption on (p_n) this implies $g = g_0$. This proves that the functions $Z_\pm(g)$ can be analytically extended beyond the boundary of the disc $\big\{g \;\big|\; |g| \le g_0\big\}$, except at g_0.

It remains to show that

$$\det(\mathbb{1} - g\Phi'(Z_+, Z_-)) \ne 0 \tag{47}$$

for $0 < |g - g_0| < \epsilon$ for some $\epsilon > 0$, since this together with the implicit function theorem proves the claim with $\vartheta = 0$. By (34) it suffices to show

$$\begin{aligned} &\frac{\partial}{\partial Z_+}\det(\mathbb{1} - g\Phi'(Z_+, Z_-))\bigg|_{Z_\pm^0} \sqrt{K_+} \\ &\quad + \frac{\partial}{\partial Z_-}\det(\mathbb{1} - g\Phi'(Z_+, Z_-))\bigg|_{Z_\pm^0} \sqrt{K_-} \ne 0. \end{aligned} \tag{48}$$

The LHS equals

$$\begin{aligned} &\Big[- g_0 a\varphi''(Z_+^0)\,(1 - g_0 b^{-1}\varphi'(Z_-^0)) - g_0^2 a^{-1} b\varphi''(Z_+^0)\varphi'(Z_-^0)\Big]\sqrt{K_+} \\ &+ \Big[- g_0 b^{-1}\varphi''(Z_-^0)\,(1 - g_0 a\varphi'(Z_+^0)) - g_0^2 a^{-1} b\varphi'(Z_+^0)\varphi''(Z_-^0)\Big]\sqrt{K_-} \end{aligned} \tag{49}$$

which obviously is < 0. The reader may also consult [10] for a general theorem on the asymptotic behavior of solutions to systems of functional equations of the type considered here. $\square$

The above result allows us to use a standard transfer theorem [16] to determine the asymptotic behavior of $Z_{N\pm}(\beta, h)$ for $N \to \infty$. We state it as follows.

Corollary 3.4. *Under the assumptions of Proposition 3.3, we have*

$$Z_{N\pm}(\beta, h) = \frac{1}{2}\sqrt{\frac{g_0 K_\pm}{\pi}}\, g_0^{-N}\, N^{-3/2}(1 + o(1)) \tag{50}$$

for $N \to \infty$, where g_0, $K_\pm > 0$ are determined by (21), (26), and (34)-(37).

3.2 The limiting measure

For $1 \le N < \infty$ and fixed β, $h \in \mathbb{R}$ we define the probability distributions $\mu_{N\pm}$ on $\Lambda_{N\pm} \subset \Lambda$ by

$$\mu_{N\pm}(\tau_s) = \frac{1}{Z_{N\pm}} e^{-H(\tau_s)}, \tag{51}$$

such that

$$\mu_N = \frac{Z_{N+}}{Z_N}\mu_{N+} + \frac{Z_{N-}}{Z_N}\mu_{N-}. \tag{52}$$

We shall need the following proposition, that can be obtained by a slight modification of the proof of Proposition 3.2 in [11], and whose details we omit.

Proposition 3.5. *Let K_R, $R \in \mathbb{N}$, be a sequence of positive numbers. Then the subset*

$$C = \bigcap_{r=1}^{\infty} \left\{\tau_s \in \Lambda \;\middle|\; |B_R(\tau)| \le K_R\right\} \tag{53}$$

of Λ is compact.

We are now ready to prove the following main result of this section.

Theorem 3.6. *Let $\beta, h \in \mathbb{R}$ and assume that the genericity condition (25) holds and that the greatest common divisor of $\left\{n \;\middle|\; p_n > 0\right\}$ is 1. Then the weak limits*

$$\mu_\pm = \lim_{N\to\infty} \mu_{N\pm} \quad \textit{and} \quad \mu = \lim_{N\to\infty} \mu_N \tag{54}$$

exist as probability measures on Λ and

$$\mu = \frac{\alpha}{1+\alpha}\,\mu_+ + \frac{1}{1+\alpha}\,\mu_-, \tag{55}$$

where α is given by (36).

Proof. The identity (55) follows immediately from (52), Corollary 3.4 and (35), provided $\mu_\pm$ exist. Hence it suffices to show that μ_+ exists (since existence of μ_- follows by identical arguments).

According to [11], it is sufficient to prove that the sequence (μ_{N+}) satisfies a certain *tightness condition* (see e.g. [7] for a definition) and that the sequence

$$\mu_{N+}(\{\tau_s \mid B_R(\tau) = \hat{\tau}, s|_{V(\hat{\tau})} = \hat{s}\}) \tag{56}$$

is convergent in $\mathbb{R}$ as $N \to \infty$, for each finite tree $\hat{\tau} \in \mathcal{T}$ and fixed spin configuration $\hat{s}$.

Tightness of (μ_{N+}): As a consequence of Proposition 3.5, this condition holds if we show that for each $\epsilon > 0$ and $R \in \mathbb{N}$ there exists $K_R > 0$ such that

$$\mu_{N+}(\{\tau_s \mid |B_R(\tau)| > K_R\}) < \epsilon, \quad N \in \mathbb{N}. \tag{57}$$

For $R = 1$ this is trivial. For $R = 2$, $k \geq 1$ we have

$$\begin{aligned}
&\mu_{N+}(\{\tau_s \mid |B_2(\tau)| = k+1\}) \\
&\quad = Z_{N+}^{-1} \sum_{N_1+\cdots+N_k=N-1} \left[a \prod_{i=1}^{k} Z_{N_i+} + a^{-1} \prod_{i=1}^{k} Z_{N_i-} \right] p_k \\
&\quad \leq k \sum_{\substack{N_1+\cdots+N_k=N-1 \\ N_1 \geq (N-1)/k}} Z_{N+}^{-1} \left[a \prod_{i=1}^{k} Z_{N_i+} + a^{-1} \prod_{i=1}^{k} Z_{N_i-} \right] p_k \\
&\quad \leq cst.\, k^{5/2} \left[Z_+(g_0)^{k-1} + Z_-(g_0)^{k-1} \right] p_k \,,
\end{aligned} \tag{58}$$

where we have used (50). The last expression tends to zero for $k \to \infty$ as a consequence of (25). This proves (57) for $R = 2$.

