Adiabatic nonlinear waves with trapped particles: Part I. General formalism

I. Y. Dodin and N. J. Fisch
Department of Astrophysical Sciences, Princeton University, Princeton, New Jersey 08544, USA
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A Lagrangian formalism is developed for a general nondissipative quasiperiodic nonlinear wave with trapped particles in collisionless plasma. The adiabatic time-averaged Lagrangian density $\mathfrak{L}$ is expressed in terms of the single-particle oscillation-center Hamiltonians; once those are found, the complete set of geometrical-optics equations is derived without referring to the Maxwell-Vlasov system. In the presence of trapped particles, $\mathfrak{L}$ does not vanish when the wave amplitude a approaches zero, which is understood from the fact that such particles carry fractions of the wave momentum and energy flux independent of a. Also, the wave action is generally not conserved then, because it can be exchanged with resonant oscillations of the trapped-particle density. The corresponding modification of the wave envelope equation is found explicitly, and the new action flow velocity is derived. Applications of these results are left to the other two papers of the series, where specific problems are addressed pertaining to properties and dynamics of waves with trapped particles.

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I. INTRODUCTION

A standard approach to describing nondissipative waves in the geometrical-optics (GO) limit is to start out with the time-averaged Lagrangian, as proposed originally by Whitham in Ref. [1]; see also [2–9]. Through that, both the nonlinear dispersion relation (NDR) and the action conservation theorem (ACT) are yielded, the latter being a particularly robust way to derive the envelope equation [10]. However, the existing models using the time-averaged Lagrangian [5, 7, 11–13] cannot account for effects caused by particles trapped in wave troughs. In particular, those effects require special treatment in that they are not necessarily perturbative, i.e., may not vanish at small amplitudes [14, 15]. Hence, describing waves such as Bernstein-Green-Kruskal (BGK) modes [16–19] has been limited to more complicated kinetic models [18, 20–24], which are specific to particular settings and generally intractable, thus obscuring the underlying physical picture. Therefore, it would be advantageous to generalize Lagrangian theories to accommodate trapped-particle effects.

It is the purpose of this paper to do so. Specifically, a Lagrangian formalism is developed here for general nondissipative quasiperiodic nonlinear waves in collisionless plasma, under the assumption that the number of trapped particles remains fixed [25]. The adiabatic time-averaged Lagrangian density $\mathfrak{L}$ is expressed in terms of the single-particle oscillation-center (OC) Hamiltonians; once those are found, the complete set of GO equations is derived without referring to the Maxwell-Vlasov system. In the presence of trapped particles, $\mathfrak{L}$ does not vanish when the wave amplitude a approaches zero, which is understood from the fact that such particles carry fractions of the wave momentum and energy flux independent of a. Also, the wave action is generally not conserved then, because it can be exchanged with resonant waves of the trapped-particle density. The corresponding modification of the wave envelope equation, or the ACT, is found explicitly for one-dimensional (1D) waves, a case in which the trapped-particle density is expressed directly in terms of the wave variables, thus providing an exact closure.

The results presented here extend our Ref. [14] in that we now (i) allow plasma parameters to vary slowly in space and time, and (ii) derive the corresponding envelope equation, or the ACT, in addition to the NDR. Applications of these results are left to Refs. [26, 27] (further referred to as Paper II and Paper III), where specific problems are addressed pertaining to properties and dynamics of waves with trapped particles.

The paper is organized as follows. In Sec. II, we derive the general form of $\mathfrak{L}$. In Sec. III, we consider 1D waves in particular and obtain the corresponding ACT and NDR. In Sec. IV, longitudinal electrostatic waves are studied as a special case. For an arbitrarily nonlinear wave, the particle OC Hamiltonian is derived, generalizing the dipole ponderomotive Hamiltonian. Then, the action density, the action flux density, and the action flow velocity are inferred. In Sec. V, we also calculate those quantities specifically in the small-amplitude limit. In Sec. VI, we summarize our main results. Some auxiliary calculations are also presented in appendixes.

II. WAVE LAGRANGIAN

In Ref. [14], we proposed the following expression for the Lagrangian spatial density of an adiabatic wave in collisionless plasma [28]:

$$\mathfrak{L} = \langle \mathfrak{L}_{\rm em} \rangle - \sum_s n_s \langle \mathcal{H}_s \rangle_{f_s}. \quad (1)$$

Here $\langle \mathfrak{L}_{\rm em} \rangle$ is the time-averaged Lagrangian density of the electromagnetic field, summation is taken over distinct species s, n_s are the corresponding average densities, and $\langle \mathcal{H}_s \rangle_{f_s}$ are the corresponding oscillation-center (OC) Hamiltonians averaged over the distributions f_s of canonical momenta $\mathbf{p}$. The formula was originally derived for homogeneous stationary waves [29], the case in which $\mathbf{p}$ and n_s are constants. What we show below is that Eq. (1) holds also in the general case, except now one needs to specify how n_s relate to the field variables.

A. Plasma Lagrangian

Consider the Lagrangian $L_\Sigma = \int \mathfrak{L}_\Sigma \, d\mathsf{V}$, with the spatial density $\mathfrak{L}_\Sigma = \mathfrak{L}_{\text{em}} + \mathfrak{L}_p$ [30]. Here $\mathfrak{L}_{\text{em}} = (E^2 - B^2)/(8\pi)$ is the field Lagrangian density, $\mathbf{E} = -\nabla\varphi - \partial_t \mathbf{A}/c$ is the electric field, $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field, φ and $\mathbf{A}$ are the scalar and vector potentials, and c is the speed of light. Also,

$$\mathfrak{L}_p = \sum_i \delta(\mathbf{x} - \mathbf{x}_i)\, L_i(\mathbf{x}, \mathbf{v}_i; \varphi, \mathbf{A}), \tag{2}$$

where the summation is taken over individual particles, and L_i are the Lagrangians of those particles; namely, $L_i = L_i^{(0)} + L_i^{(\text{int})}$, where $L_i^{(0)}$ are independent of the field, and $L_i^{(\text{int})} = (e_i/c)(\mathbf{v}_i \cdot \mathbf{A}) - e_i\varphi$. Finally, $\mathbf{x}_i(t)$ are the trajectories of individual particles, $\mathbf{v}_i = \dot{\mathbf{x}}_i$ are the corresponding velocities, and e_i are the particle charges.

