

## Adiabatic Theorem for Discrete Time Evolution

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Quantum maps, where the time evolution proceeds along discrete time, describe various important models in the studies of quantum chaos.<sup>1)</sup> Also, quantum maps describe the time evolution of quantum circuits, which play the key role in the studies of quantum computation.<sup>2)</sup> Many studies that concerns quantum maps with slow parameter assumes that the adiabatic theorem is applicable.<sup>3)</sup> Although the adiabatic time evolution is the most elementary one in the studies of quantum dynamics,<sup>4)</sup> there has been only a few works on the adiabatic theorem for quantum maps:<sup>1)</sup> One is a numerical verification by Takami<sup>5)</sup> and the other is a heuristic argument by Hogg.<sup>6)</sup> We note that the proof for slowly modulated Hamiltonians,<sup>7)</sup> whose stationary state is described by an eigenvector of a Floquet Hamiltonian,<sup>8)</sup> cannot be applicable to quantum maps in general, since the adiabatic parameter can be discontinuous in time for quantum maps.

The aim of this note is to provide a proof of the adiabatic theorem for quantum maps (“discrete adiabatic theorem”), where we employ a discrete time analog of Kato’s proof.<sup>9-11)</sup> Although we will focus on the case that the spectrum of quantum map is purely discrete and the spectral crossing is absent, various extensions should be straightforward.

We explain the adiabatic limit for quantum maps  $\hat{U}(s)$  with a slow parameter  $s$ . The adiabatic parameter evolves from  $s'$  to  $s''$  for  $N(> 0)$  steps. Let  $s_n$  be the value of  $s$  at  $n$ -th step ( $0 \leq n \leq N$ ), where  $s_0 = s'$  and  $s_N = s''$ . We assume that the interval  $\delta s_n \equiv s_n - s_{n-1}$  is  $O(N^{-1})$  as  $N \rightarrow \infty$ . Also, we assume that  $s_n$  belongs to a smooth path  $C$ . We will examine the time evolution induced by  $\hat{U}(s)$  with  $\{s_n\}_n$ . The exact time evolution operator  $\hat{U}_n$  satisfies the recursion relation

$$\hat{U}_n = \hat{U}(s_n)\hat{U}_{n-1} \quad (1)$$

for  $n > 0$  and  $\hat{U}_0 = 1$ .

We introduce assumptions for  $\hat{U}(s)$  for our proof. First,  $\hat{U}(s)$  is supposed to be unitary, and we assume that its spectrum consists of purely discrete eigenvalues  $\{e^{i\theta_j(s)}\}_j$ , where an eigenangle  $\theta_j(s)$  takes a real value for  $s \in C$ . The corresponding spectral projections  $\{\hat{P}_j(s)\}_j$  satisfy  $\hat{U}(s)\hat{P}_j(s) = e^{i\theta_j(s)}\hat{P}_j(s)$  and  $\hat{P}_j(s)\hat{P}_k(s) = \delta_{jk}\hat{P}_j(s)$ . Also, we have the resolution of unity  $\sum_j \hat{P}_j(s) = 1$ . The second assumption is

$$z_{jk}(s) \neq 1, \quad \text{where} \quad z_{jk}(s) \equiv \exp(-i\{\theta_j(s) - \theta_k(s)\}), \quad (2)$$

which correspond to the non-zero gap condition for eigenenergies. This also implies the absence of crossing of eigenangles. Hence the dimension of the  $j$ -th eigenspace is independent of  $s$ . Finally, we assume that  $\hat{P}_j(s)$  and  $\theta_j(s)$  are smooth functions of  $s$ .

The discrete adiabatic theorem for quantum map  $\hat{U}(s)$  is

that

$$\hat{P}_j(s'')\hat{U}_N\hat{P}_j(s') = \delta_{jk} + O(N^{-1}) \quad (3)$$

for  $N \rightarrow \infty$ . In the following, we will follow a “discrete analog” of the conventional proof of the adiabatic theorem.<sup>9,10)</sup> Our proof consists of two parts. One is to introduce the time evolution operator  $\hat{W}_n$  (Eq. (10)) for an interaction picture whose free evolution is the adiabatic time evolution. This is rather straightforward. The other is to examine  $\hat{W}_n$  to estimate the deviation from the adiabatic time evolution. In the latter part, we will frequently utilize a discrete analog of integration by parts:

$$\sum_{n'=1}^n (f_{n'} - f_{n'-1})g_{n'} = f_n g_n - f_0 g_1 - \sum_{n'=1}^{n-1} f_{n'}(g_{n'+1} - g_{n'}). \quad (4)$$