For $R > 2$ it is sufficient to show

$$\mu_{N+}(\{\tau_s \mid |B_{R+1}(\tau)| > K, B_R(\tau) = \hat{\tau}, s|_{V(\hat{\tau})} = \hat{s}\}) \to 0 \tag{59}$$

uniformly in N for $k \to \infty$, for fixed $\hat{\tau}$ of height R and fixed $\hat{s} \in \{\pm 1\}^{V(\hat{\tau})}$, as well as fixed $K > 0$. Let L denote the number of vertices in $\hat{\tau}$ at maximal height R. Any $\tau \in \Lambda$ with $B_R(\tau) = \hat{\tau}$ is obtained by attaching a sequence of trees $\tau_1, \ldots, \tau_S$ in Λ such that the root vertex of τ_i is identified with a vertex at maximal height in $\hat{\tau}$. We must then have

$$|\tau_1| + \cdots + |\tau_S| = |\tau| - |\hat{\tau}| \tag{60}$$

and

$$k_1 + \cdots + k_L = S, \tag{61}$$

where $k_i \geq 0$ denotes the number of trees attached to vertex v_i in $\hat{\tau}$, $i = 1, \dots, L$. For fixed $k_1, \dots, k_L$ we get a contribution to (59) equal to

$$
\begin{aligned}
& Z_{N+}^{-1} \sum_{N_1+\cdots+N_S=N-|\hat{\tau}|} \left(\prod_{i=1}^{L} (Z_{N_i \hat{s}_{v_i}})^{k_i} p_{k_i} \right) e^{-H(\hat{\tau}_{\hat{s}})} \prod_{v \in V(\hat{\tau}) \setminus \{r, v_1, \dots, v_L\}} p_{\sigma_v - 1} \\
& \quad \leq cst. \prod_{i=1}^{L} (\max Z_{\pm}^0)^{k_i} p_{k_i} \, (k_i + 1)^{5/2}
\end{aligned}
\tag{62}
$$

where the inequality is obtained as above for $R = 2$ and the constant is independent of $k_1, \dots, k_L$.

Since

$$
|B_{R+1}(\tau)| = |\hat{\tau}| + k_1 + \cdots + k_L > K \tag{63}
$$

and the number of choices of $k_1, \dots, k_L \geq 0$ for fixed $k = k_1 + \cdots + k_L$ equals

$$
\binom{k+L-1}{L-1} \leq \frac{k^{L-1}}{(L-1)!} \tag{64}
$$

the claim (59) follows from (25) and (62).

Convergence of $\mu_{N+}(\{\tau_s \mid B_R(\tau) = \hat{\tau}, s|_{V(\hat{\tau})} = \hat{s}\})$: Using the decomposition of τ into $\hat{\tau}$ with branches described above and using the arguments in the last part of the proof of Theorem 3.3 in [11] we get, with notation as above, that

$$
\begin{aligned}
& \mu_{N\pm}(\{\tau_s \mid B_R(\tau) = \hat{\tau}, s|_{V(\hat{\tau})} = \hat{s}\}) \\
& \quad \xrightarrow{N \to \infty} \frac{g_0^{|\hat{\tau}|}}{\sqrt{K_\pm}} e^{-H(\hat{\tau}_{\hat{s}})} \sum_{i=1}^{L} \sqrt{K_{\hat{s}(v_i)}} \varphi'(Z^0_{\hat{s}(v_i)}) \prod_{j \neq i} \varphi(Z^0_{\hat{s}(v_j)}),
\end{aligned}
\tag{65}
$$

provided $\hat{s}(r) = \pm 1$ (if $\hat{s}(r) = \mp 1$ the limit is trivially 0). $\square$

Introducing the notation

$$
A(\hat{s}) = \{\tau_s \mid B_R(\tau) = \hat{\tau}, \, s|_{V(\hat{\tau})} = \hat{s}\} \ ,
$$

where $\hat{\tau}$ is a finite tree of height R with spin configuration $\hat{s}$, and using (35), it follows from (65) that the $\mu_\pm$-volumes of this set are given by

$$
\mu_\pm(A(\hat{s})) = g_0^{|\hat{\tau}|} \, e^{-H(\hat{\tau}_{\hat{s}})} \sum_{i=1}^{L} \alpha^{(\hat{s}(v_i) \mp 1)/2} \, \varphi'(Z^0_{\hat{s}(v_i)}) \prod_{j \neq i} \varphi(Z^0_{\hat{s}(v_j)}),
$$

if $\hat{s}(r) = \pm 1$ and where $v_1, \dots, v_L$ are the vertices at maximal distance from the root r in $\hat{\tau}$.

The above calculations show, by similar arguments as in [11, 8], that the limiting measures $\mu_\pm$ are concentrated on trees with a single infinite path starting at r, called the *spine*, and attached to each spine vertex u_i, $i = 1, 2, 3\dots$, is a finite number k_i of finite trees, called *branches*, some of which are attached to the left and some to the right as seen from the root, cf. Fig.1.

The following corollary provides a complete description of the limiting measures $\mu_\pm$.

Corollary 3.7. *The measures $\mu_\pm$ are concentrated on the sets*

$$\bar{\Lambda}_\pm = \left\{\tau_s \in \Lambda_\pm \;\middle|\; \tau \text{ has a single spine}\right\},$$

respectively, and can be described as follows:

i) The probability that the spine vertices $u_0 = r,\ u_1, u_2, \dots, u_N$ have $k'_1, \dots, k'_N$ left branches and $k''_1, \dots, k''_N$ right branches and spin values $s_0 = \pm 1,\ s_1, s_2, \dots, s_N$, respectively, equals

$$\rho^{s_0}_{k'_1,\dots,k'_N,k''_1,\dots,k''_N}(s_0, \dots, s_N) = g_0^N e^{-H_N} \left(\prod_{i=1}^{N} (Z^0_{s_i})^{k'_i+k''_i} p_{k'_i+k''_i+1}\right) \alpha^{(s_N-s_0)/2}, \tag{66}$$

with

$$H_N = -\beta \sum_{i=1}^{N} s_{i-i}s_i - h \sum_{i=1}^{N} s_i.$$

ii) The conditional probability distribution of any finite branch τ_s at a fixed u_i, $1 \le i \le N$, given $k'_1, \dots, k'_N$, $k''_1, \dots, k''_N$, $s_0, \dots, s_N$ as above, is given by

$$\nu_{s_i}(\tau_s) = (Z^0_{s_i})^{-1}\, g_0^{|\tau|}\, e^{-H(\tau_s)} \prod_{v \in V(\tau)\setminus u_i} p_{\sigma_v - 1} \tag{67}$$

for $s(u_i) = s_i$, and 0 otherwise.

iii) The conditional distribution of the infinite branch at u_N, given $k'_1, \dots$, k'_N, $k''_1, \dots, k''_N$, $s_0, \dots, s_N$, equals μ_{s_N}.