Suppose that the electromagnetic field contains a rapidly oscillating part and consider the plasma dynamics on scales large compared to the oscillation scales. (In the presence of resonant or trapped particles, one of such scales is the period of bounce oscillations τ_b [31, Sec. 8-6].) Then, it is only the time-averaged part of the Lagrangian, $\mathsf{L}_\Sigma \equiv \int \langle \mathfrak{L}_\Sigma \rangle \, d\mathsf{V}$, that contributes to the system action. Hence L_Σ plays a role of the slow-motion Lagrangian of the system [1]. Specifically, we write

$$\langle \mathfrak{L}_\Sigma \rangle = \langle \mathfrak{L}_{\text{em}} \rangle + \langle \mathfrak{L}_p^{(p)} \rangle + \langle \mathfrak{L}_p^{(t)} \rangle. \tag{3}$$

Here $\langle \mathfrak{L}_{\text{em}} \rangle$ generally consists of two terms, $\bar{\mathfrak{L}}_{\text{em}}$ due to quasistatic fields $(\bar{\varphi}, \bar{\mathbf{A}})$ (if any) and $\langle \tilde{\mathfrak{L}}_{\text{em}} \rangle$ due to the actual wave field. The remaining terms describe contributions of passing particles and trapped particles, correspondingly, and are derived as follows.

In the case of passing particles, we separate the slow, OC motion $\bar{\mathbf{x}}_i(t)$ and the quiver motion $\tilde{\mathbf{x}}_i(t)$ and notice that

$$\begin{aligned}
&\int \langle \delta(\mathbf{x} - \bar{\mathbf{x}}_i - \tilde{\mathbf{x}}_i)\, L_i^{(p)}(\mathbf{x}, \mathbf{v}_i) \rangle \, d\mathsf{V} = \\
&\quad \int \langle \delta(\mathbf{x} - \bar{\mathbf{x}}_i)\, L_i^{(p)}(\mathbf{x} + \tilde{\mathbf{x}}_i, \bar{\mathbf{v}}_i + \tilde{\mathbf{v}}_i) \rangle \, d\mathsf{V} = \\
&\quad \int \delta(\mathbf{x} - \bar{\mathbf{x}}_i)\, \langle L_i^{(p)}(\mathbf{x} + \tilde{\mathbf{x}}_i, \bar{\mathbf{v}}_i + \tilde{\mathbf{v}}_i) \rangle \, d\mathsf{V} \equiv \\
&\qquad\qquad \int \delta(\mathbf{x} - \bar{\mathbf{x}}_i)\, \mathcal{L}_i^{(p)}(\mathbf{x}, \bar{\mathbf{v}}_i) \, d\mathsf{V}.
\end{aligned} \tag{4}$$

Here we introduced

$$\mathcal{L}_i^{(p)}(\mathbf{x}, \mathbf{v}) = \langle L_i(\mathbf{x} + \tilde{\mathbf{x}}_i, \mathbf{v} + \tilde{\mathbf{v}}_i) \rangle, \tag{5}$$

which has the meaning of a single-particle OC Lagrangian [32]. Hence, one can write

$$\langle \mathfrak{L}_p^{(p)} \rangle = \sum_i \delta(\mathbf{x} - \bar{\mathbf{x}}_i)\, \mathcal{L}_i^{(p)}(\mathbf{x}, \bar{\mathbf{v}}_i). \tag{6}$$

In the case of trapped particles we proceed similarly, except that the OC location $\bar{\mathbf{x}}_i(t)$ is now determined by the motion of the wave nodes and thus cannot serve as an independent variable. Instead, the new independent variable will be the phase $\boldsymbol{\theta}_i$ of bounce oscillations, possibly in multiple dimensions. Since these bounce oscillations are assumed adiabatic, $\mathcal{L}_i^{(t)}$ will not depend on $\boldsymbol{\theta}_i$ explicitly; rather it will depend on $\dot{\boldsymbol{\theta}}_i$ and, parametrically, on $\bar{\mathbf{x}}_i$. (Remember that the dependence on the field variables is also implied throughout the paper.) Thus, the Lagrangian of the bounce motion, henceforth also called OC Lagrangian for brevity, is given by

$$\mathcal{L}_i^{(t)}(\mathbf{x}, \dot{\boldsymbol{\theta}}_i) = \langle L_i(\mathbf{x} + \tilde{\mathbf{x}}_i, \mathbf{u} + \tilde{\mathbf{v}}_i) \rangle, \tag{7}$$

where we substituted the wave phase velocity $\mathbf{u}$ for the average velocity. This yields

$$\langle \mathfrak{L}_p^{(t)} \rangle = \sum_i \delta(\mathbf{x} - \bar{\mathbf{x}}_i)\, \mathcal{L}_i^{(t)}(\mathbf{x}, \boldsymbol{\theta}_i). \tag{8}$$

B. Routhian

Below, it will be more convenient to use canonical OC variables for particles, $(\mathbf{q}_i, \mathbf{p}_i)$. For simplicity, let us temporarily require that $\mathbf{q}_i = \bar{\mathbf{x}}_i$ for passing particles and $\mathbf{q}_i = \boldsymbol{\theta}_i$ for trapped particles; in the latter case the canonical momentum $\mathbf{p}_i$ will be the action $\mathbf{J}_i$ of the bounce oscillations. Then, let us use [33]

$$\mathcal{L}_i = \mathbf{p}_i \cdot \dot{\mathbf{q}}_i - \mathcal{H}_i, \tag{9}$$

so Eq. (3) rewrites as

$$\mathsf{L}_\Sigma = \mathsf{L} + \sum_i \mathbf{p}_i \cdot \dot{\mathbf{q}}_i, \tag{10}$$

where $\mathsf{L} = \int \mathfrak{L} \, d\mathsf{V}$, and

$$\mathfrak{L} = \langle \mathfrak{L}_{\text{em}} \rangle - \sum_i \delta(\mathbf{x} - \bar{\mathbf{x}}_i)\, \mathcal{H}_i(\mathbf{x}, \mathbf{p}_i). \tag{11}$$

Since both $\mathbf{q}_i(t)$ and $\mathbf{p}_i(t)$ are now independent functions (cf. Ref. [34, Sec. 40]), *field* equations will be insensitive to the second term in Eq. (10); i.e., for the purpose of finding *field* equations, this term can be dropped. Therefore, L plays the role of the adiabatic Lagrangian of the wave (and also of quasistatic fields, if any).