We introduce two time evolution operators that comprise the adiabatic time evolution. The first part is Kato’s geometric evolution operator  $\hat{U}_K(s, s')$ , which is supposed to satisfy so-called intertwining property

$$\hat{P}_j(s)\hat{U}_K(s, s') = \hat{U}_K(s, s')\hat{P}_j(s'). \quad (5)$$

$\hat{U}_K(s, s')$  can be expressed by a path ordered integral of an adiabatic “Hamiltonian”  $\hat{H}_K(s)$  along a segment of  $C$ :

$$\hat{U}_K(s, s') \equiv \exp_{\leftarrow} \left\{ -i \int_{s'}^s \hat{H}_K(r) dr \right\}, \quad (6)$$

where  $\exp_{\leftarrow}$  represents the path ordered exponential. Here we employ<sup>9)</sup>

$$\hat{H}_K(s) \equiv i \sum_j \left[ \frac{\partial \hat{P}_j(s)}{\partial s}, \hat{P}_j(s) \right], \quad (7)$$

which satisfy Eq. (5) as well as

$$\hat{P}_j(s)\hat{H}_K(s)\hat{P}_j(s) = 0. \quad (8)$$

The latter equation is convenient to prove the main theorem. The second part of the free evolution contains only the dynamical phase

$$\hat{U}_{D,n} \equiv \sum_j \hat{P}_j(s') \exp \left\{ i \sum_{n'=1}^n \theta_j(s_{n'}) \right\} \quad (9)$$

for  $n > 0$  and  $\hat{U}_{D,0} = 1$ .

Using the adiabatic time evolution  $\hat{U}_K(s_n, s')\hat{U}_{D,n}$  as a free evolution, we introduce the time evolution operator in the interaction picture:

$$\hat{W}_n \equiv \left\{ \hat{U}_K(s_n, s')\hat{U}_{D,n} \right\}^\dagger \hat{U}_n. \quad (10)$$

From the definition of  $\hat{U}_{D,n}$ , we have  $\hat{W}_0 = 1$ . The recursion relation for  $\hat{W}_n$  is

$$\hat{W}_n = \hat{U}_{W,n}\hat{W}_{n-1}, \quad (11)$$

where

$$\hat{U}_{W,n} \equiv \left\{ \hat{U}_K(s_n, s')\hat{U}_{D,n} \right\}^\dagger \hat{U}(s_n)\hat{U}_K(s_{n-1}, s')\hat{U}_{D,n-1}. \quad (12)$$

To prove the discrete adiabatic theorem (Eq. (3)), it is suffice to show

$$\hat{P}_j(s')\hat{W}_N\hat{P}_k(s') = \delta_{jk} + O(N^{-1}), \quad (13)$$

which will be shown in the following. We start from a differ-

ence equation of  $\hat{W}_n$

$$\hat{W}_n - \hat{W}_{n-1} = (\hat{U}_{W,n} - 1)\hat{W}_{n-1}, \quad (14)$$

which implies

$$\hat{W}_n = 1 + \sum_{n'=1}^n (\hat{U}_{W,n'} - 1)\hat{W}_{n'-1}. \quad (15)$$

To apply the discrete analog of integration by parts (Eq. (4)), we introduce

$$\hat{V}_n \equiv \sum_{n'=1}^n (\hat{U}_{W,n'} - 1) \quad (16)$$

for  $n > 0$  and  $\hat{V}_0 = 0$ . From Eqs. (15) and (4), we obtain

$$\hat{W}_n = 1 + \hat{V}_n \hat{W}_{n-1} - \sum_{n'=1}^{n-1} \hat{V}_{n'} (\hat{U}_{W,n'} - 1) \hat{W}_{n'-1}. \quad (17)$$

In the following, we will show  $\hat{U}_{W,n} - 1 = \mathcal{O}(N^{-1})$  and  $\hat{V}_n = \mathcal{O}(N^{-1})$ , which imply Eq. (13).