4 Hausdorff and spectral dimensions

In this section we determine the values of the Hausdorff and spectral dimensions of the ensemble of trees $(\mathcal{T}, \bar{\mu})$ obtained from (Λ, μ) by integrating over the spin degrees of freedom, that is

$$\bar{\mu}(A) = \mu(\{\tau_s \mid \tau \in A\})$$

for $A \subseteq \mathcal{T}$. Note that the mapping $\tau_s \to \tau$ from Λ to $\mathcal{T}$ is a contraction w. r. t. the metrics (5) and (2).

Most of the arguments in this section are based on the methods of [13], and we shall mainly focus on the novel ingredients that are needed and otherwise refer to [13] for additional details.

4.1 The annealed Hausdorff dimension

Theorem 4.1. *Under the assumptions of Theorem 3.6 the annealed Hausdorff dimension of $\bar{\mu}$ is 2 for all β, h:*

$$\bar{d}_h = \lim_{R\to\infty} \frac{\ln \langle |B_R| \rangle_{\bar{\mu}}}{\ln R} = 2\,.$$

Proof. Consider the probability distribution $\nu_\pm$ on $\{\tau_s \mid \tau \text{ is finite}\}$ given by (67) and denote by $D_R(\tau)$ the set of vertices at distance R from the root in τ. For a fixed branch T, we set

$$f_R^\pm = \langle |D_R| \rangle_{\nu_\pm}\, Z_\pm^0.$$

where $\langle \cdot \rangle_{\nu_\pm}$ denotes the expectation value w.r.t. $\nu_\pm$. Arguing as in the derivation of (21), we find

$$\begin{cases} f_R^+ = g_0\,\left(a\,\varphi'(Z_+^0)\,f_{R-1}^+ + a^{-1}\,\varphi'(Z_-^0)\,f_{R-1}^-\right) \\ f_R^- = g_0\,\left(b\,\varphi'(Z_+^0)\,f_{R-1}^+ + b^{-1}\,\varphi'(Z_-^0)\,f_{R-1}^-\right)\,, \end{cases}$$

for $R \geq 2$, and $f_1^\pm = Z_\pm^0$. Using that c, given by (32), is a left eigenvector of $g_0 \Phi_0'$ with eigenvalue 1, this implies

$$\begin{aligned} c_1\,f_R^+ + c_2\,f_R^- &= c_1\,f_{R-1}^+ + c_2\,f_{R-1}^- = \dots \\ &= c_1\,f_1^+ + c_2\,f_1^- = c_1\,Z_+^0 + c_2\,Z_-^0\,. \end{aligned}$$

Since c_1, c_2, $Z_\pm^0$, $f_R^\pm > 0$, we conclude that

$$k_1 \leq \langle |D_R| \rangle_{\nu_\pm} \leq k_2, \quad R \geq 1\,, \tag{68}$$

where k_1, k_2 are positive constants (depending on β, h). Using

$$\langle |B_R| \rangle_{\nu_\pm} = \sum_{R'=0}^{R} \langle |D_{R'}| \rangle_{\nu_\pm} \tag{69}$$

we then obtain

$$1 + k_1\,R \leq \langle |B_R| \rangle_{\nu_\pm} \leq 1 + k_2\,R, \tag{70}$$

Finally, it follows from (66) that

$$1 + R + k_1\,\frac{1}{2}R(R+1) \leq \langle |B_R| \rangle_\mu \leq 1 + R + k_2\,\frac{1}{2}R(R+1)\,, \tag{71}$$

which proves the claim. □

Remark 4.2. *By a more elaborate argument, using the methods of [13, 14], one can show that the Hausdorff dimension d_h defined by* (12) *exists and equals* 2 *almost surely, that is for all trees $\tau \in \mathcal{T}$ except for a set of vanishing $\bar{\mu}$-measure. We shall not make use of this result below and refrain from giving further details in this paper.*

4.2 The annealed spectral dimension

In this section we first establish two results needed for determining the spectral dimension. The first one is a version of a classical result, proven by Kolmogorov for Galton-Watson trees [17], on survival probabilities for $\nu_\pm$.

Proposition 4.3. *The measures $\nu_\pm$ defined by (67) fulfill*

$$\frac{k_-}{R} \leq \nu_\pm(\{\tau_s \in \Lambda \mid D_R(\tau) \neq \emptyset\}) \leq \frac{k_+}{R}, \quad R \geq 1, \tag{72}$$

where $k_\pm > 0$ are constants depending on β, h.

Proof. Let $H_R^\pm(w)$ be the generating function for the distribution of $|D_R|$ w.r.t. $\nu_\pm$,

$$H_R^\pm(w) = Z_\pm^0 \sum_{n=0}^{\infty} \nu_\pm(\{\tau_s \mid |D_R(\tau)| = n\})\, w^n. \tag{73}$$

Arguing as in the proof of (21), we have

$$\begin{cases} H_R^+ = g_0 \left(a\,\varphi(H_{R-1}^+) + a^{-1}\,\varphi(H_{R-1}^-)\right) \\ H_R^- = g_0 \left(b\,\varphi(H_{R-1}^+) + b^{-1}\,\varphi(H_{R-1}^-)\right) , \end{cases} \tag{74}$$

for $R \geq 2$, and $H_1^\pm = Z_\pm^0\, w$.

Note that

$$Z_\pm^0\, \nu_\pm(\{\tau_s \in \Lambda \mid D_R(\tau) \neq \emptyset\}) = Z_\pm^0 - H_R^\pm(0), \tag{75}$$

and that the radius of convergence for $H_R^\pm$ is ≥ 1. Also, $(H_R^\pm(0))_{R\geq 1}$ is an increasing sequence. In fact, $H_1^\pm(0) = 0$ and so $H_2^\pm(0) > 0$ by (74). Since φ is positive and increasing on $[0, \xi)$, it then follows by induction from (74) that $(H_R^\pm(0))_{R\geq 1}$ is increasing. Hence, we conclude from (74) and (21) that

$$H_R^\pm(0) \nearrow Z_\pm^0 \quad \text{for} \quad R \to \infty. \tag{76}$$

Taking R large enough and expanding $\varphi(H_R^\pm(0))$ around $Z_\pm^0$ we obtain, in matrix form,

$$\Delta_R = g_0\, \Phi_0'\, \Delta_{R-1} - \frac{g_0}{2}\, \Phi_0''\, \Delta_{R-1}^2 + O(\Delta_{R-1}^3)\,, \tag{77}$$

where

$$\Delta_R^n = \begin{pmatrix} (\Delta_R^+)^n \\ (\Delta_R^-)^n \end{pmatrix} = \begin{pmatrix} (Z_+^0 - H_R^+(0))^n \\ (Z_+^0 - H_R^+(0))^n \end{pmatrix}, \tag{78}$$

and where Φ_0', Φ_0'' are given by (27) and (30). Setting $L_R = c\,\Delta_R$, eq. (77) gives

$$L_R = L_{R-1} - \frac{g_0}{2}\,c\,\Phi_0''\,\Delta_{R-1}^2 + O(\Delta_{R-1}^3)\,. \tag{79}$$