Notice that, since $\mathsf{L} = \mathsf{L}_\Sigma - \sum_i \mathbf{p}_i \cdot \dot{\mathbf{q}}_i$, it can be considered as a Routhian of the particle-field system [34, Sec. 41], i.e., a function that acts as a Lagrangian for the field variables but as a Hamiltonian for the particle variables. As a Routhian, the wave Lagrangian was also introduced earlier in our Ref. [14]. (However, unlike in Ref. [14], here we do not perform Routh *reduction* (Appendix A); i.e., now we allow $\mathbf{p}$ to evolve.) Below, we will show how Eq. (11) corresponds to that earlier result.

C. Locally averaged densities

In Eq. (11), the summation over all particles i can be separated into (i) summation over species s, (ii) summation over $\mathbf{p}_j$ within a local elementary spatial volume $\Delta \mathsf{V}_k$, and (iii) summation over all $\Delta \mathsf{V}_k$. Specifically, let us choose the elementary volumes large enough such that both $\mathcal{H}_s$ and the densities n_s vary little [35] within $\Delta \mathsf{V}_k$. Then,

$$\begin{aligned}
&\sum_i \delta(\mathbf{x}-\bar{\mathbf{x}}_i)\,\mathcal{H}_i(\mathbf{x},\mathbf{p}_i) \\
&\quad = \sum_s \sum_k \delta(\mathbf{x}-\bar{\mathbf{x}}_k) \sum_j \mathcal{H}_s(\bar{\mathbf{x}}_k,\mathbf{p}_j) \\
&\quad = \sum_s \sum_k \delta(\mathbf{x}-\bar{\mathbf{x}}_k)\, n_s(\bar{\mathbf{x}}_k,t) \langle \mathcal{H}_s(\bar{\mathbf{x}}_k,\mathbf{p})\rangle_{f_s} \Delta \mathsf{V}_k \\
&\quad = \sum_s n_s(\mathbf{x},t)\langle \mathcal{H}_s(\mathbf{x},\mathbf{p})\rangle_{f_s}. \qquad (12)
\end{aligned}$$

This puts $\mathfrak{L}$ in the anticipated form, Eq. (1), or

$$\mathfrak{L} = \langle \mathfrak{L}_{\rm em}\rangle - \sum_s n_s^{(p)}\langle \mathcal{H}_s^{(p)}\rangle_{f_s} - \sum_s n_s^{(t)}\langle \mathcal{H}_s^{(t)}\rangle_{f_s}. \qquad (13)$$

From now on, the specific canonical variables will not matter; i.e., further canonical transformations are allowed in $\mathcal{H}_s^{(p)}$ and $\mathcal{H}_s^{(t)}$, if necessary.

D. Independent variables

We are now to choose the independent variables that will describe the field. [Quasistatic fields, if any, can be described by $(\bar{\varphi}, \bar{\mathbf{A}})$ as usual and thus will not be considered explicitly.] Suppose that the wave is characterized by a smooth envelope $a(\mathbf{x},t)$, arbitrarily normalized. Also suppose that the wave field, while not necessarily monochromatic, oscillates rapidly with some canonical phase ξ, the period being 2π. Hence, the local temporal and spatial periods can be defined as $T = 2\pi/\omega$ and $\lambda = 2\pi/k$, such that $u = \omega/k$ is the phase speed, $\mathbf{u} = u\mathbf{k}/k$,

$$\omega = -\partial_t \xi, \quad \mathbf{k} = \nabla \xi, \qquad (14)$$

and, in particular,

$$\partial_t k_i + \partial_i \omega = 0, \quad \partial_j k_i - \partial_i k_j = 0. \qquad (15)$$

where we introduced $\partial_i \equiv \partial_{x_i}$. The function $\mathfrak{L}$ will then depend on $(\partial_t \xi, \nabla \xi)$ [36] but not on ξ, for it describes the dynamics on scales τ and λ such that $\tau \gg T$ and $\ell \gg \lambda$ [1]. (In the presence of trapped particles, we also require $\tau \gg \tau_b$ and $\ell \gg u\tau_b$.)

The question that remains is how to treat n_s when varying $\mathfrak{L}$. For passing particles, the OC densities $n_s^{(p)}$ are determined by $\bar{\mathbf{x}}_i = \mathbf{q}_i$, which are independent variables; thus, $n_s^{(p)}(\mathbf{x},t)$ are also independent of the field variables. For trapped particles, however, $n_s^{(t)}(\mathbf{x},t)$ are determined by $\bar{\mathbf{x}}_i$ which are tied to the wave troughs; thus, $n_s^{(p)}(\mathbf{x},t)$ is connected with the wave phase. In particular, for 1D waves this connection can be implemented as an exact closure, which is done as follows.

III. ONE-DIMENSIONAL WAVES WITH TRAPPED PARTICLES

A. Extended Lagrangian

First of all, notice that, in a 1D system, trapped particles travel at the wave phase velocity u. Hence, the corresponding continuity equations read as

$$\partial_t \mathfrak{n}_s + \partial_x(\mathfrak{n}_s u) = 0, \qquad (16)$$

where we introduced $\mathfrak{n}_s \equiv n_s^{(t)}$ to shorten the notation. One can embed Eq. (16) in the formalism by considering a new, extended Lagrangian density

$$\Lambda = \mathfrak{L} + \sum_s \mu_s \big[\partial_t \mathfrak{n}_s + \partial_x(\mathfrak{n}_s u)\big]. \qquad (17)$$

Here μ_s are Lagrange multipliers [37], i.e., new independent functions of (t,x), yet to be found. In particular, varying Λ with respect to $\mathfrak{n}_s$ yields

$$\partial_t \mu_s + u\,\partial_x \mu_s + \langle \mathcal{H}_s^{(t)}\rangle_{f_s} = 0, \qquad (18)$$

whereas Eq. (16) flows from varying Λ with respect to μ_s.