We examine  $\hat{P}_j(s')(\hat{U}_{W,n} - 1)\hat{P}_k(s')$  for  $j \neq k$ :

$$\hat{P}_j(s')(\hat{U}_{W,n} - 1)\hat{P}_k(s') = Z_{n-1,jk} \hat{R}_{n,jk}, \quad (18)$$

where

$$Z_{n,jk} \equiv \exp \left\{ -i \sum_{n'=1}^n [\theta_j(s_{n'}) - \theta_k(s_{n'})] \right\} \quad (19)$$

and

$$\hat{R}_{n,jk} \equiv \left\{ \hat{U}_K(s_n, s') \right\}^\dagger \hat{P}_j(s_n) \hat{P}_k(s_{n-1}) \hat{U}_K(s_{n-1}, s'). \quad (20)$$

From the smoothness of  $\hat{P}_j(s)$ , we have

$$\hat{P}_j(s_n) = \hat{P}_j(s_{n-1}) + \hat{P}'_j(s_{n-1})\delta s_n + \mathcal{O}((\delta s_n)^2), \quad (21)$$

which implies

$$\hat{R}_{n,jk} = \mathcal{O}(N^{-1}) \quad \text{for } j \neq k. \quad (22)$$

Hence we obtain  $\hat{P}_j(s')(\hat{U}_{W,n} - 1)\hat{P}_k(s') = \mathcal{O}(N^{-1})$ . On the other hand, the ‘‘diagonal’’ part of  $\hat{U}_{W,n} - 1$  is

$$\begin{aligned} & \hat{P}_j(s')(\hat{U}_{W,n} - 1)\hat{P}_j(s') \\ &= \left\{ \hat{U}_K(s_n, s') \right\}^\dagger \hat{P}_j(s_n) \left\{ 1 - \hat{U}_K(s_n, s_{n-1}) \right\} \hat{P}_j(s_{n-1}) \hat{U}_K(s_{n-1}, s'). \end{aligned} \quad (23)$$

From Eqs. (6) and (8), we have  $1 - \hat{U}_K(s_n, s_{n-1}) = \mathcal{O}((\delta s_n)^2)$ . Hence we conclude  $\hat{P}_j(s')(\hat{U}_{W,n} - 1)\hat{P}_j(s') = \mathcal{O}(N^{-2})$ , which is much smaller than the off-diagonal components of  $(\hat{U}_{W,n} - 1)$ .

Next, we examine  $\hat{V}_n$ . The diagonal part is  $\hat{P}_j(s')\hat{V}_n\hat{P}_j(s') = \sum_{n'=1}^n \hat{P}_j(s')(\hat{U}_{W,n'} - 1)\hat{P}_j(s') = \mathcal{O}(n/N^2)$ . Hence, we have  $\hat{P}_j(s')\hat{V}_n\hat{P}_j(s') = \mathcal{O}(N^{-1})$  for  $0 < n \leq N$ . On the other hand,

our estimation of the off-diagonal part requires to take into account the destructive quantum interference effect induced by the dynamical phase factors. Namely, our task is to examine the following oscillatory summation

$$\hat{P}_j(s')\hat{V}_n\hat{P}_k(s') = \sum_{n'=1}^n Z_{n'-1,jk} \hat{R}_{n',jk} \quad (24)$$

for  $j \neq k$ . To apply Eq. (4) to Eq. (24), we examine the difference of  $Z_{n,jk}$ :

$$Z_{n,jk} - Z_{n-1,jk} = Z_{n-1,jk} \{ z_{jk}(s_n) - 1 \} \quad (25)$$

(see, Eqs. (2) and (19)). From the non-crossing condition for the eigenangle (Eq. (2)), we have

$$Z_{n-1,jk} = \frac{Z_{n,jk} - Z_{n-1,jk}}{z_{jk}(s_n) - 1}. \quad (26)$$

From Eq. (24), we have

$$\begin{aligned} & \hat{P}_j(s')\hat{V}_n\hat{P}_k(s') \\ &= \frac{Z_{n,jk} \hat{R}_{n,jk}}{z_{jk}(s_n) - 1} - \frac{\hat{R}_{1,jk}}{z_{jk}(s_1) - 1} + \sum_{n'=1}^{n-1} Z_{n',jk} \hat{R}_{n',jk}^{(2)} \end{aligned} \quad (27)$$

where

$$\hat{R}_{n,jk}^{(2)} \equiv \frac{\hat{R}_{n+1,jk}}{z_{jk}(s_{n+1}) - 1} - \frac{\hat{R}_{n,jk}}{z_{jk}(s_n) - 1}. \quad (28)$$

The first two terms in the right-hand side of Eq. (27) is  $\mathcal{O}(N^{-1})$  from Eqs. (22) and (2). It is straightforward to see  $R_n^{(2)} = \mathcal{O}(N^{-2})$  from Eq. (2) and the smoothness of  $\hat{P}_j(s)$  and  $\theta_j(s)$ . These estimations imply  $\hat{P}_j(s')\hat{V}_n\hat{P}_k(s') = \mathcal{O}(N^{-1})$  for  $j \neq k$ . Thus we have shown Eq. (13). This completes the proof of the discrete adiabatic theorem.

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