From this we deduce that there exists $R_0 > 0$ such that

$$L_{R-1} - A_- L_{R-1}^2 \le L_R \le L_{R-1} - A_+ L_{R-1}^2, \qquad R \ge R_0, \tag{80}$$

where $A_\pm = A_\pm(\beta, h)$ are constants. Hence, it follows that

$$\frac{1}{L_{R-1}} + B_- \le \frac{1}{L_{R-1}}\frac{1}{1 - A_- L_{R-1}} \le \frac{1}{L_R} \le \frac{1}{L_{R-1}}\frac{1}{1 - A_+ L_{R-1}} \le \frac{1}{L_{R-1}} + B_+,$$

for $R \ge R_0$, where $B_\pm > 0$ are constants. This implies

$$B_- R + C_- \le \frac{1}{L_R} \le B_+ R + C_+ \tag{81}$$

for suitable constants $C_\pm$. Evidently, this proves that

$$\frac{D_-}{R} \le Z_\pm^0 - H_R^\pm(0) \le \frac{D_+}{R}, \quad R \ge 1, \tag{82}$$

where $D_\pm > 0$ are constants, which together with (75) proves the claim. $\square$

We also note the following generalization of Lemma 4 in [13].

Lemma 4.4. *Suppose $u : \Lambda \to \mathbb{C}$ is a bounded function depending only on $\tau_s \in \Lambda$ through the ball $B_R(\tau)$ and the spins in $B_R(\tau)$, except those on its boundary, for some $R \ge 1$. Moreover, define the function $E_R : \Lambda \to \mathbb{R}$ by*

$$E_R(\tau_s) = \sum_{v \in D_R(\tau)} \frac{\sqrt{K_{s_v}}}{Z_{s_v}^0}, \tag{83}$$

with the convention $E_R(\tau_s) = 0$ if $D_R(\tau) = \emptyset$. Then

$$\int_\Lambda u(\tau_s) d\mu_\pm(\tau_s) = \frac{Z_\pm^0}{\sqrt{K_\pm}} \int_\Lambda u(\tau_s) E_R(\tau_s) d\nu_\pm(\tau_s). \tag{84}$$

Proof. Using (66-67) we may evaluate the LHS of (84) and get

$$\sum_{\tau_s \in \Lambda(R)} u(\tau_s)\, g_0^{|\tau|}\, e^{-H(\tau_s)}\, \alpha^{(s(v_R) - s_0)/2} \prod_{v \in V(\tau) \setminus r} p_{\sigma_v - 1}\,, \tag{85}$$

where $\Lambda(R)$ denotes the set of finite rooted trees in Λ with one marked vertex w_R of degree 1 at distance R from the root, and v_R is the neighbor of w_R.

On the other hand, the integral on the RHS can be written as

$$\frac{1}{Z_\pm^0} \sum_{\tau_s \in \Lambda(R)} u(\tau_s)\, g_0^{|\tau|} e^{-H(\tau_s)} \frac{\sqrt{K_{s(v_R)}}}{Z_{s(v_R)}^0} Z_{s(v_R)}^0 \prod_{v \in V(\tau) \setminus r} p_{\sigma_v - 1}. \tag{86}$$

By comparing the two expressions the identity (84) follows. $\square$

As a consequence of this result we have the following lemma.

Lemma 4.5. *There exist constants* $c_\pm > 0$ *such that*

$$\left\langle |B_R|^{-1} \right\rangle_{\mu_\pm} \leq c_\pm R^{-2} \tag{87}$$

Proof. Define, for fixed $R \geq 1$, the function

$$u(\tau) = \begin{cases} |D_R(\tau)|^{-1} & \text{if } D_R(\tau) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{88}$$

Then $u(\tau)$ fulfills the assumptions of Lemma 4.4 for this value of R. Hence

$$\begin{aligned} \left\langle |D_R(\tau)|^{-1} \right\rangle_{\mu_\pm} &= \frac{Z^0_\pm}{\sqrt{K_\pm}} \sum_{\tau_s : D_R(\tau) \neq \emptyset} |D_R(\tau)|^{-1}\, E(\tau_s)\, e^{-H(\tau_s)} \prod_{v \in V(\tau) \setminus r} p_{\sigma_v - 1} \\ &\leq c'_\pm \sum_{\tau_s : D_R(\tau) \neq \emptyset} e^{-H(\tau_s)} \prod_{v \in V(\tau) \setminus r} p_{\sigma_v - 1} \leq \frac{c''_\pm}{R}, \end{aligned} \tag{89}$$

where Proposition 4.3 is used in the last step. Combining this fact with Jensen's inequality, we obtain

$$\begin{aligned} \left\langle |B_R|^{-1} \right\rangle_{\mu_\pm} &= \left\langle \frac{1}{|D_1| + \cdots + |D_R|} \right\rangle_{\mu_\pm} \\ &\leq R^{-1} \left\langle (|D_1||D_2| \cdots |D_R|)^{-1/R} \right\rangle_{\mu_\pm} \\ &\leq R^{-1} \prod_{i=1}^{R} \left\langle |D_i|^{-1} \right\rangle_{\mu_\pm}^{1/R} \\ &\leq c''_\pm (R!)^{-1/R} \leq c_\pm R^{-2}. \end{aligned} \tag{90}$$

□

Returning to the spectral dimension, let us define, with the notation of subsection 2.5, the generating function for return probabilities of the simple random walk on a tree τ by

$$Q_\tau(x) = \sum_{t=0}^{\infty} (1-x)^{\frac{t}{2}} \pi_t(\tau, r)\,, \tag{91}$$

and set

$$Q(x) = \langle Q_\tau(x) \rangle_{\bar{\mu}}\ . \tag{92}$$

The annealed spectral dimension as defined by (15) is related to the singular behavior of the function $Q(x)$ as follows. First, note that if $\bar{d}_s$ exists, we have

$$\langle \pi_t(\tau, r) \rangle_{\bar{\mu}} \sim t^{-\frac{\bar{d}_s}{2}}, \quad t \to \infty\,. \tag{93}$$

For $\bar{d}_s < 2$, this implies that $Q(x)$ diverges as

$$Q(x) \sim x^{-\gamma}, \quad \text{as} \quad x \to 0, \tag{94}$$

where

$$\gamma = 1 - \frac{\bar{d}_s}{2}\,. \tag{95}$$

We shall take (94) and (95) as the definition of $\bar{d}_s$ and prove (94) with $\gamma = \frac{1}{3}$ by establishing the estimates

$$\underline{c}\, x^{-1/3} \le Q(x) \le \bar{c}\, x^{-1/3} \tag{96}$$

for x sufficiently small, where $\underline{c}$ and $\bar{c}$ are positive constants, that may depend on β, h.