Further, notice that

$$\hat{\Lambda} = \mathfrak{L} - \sum_s \mathfrak{n}_s \big[\partial_t \mu_s + (\omega/k)\,\partial_x \mu_s\big] \qquad (19)$$

is a Lagrangian density equivalent to Λ, yet with an advantage that $\hat{\Lambda}$ depends on only [36] the *first*-order derivatives of ξ [cf. Eq. (14)]:

$$\hat{\Lambda} = \hat{\Lambda}(a, \partial_t \xi, \partial_x \xi, \mathfrak{n}, \partial_t \mu, \partial_x \mu). \qquad (20)$$

Then, varying $\hat{\Lambda}$ with respect to ξ is as usual and yields $\partial_t \hat{\Lambda}_\omega - \partial_x \hat{\Lambda}_k = 0$ [1]. (We henceforth use indexes ω, k, and a to denote the corresponding partial derivatives.) On the other hand,

$$\hat{\Lambda}_\omega = \mathfrak{L}_\omega - \frac{1}{k}\sum_s \mathfrak{n}_s\,\partial_x \mu_s, \qquad (21)$$

$$\hat{\Lambda}_k = \mathfrak{L}_k + \frac{\omega}{k^2}\sum_s \mathfrak{n}_s\,\partial_x \mu_s, \qquad (22)$$

where the derivatives are taken, in particular, at fixed $\mathfrak{n}$. Thus, one obtains

$$\partial_t \mathfrak{L}_\omega - \partial_x \mathfrak{L}_k = \sum_s M_s, \qquad (23)$$

$$M_s = \partial_t(\sigma_s\,\partial_x \mu_s) + \partial_x(\sigma_s u\,\partial_x \mu_s). \qquad (24)$$

Here we introduced $\sigma_s = \mathfrak{n}_s/k$, which is proportional to the number of trapped particles (of type s) within one wavelength. For adiabatic waves this number is constant in the frame moving with the phase velocity; i.e.,

$$\partial_t \sigma_s + u\, \partial_x \sigma_s = 0, \tag{25}$$

which is also obtained from Eqs. (15) and (16). Then,

$$M_s = \sigma_s \big[\partial^2_{xt}\mu_s + \partial_x (u\, \partial_x \mu_s)\big] = -\sigma_s\, \partial_x \langle \mathcal{H}_s^{(t)} \rangle_{f_s}, \tag{26}$$

where we used Eq. (18). Hence, one gets

$$\partial_t \mathfrak{L}_\omega - \partial_x \mathfrak{L}_k = -\sum_s \sigma_s\, \partial_x \langle \mathcal{H}_s^{(t)} \rangle_{f_s}, \tag{27}$$

or, equivalently,

$$\partial_t \mathfrak{L}_\omega + \partial_x \Big[-\mathfrak{L}_k + \sum_s \sigma_s \langle \mathcal{H}_s^{(t)} \rangle_{f_s} \Big] = \sum_s \langle \mathcal{H}_s^{(t)} \rangle_{f_s}\, \partial_x \sigma_s. \tag{28}$$

Further notice that $(\mathfrak{L}_k)_\sigma = (\mathfrak{L}_k)_\mathfrak{n} + \sum_k \sigma_s (\partial_{\mathfrak{n}_s} \mathfrak{L})_k$, so

$$(\mathfrak{L}_k)_\sigma = (\mathfrak{L}_k)_\mathfrak{n} - \sum_k \sigma_s \langle \mathcal{H}_s^{(t)} \rangle_{f_s}, \tag{29}$$

where we substituted $\mathfrak{n}_s = \sigma_s k$, in the left-hand side; similarly, $(\mathfrak{L}_\omega)_\sigma = (\mathfrak{L}_\omega)_\mathfrak{n}$. (Here the external subindexes show variables kept fixed at differentiation.) Thus, it is convenient to consider $\mathfrak{L}$ as a function of σ_s rather than of $\mathfrak{n}_s$, specifically as follows.

B. Action conservation and wave dispersion

From now on, let us consider $\mathfrak{L}$ as [36]

$$\mathfrak{L} = \mathfrak{L}(a, \partial_t \xi, \partial_x \xi, \sigma). \tag{30}$$

Using Eq. (29), one can hence write Eq. (28) as

$$\partial_t \mathfrak{L}_\omega - \partial_x \mathfrak{L}_k = \sum_s \langle \mathcal{H}_s^{(t)} \rangle_{f_s}\, \partial_x \sigma_s. \tag{31}$$

Equation (31) represents a generalization of the well-known ACT for 1D waves [1], reproduced in the limit $\sigma_s = 0$ (also see Appendix B). Thus, we interpret

$$\mathcal{I} = \mathfrak{L}_\omega, \quad \mathcal{J} = -\mathfrak{L}_k \tag{32}$$

as the new action density and the new action flux density, correspondingly. Notice, however, that Eq. (31), or

$$\partial_t \mathcal{I} + \partial_x \mathcal{J} = \sum_s \langle \mathcal{H}_s^{(t)} \rangle_{f_s}\, \partial_x \sigma_s, \tag{33}$$

does not have a conservative form in general, due to the nonzero right-hand side. This is because the wave of the trapped-particle density is, by definition, always resonant with the electric field, so the two can exchange quanta whenever σ_s are modulated. Interestingly, the effect of σ-waves, which are described by Eq. (25), is similar to the effect of entropy waves on magnetohydrodynamic oscillations reported in Refs. [38, 39].

Finally, we can complement the ACT with the NDR, by varying $\mathfrak{L}$ with respect to the wave amplitude. Since $\mathfrak{n}_s$ are independent of a, the NDR has the same general form as for waves without trapped particles, reading as

$$\mathfrak{L}_a = 0. \tag{34}$$

IV. LONGITUDINAL WAVES

Now let us consider the specific case of 1D longitudinal waves as a paradigmatic example. To do so, we will need to construct the Lagrangian density $\mathfrak{L}$ for such waves, which, in turn, requires calculating the single-particle OC Hamiltonians $\mathcal{H}_s$ first. For linear waves, this is done in Appendix B. For nonlinear waves, this also can be done straightforwardly, at least to the zeroth order in τ^{-1} and ℓ^{-1}. Namely, we proceed as follows.

A. Single-particle OC Hamiltonians

Consider the Lagrangian of a single particle in a stationary homogeneous electrostatic wave in nonmagnetized plasma. It is only the 1D motion along the wave field $\tilde{\mathbf{E}}$ that matters for us; thus, we take

$$L = mv^2/2 - e\tilde{\varphi}(x - ut), \tag{35}$$

where m and e are the particle mass and charge, v is the velocity in the laboratory frame K, and the potential $\tilde{\varphi}$ is periodic yet not necessarily sinusoidal. (A quasistatic potential $\bar{\varphi}$ can be included straightforwardly and will not be discussed here explicitly.) Rewrite L as

$$L = mu^2/2 + muw + mw^2 - \mathcal{E}, \tag{36}$$

where $y = x - ut$, so $\dot{y} \equiv w = v - u$, and also

$$\mathcal{E} \equiv mw^2/2 + e\tilde{\varphi}(y), \tag{37}$$

which is the energy in the moving frame $\hat{K}$ where $\tilde{\varphi}$ is static. Then, time-averaging yields

$$\langle L \rangle = mu^2/2 + mu\langle w \rangle + m\langle w^2 \rangle - \mathcal{E}, \tag{38}$$

where we used that $\mathcal{E}$ is conserved on the oscillation scale.