Theorem 4.6. *Under the assumptions of Theorem 3.6, the annealed spectral dimension of* $(\mathcal{T}, \bar{\mu})$ *is*

$$\bar{d}_s = \frac{4}{3}\,.$$

Proof. We first prove the lower bound in (96).

Let $R \ge 1$ be fixed and consider the spine vertices $u_0, u_1, \dots, u_R$ with given spin values $s_0, \dots, s_R$ and branching numbers $k'_1, \dots, k'_R$, $k''_1, \dots, k''_R \ge 0$ as in Corollary 3.7. The conditional probability that a given branch at u_j has length $\ge R$ is bounded by $\frac{c}{R}$ by Proposition 4.3. Hence, the conditional probability that at least one of the $k'_j + k''_j$ branches at u_j has height $\ge R$ is bounded by $(k'_j + k''_j)\frac{c}{R}$. Using Corollary 3.7 and summing over $k'_1, \dots, k'_R$, $k''_1, \dots, k''_R$, we get that the conditional probability q_R that at least one branch at u_j is of height $\ge R$, given $s_0, \dots, s_R$, is bounded by

$$\frac{1}{1+\alpha}\, g_0^R\, e^{-H_R} \prod_{\substack{i=1 \\ i \neq j}}^{R} \varphi'(Z^0_{s_i})\, \varphi''(Z^0_{s_j}) \alpha^{(s_R+1)/2}\, \frac{c}{R} \le \frac{c'}{R}. \tag{97}$$

Using that the distributions of the branches at different spine vertices are independent for given $s_0, \dots, s_R$, it follows that the conditional probability that no branch at $u_1, \dots, u_R$ has length $\ge R$, for given $s_0, \dots, s_R$, is bounded from below by

$$(1 - q_R)^R \ge \left(1 - \frac{c'}{R}\right)^R \ge e^{-c' + O(R^{-1})}. \tag{98}$$

Denoting this conditioned event by $\mathcal{A}_R$, it follows from Lemmas 6 and 7 in [13] that the conditional expectation of $Q_\tau(x)$, given $s_0, s_1, \ldots, s_R$, is

$$
\begin{aligned}
&\geq e^{c'+O(R^{-1})} \left\langle \left(\frac{1}{R} + Rx + \sum_{T\subset\tau}^{R} x\,|T| \right)^{-1} \right\rangle_R \\
&\geq e^{c'+O(R^{-1})} \left(\frac{1}{R} + Rx + x \left\langle \sum_{T\subset\tau}^{R} |T| \right\rangle_R \right)^{-1} .
\end{aligned}
\tag{99}
$$

Here $\langle\cdot\rangle_R$ denotes the conditional expectation value w.r.t. μ on $\mathcal{A}_R$ and $\sum_{T\subset\tau}^{R}$ the sum over all branches T of τ attached to vertices on the spine at distance $\leq R$ from the root. We have

$$
\begin{aligned}
\left\langle \sum_{T\subset\tau}^{R} |T| \right\rangle_R &= \sum_{i=1}^{R} \left\langle |B_R^i(\tau)| \right\rangle_R \\
&\leq \sum_{i=1}^{R} \mu(\mathcal{A}_R \mid s_0, \ldots, s_R)^{-1} \left\langle |B_R^i| \right\rangle_\mu \\
&\leq e^{c'+O(R^{-1})} \sum_{i=1}^{R} \langle |B_R| \rangle_{\nu_{s_i}} \leq C\,R^2,
\end{aligned}
\tag{100}
$$

where (70) is used in the last step.

This bound being independent of $s_0, \ldots, s_R$ we have proven that

$$
Q(x) \geq cst. \left(\frac{1}{R} + Rx + CR^2 x \right)^{-1} \tag{101}
$$

and consequently, choosing $R \sim x^{-\frac{1}{3}}$, it follows that

$$
Q(x) \geq \underline{c}\, x^{-\frac{1}{3}}. \tag{102}
$$

As concerns the upper bound in (96), it follows by an argument identical to the one in [13] on p.1245–50 by using Lemma 4.5. $\square$

5 Absence of spontaneous magnetization

Using the characterization of the measure $\mu^{(\beta,h)}$ established in Section 3 and that $\bar{d}_h = 2$, we are now in a position to discuss the magnetization properties of generic Ising trees in some detail. In view of the fact that the trees have a single spine, we distinguish between the magnetization on the spine and the bulk magnetization. In subsection 5.1 we show that the former can be expressed in terms of an effective Ising model on the half-line $\{0, 1, 2, \ldots\}$. The bulk magnetization is discussed in subsection 5.2

5.1 Magnetization on the spine

The following result is crucial for the subsequent discussion.

Proposition 5.1. *Under the assumptions of Theorem 3.6, the functions $Z^0_\pm$ are smooth functions of β, h.*

Proof. In Section 3.1 we have shown that $Z_\pm(\beta, h, g)$ is a solution to the equation

$$F(Z_+, Z_-, g) = 0$$

where F is defined in (23), and that

$$Z^0_\pm(\beta, h) = Z_\pm(g_0(\beta, h), \beta, h) \tag{103}$$

is a solution to

$$\begin{cases} F(Z^0_+, Z^0_-, g_0) = 0 \\ \det(\mathbb{1} - g_0\Phi'(Z^0_+, Z^0_-)) = 0\,, \end{cases} \tag{104}$$

considered as three equations determining (Z^0_+, Z^0_-, g_0) implicitly as functions of (β, h). Hence, defining $G:\ (-R, R)^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$G(Z^0_+, Z^0_-, g_0, \beta, h) = \begin{pmatrix} F(Z^0_+, Z^0_-, g_0) \\ \det(\mathbb{1} - g_0\Phi'(Z^0_+, Z^0_-)) \end{pmatrix}$$

it suffices to show that its Jacobian J w.r.t. (Z^0_+, Z^0_-, g_0) is regular at $(Z^0_+(\beta, h), Z^0_-(\beta, h), g_0(\beta, h))$. We have