To proceed, it is convenient to introduce the angle θ and the action J of the oscillations in $\hat{K}$; in partucular,

$$J = \frac{m}{2\pi} \oint w\, dy. \tag{39}$$

Then the oscillation period can be expressed in terms of the corresponding canonical frequency Ω, yielding $\langle w \rangle =$

$\varsigma\Omega/k$, where $\varsigma \equiv \operatorname{sgn}\langle w\rangle = \operatorname{sgn} w$ (for trapped particles $\varsigma = 0$), and $m\langle w^2\rangle = J\Omega$. Then,

$$\langle L\rangle = muV - mu^2/2 + \pounds, \tag{40}$$

where we used that the average velocity $V \equiv \langle v\rangle$ equals

$$V = u + \varsigma\Omega/k \tag{41}$$

and introduced $\pounds = J\Omega - \mathcal{E}$.

In particular, notice the following. Since the generating function of the transformation $(y, mw) \to (\theta, J)$ clearly does not depend on time explicitly, $\mathcal{E}$ acts as a Hamiltonian in (θ, J)-representation (cf. Ref. [14]), and thus $\pounds$ is the corresponding Lagrangian; then,

$$\partial_J\mathcal{E} = \Omega, \quad \partial_J\pounds = J\partial_J\Omega. \tag{42}$$

[Partial derivatives are used because $\mathcal{E}$ and $\pounds$ can also depend parametrically on (a, ω, k); cf. Sec. IV B.]

First, let us consider a passing particle. In this case, for the canonical momentum $\mathbf{p}$ one can take $P \equiv \partial_V\langle L\rangle$ [32, 40]. Assuming $J = J(V)$, one thereby obtains $P = mu + \partial_J\pounds\,\partial_V J$, or, using Eq. (42), $P = mu + J\,\partial_V\Omega$. Hence, from Eq. (41), we get

$$P = mu + \varsigma kJ. \tag{43}$$

Then the OC Hamiltonian $\mathcal{H} = PV - \langle L\rangle$ reads as

$$\mathcal{H} = \mathcal{E} + Pu - mu^2/2. \tag{44}$$

In particular, notice that Eq. (44) can be understood as a generalization of the nonrelativistic dipole ponderomotive Hamiltonian [Eq. (B1), with Φ_s from Eq. (B13)] to the case of fully nonlinear particle motion in an arbitrary longitudinal electrostatic wave.

In case of a trapped particle, the average coordinate is fixed, yielding $V = u$, so now those are (θ, J) that we choose to serve as $(\mathbf{q}, \mathbf{p})$. Hence, $\mathcal{H} = J\Omega - \langle L\rangle$, and thus

$$\mathcal{H} = \mathcal{E} - mu^2/2. \tag{45}$$

B. Parametrization. Wave Lagrangian

Although a can be defined as an arbitrary measure of the field amplitude, for the purpose of this paper it is convenient to introduce it specifically as the amplitude of the wave electric field $\tilde{E}$. Hence, we can write Eq. (13) explicitly as

$$\mathfrak{L} = \bar{\mathfrak{L}}_{\text{em}} + \frac{a^2}{16\pi} - \sum_s n_s^{(p)}\langle\mathcal{H}_s^{(p)}\rangle_{f_s} - \sum_s n_s^{(t)}\langle\mathcal{H}_s^{(t)}\rangle_{f_s}, \tag{46}$$

with $\mathcal{H}_s^{(p)}$ to be taken from Eq. (44), and $\mathcal{H}_s^{(t)}$ to be taken from Eq. (45).

Notice also that, once we have adopted $a \equiv \tilde{E}$, the bounce-motion energy $\mathcal{E}$ can depend parametrically on a and k but not on ω, because the particle motion in $\hat{K}$ is entirely determined by the spatial structure of the wave potential, which is static there. In other words,

$$\mathcal{E} = \mathcal{E}(J, a, k). \tag{47}$$

Yet note that $J = J(P, \omega, k)$ for passing particles [see Eq. (43)], whereas for trapped particles J is an independent variable. In particular, this yields the following equalities that we will use below. First of all,

$$\partial_\omega\mathcal{E}^{(p)} = \partial_J\mathcal{E}^{(p)}\,\partial_\omega J^{(p)} = -m\varsigma\Omega/k^2, \tag{48}$$

where we substituted Eq. (42) for $\partial_J\mathcal{E}$ and Eq. (43) for $\partial_\omega J(P, \omega, k)$; also, $\partial_\omega\mathcal{E}^{(t)} = 0$. Hence, one obtains

$$\partial_\omega\mathcal{H}^{(p)} = (P - mV)/k, \quad \partial_\omega\mathcal{H}^{(t)} = -mu/k. \tag{49}$$

C. Action density

Now we can calculate the wave action density $\mathcal{I}$ [Eq. (32)], namely, as follows. Since the first two terms in Eq. (46) are independent of ω and k, one gets

$$\mathcal{I} = \mathcal{I}^{(p)} + \mathcal{I}^{(t)}, \quad \mathcal{I}^{(b)} = -\sum_s n_s^{(b)}\langle\partial_\omega\mathcal{H}_s^{(b)}\rangle_{f_s}, \tag{50}$$

with $b = p, t$. Hence, Eq. (49) yields

$$\mathcal{I}^{(p)} = k^{-1}\sum_s n_s^{(p)}\,\langle m_s V - P\rangle_{f_s}, \tag{51}$$

$$\mathcal{I}^{(t)} = k^{-1}\sum_s n_s^{(t)} m_s u. \tag{52}$$

One may say that $\mathcal{I}^{(p)}$ is proportional to (the density of) the *ponderomotive* momentum carried by passing particles (cf. Appendix B 2), and $\mathcal{I}^{(t)}$ is proportional to the *kinetic* momentum carried by trapped particles. Also, notice that the two can be combined as

$$k\mathcal{I} = \sum_s n_s m_s\langle V\rangle_{f_s} - \sum_s n_s^{(p)}\langle P\rangle_{f_s}. \tag{53}$$

The right-hand side here equals the difference between the system total kinetic momentum [$n_s \equiv n_s^{(p)} + n_s^{(t)}$ being the total density of species s] less the momentum stored in particles, i.e., that of the untrapped population. Therefore, by definition, $k\mathcal{I}$ represents the wave total momentum, in agreement with Ref. [2, Sec. 15.4] (see also Appendix B in Paper III). Notice that a part of this momentum [namely, $k\mathcal{I}^{(t)}$] does not vanish at small a, which is due to the fact that it is stored in the trapped-particle translational motion with velocity $u = \omega/k$, independent of the field amplitude.

D. Action flow

The action flux density $\mathcal{J}$ [Eq. (32)] can be found similarly from Eq. (46) and reads as $\mathcal{J} = \mathcal{J}^{(p)} + \mathcal{J}^{(t)}$, where

$$\mathcal{J}^{(b)} = \sum_s n_s^{(b)}\,\partial_k\langle\mathcal{H}_s^{(b)}\rangle_{f_s}, \tag{54}$$

again with $b = p, t$. In particular, notice that $\mathcal{J}$ does not vanish at small a either. This is because $\mathcal{J} = \Pi/\omega$, where Π is the energy flux density [2, Sec. 15.4] (see also Appendix B in Paper III), a part of which, carried by trapped particles, is independent of a.

Notice also that $v_{\mathcal{I}} \equiv \mathcal{J}/\mathcal{I}$ has the meaning of the action flow velocity and reads as

$$v_{\mathcal{I}} = \frac{\mathcal{J}^{(p)} + \mathcal{J}^{(t)}}{\mathcal{I}^{(p)} + \mathcal{I}^{(t)}}. \tag{55}$$

Introduce $\varrho \equiv \mathcal{I}^{(t)}/\mathcal{I}^{(p)}$ and also $v_{\mathcal{I}}^{(b)} = \mathcal{J}^{(b)}/\mathcal{I}^{(b)}$. Then,

$$v_{\mathcal{I}} = \frac{v_{\mathcal{I}}^{(p)} + \varrho v_{\mathcal{I}}^{(t)}}{1 + \varrho}. \tag{56}$$

V. DISCUSSION

Now that we have developed the general formalism, it is instructive to consider 1D longitudinal electrostatic waves at small a in particular. Then, one can use Eq. (B1) for $\mathcal{H}_s^{(p)}$ [41], so

$$\mathfrak{L} \approx \mathfrak{L}^{(0)} + \frac{\epsilon a^2}{16\pi} - \sum_s n_s^{(t)} \langle \mathcal{H}_s^{(t)} \rangle_{f_s}, \tag{57}$$

where $\mathfrak{L}^{(0)}$ is independent of the field variables, and ϵ is the longitudinal dielectric function (Appendix B). Also, $\mathcal{I}^{(p)} \approx \epsilon_\omega |\tilde{E}|^2/(16\pi)$, and thus $v_{\mathcal{I}}^{(p)}$ equals the linear group velocity $v_{g0} = \omega_k$ (Appendix B 2). Further, let us neglect $\mathcal{E}_s^{(t)}$ compared to $m_s u^2/2$ in Eq. (45). Then, $\mathcal{H}_s^{(t)} \approx -m_s\omega^2/(2k^2)$, so one obtains

$$\mathcal{J}^{(t)} \approx \sum_s n_s^{(t)}\, m_s u^2/k = u\mathcal{I}^{(t)}, \tag{58}$$

which is independent of a, as expected. This yields $v_{\mathcal{I}}^{(t)} \approx u$, and, therefore,

$$v_{\mathcal{I}} \approx \frac{v_{g0} + \varrho u}{1 + \varrho}. \tag{59}$$

Equation (59) should not be confused with a similar expression in Ref. [42] derived for what is called there the nonlinear group velocity. The effects addressed in Ref. [42] result in a nonconservative form of the envelope equation, i.e., violate the ACT. Hence, they *by definition* constitute wave damping and cannot be discriminated from nondissipative propagation of a signal [43], contrary to the statement made in that paper. Rather, Ref. [42] seems to describe in detail the effects reported in Refs. [44–48], albeit only at small enough number of trapped particles (Paper III), because $\mathfrak{L}$ is assumed vanishing at zero a [49].

Unlike in Ref. [42], our formulation in its present form does not account for collisionless dissipation (except at inhomogeneous σ_s), so the difference between $v_{\mathcal{I}}$ and v_{g0} is entirely due to adiabatic effects here [50]. In addition to the action flow velocity $v_{\mathcal{I}}$ that we presented, one can also define the energy flow and momentum flow velocities [1]. For a nonlinear wave, those will be different from each other and from the true nonlinear group velocity v_g, at which the modulation impressed on a wave propagates adiabatically. In Paper III, we will discuss how this true v_g is affected by trapped particles in particular.

VI. SUMMARY

In this paper, a Lagrangian formalism is developed for general nondissipative quasiperiodic nonlinear waves in collisionless plasma, under the assumption that the number of trapped particles remains fixed. Specifically, the time-averaged adiabatic Lagrangian density is derived in the following form:

$$\mathfrak{L} = \langle \mathfrak{L}_{\text{em}} \rangle - \sum_s n_s \langle \mathcal{H}_s \rangle_{f_s}. \tag{60}$$

Here $\langle \mathfrak{L}_{\text{em}} \rangle$ is the time-averaged Lagrangian density of the electromagnetic field, summation is taken over distinct species s, n_s are the corresponding average densities, and $\langle \mathcal{H}_s \rangle_{f_s}$ are the corresponding OC Hamiltonians averaged over the distributions f_s of canonical momenta. Once $\mathcal{H}_s$ are found, the complete set of GO equations is derived without referring to the Maxwell-Vlasov system.

For the first time, the average Lagrangian accounts also for particles trapped by the wave. In particular, 1D waves are considered, in which case $\mathfrak{L} = \mathfrak{L}(a, \omega, k, \sigma)$; here ω and k are the wave local frequency and the wave number, and $\sigma_s \equiv n_s^{(t)}/k$ are proportional to the number of trapped particles within a wavelength. Correspondingly, the GO equations are summarized as follows. The first one is the consistency condition, $\partial_t k + \partial_x \omega = 0$, due to $\omega = -\partial_t \xi$ and $k = \partial_x \xi$; here ξ is the wave canonical phase. The second one is the NDR, given by $\mathfrak{L}_a = 0$. The third GO equation represents a modified ACT,

$$\partial_t \mathcal{I} + \partial_x \mathcal{J} = \sum_s \mathcal{H}_s^{(t)}\, \partial_x \sigma_s, \tag{61}$$

with $\mathcal{I} \equiv \mathfrak{L}_\omega$ and $\mathcal{J} \equiv -\mathfrak{L}_k$ being the action density and the action flux density, correspondingly. Because of the source term on the right-hand side, the wave action may not be conserved then, due to the fact that it can be exchanged with resonant waves of the trapped-particle density (σ-waves).