$$J = \begin{pmatrix} \mathbb{1} - g_0\Phi'(Z^0_+, Z^0_-) & & -\Phi(Z^0_+, Z^0_-) \\ A_+ & A_- & B \end{pmatrix}\,,$$

where

$$A_\pm = \frac{\partial}{\partial Z^0_\pm}\det(\mathbb{1} - g_0\Phi'(Z^0_+, Z^0_-))\,, \qquad B = \frac{\partial}{\partial g_0}\det(\mathbb{1} - g_0\Phi'(Z^0_+, Z^0_-))$$

are readily calculated and equal

$$\begin{aligned} A_+ &= -g_0\, a\,\varphi''(Z^0_+)\,(1 - g_0\, b^{-1}\varphi'(Z^0_-)) - g_0^2\, a^{-1}\, b\,\varphi''(Z^0_+)\varphi'(Z^0_-)\,, \\ A_- &= -g_0\, b^{-1}\,\varphi''(Z^0_-)\,(1 - g_0\, a\varphi'(Z^0_+)) - g_0^2\, a^{-1}\, b\,\varphi'(Z^0_+)\varphi''(Z^0_-)\,, \\ B &= -a\varphi'(Z^0_+) - b^{-1}\varphi'(Z^0_-) + 2g_0\,(ab^{-1} - a^{-1}b)\,\varphi'(Z^0_+)\varphi'(Z^0_-)\,. \end{aligned}$$

Using eqs. (104) and (36), we get

$$\det J = (Z^0_+\, b\,\varphi'(Z^0_+) + g_0^{-1}\, Z^0_-\,(1 - g_0 a\varphi'(Z^0_+)))\begin{vmatrix} 1 & -\alpha \\ A_+ & A_- \end{vmatrix} < 0\,,$$

since clearly $A_\pm < 0$ and $\alpha > 0$ by Remark 3.2. This proves the claim. ☐

We can now establish the following result for the single site magnetization on the spine.

Theorem 5.2. *Under the assumptions of Theorem 3.6, the probability* $\mu^{(\beta,h)}(\{s_v = +1\})$ *is a smooth function of* β, h *for any spine vertex* v. *In particular, there is no spontaneous magnetization in the sense that*

$$\lim_{h\to 0} \mu^{(\beta,h)}(\{s_v = +1\}) = \frac{1}{2} \,. \tag{105}$$

Proof. For the root vertex r, we have by eq. (55) that

$$\mu^{(\beta,h)}(\{s(r) = +1\}) = \frac{\alpha(\beta,h)}{1+\alpha(\beta,h)} \,, \tag{106}$$

where $\alpha(\beta,h)$ is given by (36) and is a smooth function of β, h by Proposition 5.1. Hence, to verify (105) for $v = r$ it suffices to note that $\alpha(\beta,0) = 1$, since $a = b^{-1}$ and $Z^0_+ = Z^0_-$ for $h = 0$.

Now, assume $v = u_N$ is at distance N from the root, and define

$$p_{ij} = \mu_i(\{s_v = j\}) \, \frac{\alpha^{\frac{1+i}{2}}}{1+\alpha} \,, \tag{107}$$

for $i,\, j \in \{\pm 1\}$, where we use ± 1 and $\pm$ interchangeably. From eq. (66) follows that

$$\begin{aligned} \mu_{s_0}(\{s_v = s_N\}) &= \sum_{\substack{k'_i,k''_i \ge 0 \\ s_1,\ldots,s_{N-1}}} \rho^{s_0}_{k'_1,\ldots,k'_N,k''_1,\ldots,k''_N}(s_0,\ldots,s_N) \\ &= \sum_{s_1,\ldots,s_{N-1}} \prod_{i=1}^{N} g_0[\Phi'(Z^0_+,Z^0_-)]_{s_{i-1}s_i} \, \alpha^{\frac{s_N - s_0}{2}} \\ &= [(g_0\,\Phi'(Z^0_+,Z^0_-))^N]_{s_0 s_N} \, \alpha^{\frac{s_N - s_0}{2}} \,, \end{aligned} \tag{108}$$

where we have used that the matrix elements of $\Phi'(Z^0_+, Z^0_-)$ are given by

$$[\Phi'(Z^0_+,Z^0_-)]_{s_{i-1}s_i} = e^{\beta s_{i-i} s_i + h s_i} \varphi'(Z^0_{s_i}) \,. \tag{109}$$

Hence, substituting into (107) we have

$$p_{ij} = \left[(g_0 \Phi'(Z^0_+,Z^0_-))^N\right]_{ij} \frac{\alpha^{\frac{1+j}{2}}}{1+\alpha} . \tag{110}$$

By Proposition 5.1, all factors on the RHS of (110) are smooth functions of β, h, and by (55) we have

$$\mu^{(\beta,h)}(\{s_v = j\}) = p_{+j} + p_{-j} \,. \tag{111}$$

Eq. (105) is now obtained from (110) by noting again that for $h = 0$ we have $\alpha = 1$ and hence $c_1 = c_2$, which by (31) gives

$$\begin{aligned} p_{+j} + p_{-j} &= \left[(1 \quad 1)(g_0\, \Phi'(Z^0, Z^0))^N\right]_j \frac{1}{2} \\ &= (1 \quad 1)_j \frac{1}{2} = \frac{1}{2}\,. \end{aligned} \tag{112}$$

□

The preceding proof together with (66) shows that the distribution of spin variables $s_0, \ldots, s_N$ on the spine can be written in the form

$$\rho(s_0, \ldots, s_N) = e^{-H'_N(s_0,\ldots,s_N)} \left(g_0^2\, \varphi'(Z_+^0)\varphi'(Z_-^0)\right)^{N/2} \frac{\sqrt{\alpha}}{1+\alpha} \tag{113}$$

where

$$H'_N(s_0, \ldots, s_N) = -\beta \sum_{i=1}^{N} s_{i-1}s_i - h' \sum_{i=1}^{N} s_i - \frac{s_N}{2} \log \alpha \tag{114}$$

and

$$h' = h + \frac{1}{2} \ln \frac{\varphi'(Z_+^0)}{\varphi'(Z_-^0)}\,. \tag{115}$$

Since $\rho(s_0, \ldots, s_N)$ is normalized, the expectation value w.r.t. μ of a function $f(s_0, \ldots, s_{N-1})$ hence coincides with the expectation value w. r. t. the Gibbs measure of the Ising chain on $[0, N]$, with Hamiltonian given by (114) and (115). In particular, we have that the mean magnetization on the spine vanishes in the absence of an external magnetic field, since h' is a smooth function of h, by Proposition 5.1, and vanishes for $h = 0$ (see e.g. [5] for details about the 1d Ising model).