For longitudinal electrostatic waves, considered as a paradigmatic case, $\mathfrak{L}$ is calculated explicitly [Eq. (46)], including expressions for $\mathcal{H}_s$ [Eqs. (44), (45)], which generalize the dipole ponderomotive Hamiltonian [Eqs. (B1), (B13)]. It is shown that $\mathfrak{L}$, $\mathcal{I}$, and $\mathcal{J}$ do not vanish at small a, which is understood from the fact that trapped particles carry fractions of the wave momentum density $\rho = k\mathfrak{L}_\omega$ and the energy flux density $\Pi = -\omega\mathfrak{L}_k$ independent of a. Particularly, in the limit of small a, the action flow velocity is obtained,

$$v_{\mathcal{I}} \approx \frac{v_{g0} + \varrho u}{1+\varrho}, \tag{62}$$

where $v_{g0} = \omega_k$ is the linear group velocity, u is the wave phase velocity, and $\varrho = \mathcal{I}^{(t)}/\mathcal{I}^{(p)}$. Equation (62) should not be confused with a seemingly akin formula in Ref. [42], because here the difference between $v_{\mathcal{I}}$ and v_{g0} is entirely due to adiabatic effects, unlike in Ref. [42].

Applications of these results are left to Papers II and III, where specific problems are addressed pertaining to properties and dynamics of waves with trapped particles.

VII. ACKNOWLEDGMENTS

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Appendix A: Routh reduction

In this appendix, we restate the concept of Routh reduction [51], complementing the derivation of the wave Lagrangian that we reported earlier in Ref. [14].

Consider a dynamical system described by generalized coordinates $\mathbf{q} \equiv (q^1, \dots q^N)$, so the corresponding Lagrangian has a form $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, t being the time. The original least-action principle is then formulated as follows [34, Sec. 2]: among trajectories $\mathbf{q}(t)$ starting at $\mathbf{q}_1$ at time t_1 and ending at $\mathbf{q}_2$ at time t_2, realized is the one on which the action $S = \int_{t_1}^{t_2} L\, dt$ is minimal. Since the general variation of S reads as [33]

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \cdot \delta \mathbf{q}\, dt + \mathbf{p} \cdot \delta \mathbf{q} \bigg|_{t_1}^{t_2}, \tag{A1}$$

with $\mathbf{p} \equiv \partial_{\dot{\mathbf{q}}} L$, and $\delta \mathbf{q}(t_{1,2}) = 0$ due to $\mathbf{q}_{1,2}$ being fixed, one thereby obtains the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}. \tag{A2}$$

For simplicity, we henceforth consider a 2D system, with $\mathbf{q} = (\theta, x)$, the extension to a larger number of dimensions being straightforward. Suppose, in particular, that θ is cyclic, i.e., does not enter L explicitly. Then the momentum J canonically conjugate to θ is conserved, so we can use the equality $J = \partial_{\dot{\theta}} L$ to express $\dot{\theta}$ as

$$\dot{\theta} \equiv \Omega(J, x, \dot{x}, t). \tag{A3}$$

Hence, S can be understood as a functional of $x(t)$ only.

Now consider the set of trajectories $x(t)$ starting at given x_1 at time t_1 and ending at given x_2 at time t_2, while θ_1 and θ_2 are arbitrary [albeit connected through Eq. (A3)]. Suppose that $\bar{x}(t)$ satisfies Eq. (A2) and consider the linear variation of $S[x(t)]$ with respect to $\delta x(t) = x(t) - \bar{x}(t)$. Then, from Eq. (A1), one obtains

$$\delta S = J\, \delta\theta \bigg|_{t_1}^{t_2}. \tag{A4}$$

Since S is thereby *not* minimized on $\bar{x}(t)$ within these variation procedure, consider another, "reduced" action

$$\hat{S} = S - \int_{t_1}^{t_2} J\dot{\theta}\, dt. \tag{A5}$$

Here, the latter term can also be put as $\int_{\theta_1}^{\theta_2} J\, d\theta$, where depending on $x(t)$ are only the integration limits. Therefore, its variation around $\bar{x}(t)$ equals $J\, \delta\theta\, |_{t_1}^{t_2}$, thus yielding $\delta \hat{S}[\bar{x}(t)] = 0$. Then a new variational principle can be formulated as follows: among trajectories $x(t)$ starting at x_1 at t_1 and ending at x_2 at t_2, *with arbitrary θ_1 and θ_2*, realized is $\bar{x}(t)$ on which $\hat{S}$ is minimal.

Notice further that Eq. (A5) rewrites as $\hat{S} = \int_{t_1}^{t_2} R\, dt$, with the equivalent Lagrangian

$$R(x, \dot{x}, t) = L(x, \dot{x}, t, \Omega) - J\Omega \tag{A6}$$

[where $\Omega = \Omega(J, x, \dot{x}, t)$], also known as Routhian. Hence, the motion equation that flows from the new variational principle reads as

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{x}} = \frac{\partial R}{\partial x}, \tag{A7}$$

also in agreement with the general Routh equations [34, Sec. 41]. Since R is independent of θ, the system phase space is effectively reduced now, and x-motion decouples, which is what constitutes the Routh reduction. For plasma physics applications of this technique, see Refs. [32, 40, 52–55].

Appendix B: Linear waves

Here, we show how the known GO equations for linear electromagnetic waves [which have $n^{(t)} = 0$] follow from the general Lagrangian formalism discussed in Sec. II.

1. General electromagnetic waves

Let us take $\tilde{\mathbf{E}}, \tilde{\mathbf{B}} \propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$, and employ the dipole approximation for $\mathcal{H}_s$ [52, 53], namely,

$$\mathcal{H}_s = \mathcal{H}_s^{(0)} + \Phi_s, \quad \Phi_s = -\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\alpha}}_s \cdot \tilde{\mathbf{E}}/4, \tag{B1}$$

where $\mathcal{H}_s^{(0)}$ is some function of the particle canonical momenta, Φ_s is the ponderomotive potential, and $\hat{\boldsymbol{\alpha}}_s$ is the linear polarizability. Then, since

$$1 + 4\pi \sum_s n_s \langle \hat{\boldsymbol{\alpha}}_s \rangle_{f_s} = \hat{\boldsymbol{\epsilon}}(\omega, \mathbf{k}), \tag{B2}$$

where $\hat{\boldsymbol{\epsilon}}$ is the linear dielectric tensor, one obtains

$$\mathfrak{L} = \mathfrak{L}^{(0)} + \frac{1}{16\pi}\left(\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}} - |\tilde{\mathbf{B}}|^2\right), \tag{B3}$$