We state this result as follows.

Corollary 5.3. *Under the assumptions of Theorem 3.6, the mean magnetization on the spine vanishes as $h \to 0$, i.e.*

$$\lim_{h\to 0} \lim_{N\to\infty} \left\langle \frac{s_0 + \cdots + s_{N-1}}{N} \right\rangle_{\beta,h} = 0. \tag{116}$$

5.2 Mean magnetization

For the mean magnetization on the full infinite tree, defined in Sec. 2.3, we have the following result, which requires some additional estimates in combination with Proposition 5.1.

Theorem 5.4. *Under the assumptions of Theorem 3.6, the mean magnetization vanishes for $h \to 0$, i.e.*

$$\lim_{h\to 0} M(\beta, h) = 0\,, \quad \beta \in \mathbb{R}\,,$$

where $M(\beta, h)$ is defined by (10)-(11).

Proof. Consider the measure $\nu_\pm$ given by (67) and, for a given finite branch T, let $S_R(T)$ denote the sum of spins at distance R from the root of T. Setting

$$m_R^\pm = Z_\pm^0 \langle S_R \rangle_{\nu_\pm} \tag{117}$$

it follows, by decomposing T according to the spin and the degree of the vertex closest to the root, that

$$\begin{cases} m_R^+ = g_0\, \big(a\,\varphi'(Z_+^0)\, m_{R-1}^+ + a^{-1}\,\varphi'(Z_-^0)\, m_{R-1}^-\big) \\ m_R^- = g_0\, \big(b\,\varphi'(Z_+^0)\, m_{R-1}^+ + b^{-1}\,\varphi'(Z_-^0)\, m_{R-1}^-\big)\,, \end{cases} \tag{118}$$

for $R \geq 1$, and $m_0^\pm = \pm Z_\pm^0$. In matrix notation these recursion relations read

$$m_R = g_0 \Phi_0'\, m_{R-1}, \tag{119}$$

which, upon multiplication by the left eigenvector c of $g_0\Phi_0'$, leads to

$$c\, m_R = g_0\, c\, \Phi_0'\, m_{R-1} = c\, m_{R-1}, \tag{120}$$

and hence

$$c_1\, m_R^+ + c_2\, m_R^- = c_1\, Z_+^0 - c_2\, Z_-^0, \qquad R \geq 0\,. \tag{121}$$

Now, fix $N \geq 1$ and let $U_{R,N}$ denote the sum of all spins at distance $R \geq 1$ from the N'th spine vertex u_N in the branches attached to u_N. The conditional expectation of $U_{R,N}$, given $s_0, s_1, \dots, s_N$, then only depends on s_N, and its value is obtained from Corollary 3.7 by summing over $k_N', k_N'' \geq 0$, which yields

$$\begin{aligned} \left(\sum_{k=0}^{\infty} (Z_{s_N}^0)^k (k+1) p_{k+1}\right)^{-1} & \sum_{k=0}^{\infty} (Z_{s_N}^0)^{k-1} p_{k+1} k(k+1)\, m_R^{s_N} \\ &= \varphi'(Z_{s_N}^0)^{-1} \varphi''(Z_{s_N}^0)\, m_R^{s_N} \equiv d_R^{s_N}\,. \end{aligned} \tag{122}$$

Using the matrix representation (110) for p_{ij}, this gives

$$\langle U_{R,N} \rangle_{\beta,h} = \frac{1}{1+\alpha} \begin{pmatrix} 1 & 1 \end{pmatrix} (g_0 \Phi'(Z_+^0, Z_-^0))^N \begin{pmatrix} \alpha\, d_R^+ \\ d_R^- \end{pmatrix}\,. \tag{123}$$

As pointed out in Remark 3.2, the matrix $g_0\Phi_0'$ has a second left eigenvalue λ such that $|\lambda| < 1$. Let (e_1, e_2) be a smooth choice of eigenvectors corresponding to λ as a function of (β, h), e.g.

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} g_0 b \varphi'(Z_-^0) \\ g_0 b^{-1} \varphi'(Z_-^0) - 1 \end{pmatrix}, \tag{124}$$

and write

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + B \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\,. \tag{125}$$

From (123) we then have

$$\begin{aligned}\langle U_{R,N}\rangle_{\beta,h} &= \frac{1}{1+\alpha}\,\left(A(c_1 \quad c_2) + B\lambda^N(e_1 \quad e_2)\right)\begin{pmatrix}\alpha\, d_R^+ \\ d_R^-\end{pmatrix} \\ &= \frac{A}{1+\alpha}(c_1\alpha d_R^+ + c_2 d_R^-) + \lambda^N\frac{B}{1+\alpha}(e_1\alpha d_R^+ + e_2 d_R^-)\,,\end{aligned} \tag{126}$$

and from (the proof of) Theorem 5.2 it follows that $A \to \tilde{c}^{-1}$ and $B \to 0$ for $h \to 0$, where $\tilde{c} = c_1(\beta, 0) = c_2(\beta, 0)$.

Next, note that $|d_R^\pm|, R \geq 1$, are bounded by a constant $C_1 = C_1(\beta, h)$ as a consequence of (68), and that

$$\begin{aligned}\langle M_R(\beta,h)\rangle_{\beta,h} &\leq \langle|B_R|\rangle^{-1}_{\beta,h}\sum_{R',N\leq R}\left|\langle U_{R',N}\rangle_{\beta,h}\right| \\ &\leq C_2 R^{-2}\sum_{R',N\leq R}\left|\langle U_{R',N}\rangle_{\beta,h}\right|\end{aligned} \tag{127}$$

for some constant $C_2 = C_2(\beta, h)$ by (71). It now follows from (126) that

$$\left|\langle M_R(\beta,h)\rangle_{\beta,h}\right| \leq \frac{A\,C_2}{R(1+\alpha)}\sum_{R'=1}^{R}(c_1\alpha d_R^+ + c_2 d_R^-) + R^{-1}\,B\,C_1\,C_2\,\max\{e_1, e_2\}. \tag{128}$$

Obviously, the second term on the RHS vanishes in the limit $R \to \infty$. Rewriting the summand in the first term on the RHS as

$$\begin{aligned}c_1\alpha d_R^+ + c_2 d_R^- &= c_1\,\alpha\, m_R^+\varphi'(Z_+^0)^{-1}\varphi''(Z_+^0) + c_2\, m_R^-\varphi'(Z_-^0)^{-1}\varphi''(Z_-^0) \\ &= (c_1 m_R^+ + c_2 m_R^-)\,\varphi'(Z^0)^{-1}\varphi''(Z^0) \\ &+ c_1 m_R^+\left[\alpha\varphi'(Z_+^0)^{-1}\varphi''(Z_+^0) - \varphi'(Z^0)^{-1}\varphi''(Z^0)\right] \\ &+ c_2 m_R^-\left[\varphi'(Z_-^0)^{-1}\varphi''(Z_-^0) - \varphi'(Z^0)^{-1}\varphi''(Z^0)\right]\,,\end{aligned} \tag{129}$$

we see the last two terms in this expression tend to 0 uniformly in R as $h \to 0$ by continuity of $Z_\pm^0$, g_0 and boundedness of $|m_R^\pm|$, and the same holds for the first term as a consequence of (121) and continuity of c_1, c_2, $Z_\pm^0$ and g_0. In conclusion, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\langle M_R(\beta,h)\rangle_\mu\right| \leq \epsilon\frac{A\,C_2}{1+\alpha} + C'\,R^{-1}\,, \tag{130}$$

if $|h| < \delta$, where C' is a constant. This completes the proof of the theorem. □

Remark 5.5. *A natural alternative to the mean magnetization as defined by* (10)-(11) *is the quantity*

$$\bar{M}(\beta,h) = \limsup_{R\to\infty}\bar{M}_R(\beta,h)\,,$$

where

$$M_R(\beta, h) = \left\langle |B_R(\tau)|^{-1} \sum_{v \in B_R(\tau)} s_v \right\rangle_{\beta,h} .$$

It is natural to conjecture that $\lim_{h\to 0} \bar{M}(\beta, h) = 0$ *holds for generic Ising trees.*

6 Conclusions

The statistical mechanical models on random graphs considered in this paper possess two simplifying features, beyond being Ising models, the first being that the graphs are restricted to be trees and the second that they are generic, in the sense of (25). Relaxing the latter condition might be a way of producing models with different magnetization properties from the ones considered here. Infinite non-generic trees having a single vertex of infinite degree have been investigated in [19, 20], but it is unclear whether a non-trivial coupling to the Ising model is possible. A different question is whether validity of the genericity condition (25) for $h = 0$ implies its validity for all $h \in \mathbb{R}$. The arguments presented in Section 3.1 only show that the domain of genericity in the (β, h)-plane is an open subset containing the β-axis, and thus leaves open the possibility of a transition to non-generic behavior at the boundary of this set.

Coupling the Ising model to other ensembles of infinite graphs represents a natural object of future study. In particular, models of planar graphs may be tractable. The so-called uniform infinite causal triangulations of the plane are known to be closely related to planar trees [14, 21], and a quenched version of this model coupled to the Ising model without external field has been considered in [21], and found to have a phase transition. Analysis of the non-quenched version, analogous to the models considered in the present paper, seem to require developing new techniques. Surely, this is also the case for other planar graph models such as the uniform infinite planar triangulation [3] or quadrangulation [8].

Acknowledgments

This work is supported by the Danish Agency for Science, Technology and Innovation through the Geometry and Mathematical Physics School (GE-OMAPS), by the NordForsk researcher network in Random Geometry, and by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation.

References

[1] J. Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry: a statistical field theory approach*. Cambridge University Press, Cambridge, 1997.

[2] J. Ambjørn, B. Durhuus, T. Jonsson, and G. Thorleifsson. Matter fields with $c > 1$ coupled to 2d gravity. *Nucl. Phys. B*, 398:568–592, 1993.

[3] O. Angel and O. Schramm. Uniform infinite planar triangulations. *Comm. Math. Phys.*, 241:191–213, 2003.

[4] C. F. Baillie, D. A. Johnston, and J. P. Kownacki. Ising spins on thin graphs. *Nucl. Phys. B*, 432:551–570, 1994.

[5] R. Baxter. *Exactly solved models in statistical mechanics*. Academic Press, 1982. `http://tpsrv.anu.edu.au/Members/baxter/book`.

[6] D. Benedetti and R. Loll. Quantum gravity and matter: counting graphs on causal dynamical triangulations. *Gen. Relativ. Gravit.*, 39:863–898, 2007.

[7] P. Billingsley. *Convergence of probability measures*. John Wiley and Sons, 1968.

[8] P. Chassaing and B. Durhuus. Local limit of labeled trees and expected volume growth in a random quadrangulation. *Ann. Probab.*, 34:879–917, 2006.

[9] C. Domb. On the theory of cooperative phenomena in crystals. *Adv. Phys.*, 9(34):149–244, 1960.

[10] M. Drmota. *Random Trees*. Springer, Wien-New York, 2009.

[11] B. Durhuus. Probabilistic aspects of infinite trees and surfaces. *Acta. Phys. Pol. B*, 34(10):4795–4811, 2006.

[12] B. Durhuus, T. Jonsson, and J. Wheater. Random walks on combs. *J. Phys. A*, 39:1009–1038, 2006.

[13] B. Durhuus, T. Jonsson, and J. Wheater. The spectral dimension of generic trees. *J. Stat. Phys.*, 128:1237–1260, 2007.

[14] B. Durhuus, T. Jonsson, and J. Wheater. On the spectral dimension of causal triangulations. *J. Stat. Phys.*, 139:859–881, 2010.

[15] T. P. Eggarter. Cayley trees, the Ising problem and the thermodynamic limit. *Phys. Rev. B*, 9(7):2989–2992, 1974.

[16] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009. `http://algo.inria.fr/flajolet/Publications/books.html`.

[17] T. E. Harris. *The theory of branching processes*. Springer-Verlag, Berlin-Gottingen-Heidelberg, 1963.

[18] E. Ising. Beitrag zur theorie des ferromagnetismus. *Z. Phys.*, 31:253–258, 1925.

[19] T. Jonsson and S. O. Stefánsson. Appearance of vertices of infinite order in a model of random trees. *J. Phys. A, Math. Theor.*, 42(485006), 2009.

[20] T. Jonsson and S. O. Stefánsson. Condensation in nongeneric trees. *J. Stat. Phys.*, 142:277–313, 2011.

[21] M. Krikun and A. Yambartsev. Phase transition for the Ising model on the Critical Lorentzian triangulation. 2008, arXiv:0810.2182v1.

[22] W. Lenz. Beiträge zum verstndnis der magnetischen eigenschaften in festen körpern. *Physik. Z.*, 21:613–615, 1920.

[23] E. Müller-Hartmann and J. Zittartz. New type of phase transition. *Phys. Rev. Lett.*, 33:893–897, 1974.

[24] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev.*, 65(3–4):117–149, 1944.

[25] H. E. Stanley. *Introduction to phase transition and critical phenomena*. Clarendon Press, Oxford, 1971.