where the term

$$\mathfrak{L}^{(0)} = \bar{\mathfrak{L}}_{\text{em}} - \sum_s n_s \langle \mathcal{H}_s^{(0)} \rangle_{f_s} \tag{B4}$$

is independent of the wave variables. Now let us introduce the wave amplitude a via $\tilde{\mathbf{E}} = a\mathbf{e}$ for the electric field envelope, where $\mathbf{e}$ determines polarization; hence, $\tilde{B} = |\mathbf{n} \times \mathbf{e}|a$, where $\mathbf{n} \equiv c\mathbf{k}/\omega$, and c is the speed of light. Then,

$$\mathfrak{L} = \mathfrak{L}^{(0)} + \frac{a^2}{16\pi}\,\mathfrak{D}(\omega, \mathbf{k}), \tag{B5}$$

where we introduced

$$\mathfrak{D}(\omega, \mathbf{k}) = \mathbf{e}^* \cdot \hat{\boldsymbol{\epsilon}} \cdot \mathbf{e} - |\mathbf{n} \times \mathbf{e}|^2. \tag{B6}$$

Varying the wave Lagrangian with respect to the amplitude a yields the dispersion relation $\mathfrak{L}_a = 0$, or

$$\mathfrak{D}(\omega, \mathbf{k}) = 0, \tag{B7}$$

which coincides with the known dispersion relation at prescribed $\mathbf{e}$ [31, Sec. 1-3]. In fact, the *vector* equation,

$$\hat{\boldsymbol{\epsilon}} \cdot \mathbf{e} + \mathbf{n} \times (\mathbf{n} \times \mathbf{e}) = 0, \tag{B8}$$

also can be recovered, namely, by varying $\mathfrak{L}$ with respect to $\mathbf{e}^*$. [One could, of course, vary $\mathfrak{L}$ also with respect to $\tilde{\mathbf{E}}^*$ and get Eq. (B8) immediately.]

Now let us vary $\mathfrak{L}$ with respect to the wave phase ξ, with Eq. (14) taken into account. Like for any other Lagrangian density of the form $\mathfrak{L}(a, \omega, \mathbf{k})$, one obtains then [1]

$$\partial_t \mathfrak{L}_\omega - \nabla \cdot \mathfrak{L}_{\mathbf{k}} = 0. \tag{B9}$$

The quantity $\mathcal{I} \equiv \mathfrak{L}_\omega$ can be written as

$$\begin{aligned}\mathcal{I} &= \frac{a^2}{16\pi}\,(\mathbf{e}^* \cdot \hat{\boldsymbol{\epsilon}}_\omega \cdot \mathbf{e}) + \frac{a^2}{8\pi\omega}\,|\mathbf{n} \times \mathbf{e}|^2 \\ &= \frac{a^2}{16\pi\omega}\left[\mathbf{e}^* \cdot (\omega \hat{\boldsymbol{\epsilon}}_\omega) \cdot \mathbf{e} + \mathbf{e}^* \cdot \hat{\boldsymbol{\epsilon}} \cdot \mathbf{e} + |\mathbf{n} \times \mathbf{e}|^2\right] \\ &= \frac{1}{16\pi\omega}\left[\tilde{\mathbf{E}}^* \cdot \partial_\omega(\hat{\boldsymbol{\epsilon}}\omega) \cdot \tilde{\mathbf{E}} + |\tilde{\mathbf{B}}|^2\right] \equiv \frac{\varepsilon}{\omega},\end{aligned} \tag{B10}$$

where we used Eq. (B7) and introduced ε for the linear-wave energy density [52]; thus, $\mathcal{I}$ equals the linear-wave action density. Also, $\boldsymbol{\mathcal{J}} \equiv -\mathfrak{L}_{\mathbf{k}}$ can be put as

$$\boldsymbol{\mathcal{J}} = -\frac{a^2}{16\pi}\,\mathfrak{D}_{\mathbf{k}} = \frac{a^2}{16\pi}\,\mathfrak{D}_\omega\left(-\frac{\mathfrak{D}_{\mathbf{k}}}{\mathfrak{D}_\omega}\right) = \mathbf{v}_{g0}\mathcal{I}, \tag{B11}$$

where we used $-\partial_{\mathbf{k}}\mathfrak{D}/\partial_\omega\mathfrak{D} = \omega_{\mathbf{k}}$ [from Eq. (B7)], the latter being the linear group velocity $\mathbf{v}_{g0}$; thus, $\boldsymbol{\mathcal{J}}$ is the linear-wave action flux density. Hence Eq. (B9) rewrites as

$$\partial_t \mathcal{I} + \nabla \cdot (\mathbf{v}_{g0}\mathcal{I}) = 0, \tag{B12}$$

in agreement with the linear ACT [2, Sec. 11.7].

Equations (14), (B7), and (B12) represent a complete set of equations describing nondissipative linear electromagnetic waves in the GO approximation (cf. Ref. [2, Chaps. 14, 15]). As one can see from the above calculation, the Maxwell's equations and the Vlasov equation *per se* are not needed to derive these equations [56].

2. Longitudinal electrostatic waves

Finally, let us consider longitudinal electrostatic waves in somewhat more detail. In this case, for the longitudinal polarizability of an individual particle with OC velocity $\mathbf{V}$, we take $\alpha_s = -e_s^2/[m_s(\omega - \mathbf{k}\cdot\mathbf{V})^2]$ [40], where e_s and m_s are the particle charge and mass, respectively; in particular, this corresponds to

$$\Phi_s = \frac{e_s^2|\tilde{\mathbf{E}}|^2}{4m_s(\omega - \mathbf{k}\cdot\mathbf{V})^2} \tag{B13}$$

(cf. Refs. [40, 57, 58]). Then, the longitudinal dielectric function,

$$\epsilon = 1 + 4\pi \sum_s n_s \langle \alpha_s \rangle_{f_s}, \tag{B14}$$

can be written as (cf. Ref. [24])

$$\epsilon = 1 - \sum_s \omega_{ps}^2 \int_{-\infty}^{\infty} \frac{f_s(V_x)}{(\omega - kV_x)^2}\,dV_x, \tag{B15}$$

where $\omega_{ps}^2 = 4\pi n_s e_s^2/m_s$, and $f_s(V_x)$ are the distributions of the particle longitudinal velocities V_x, normalized such that $\int_{-\infty}^{\infty} f_s(V_x)\,dV_x = 1$. By definition, a linear wave has no trapped particles, so $f_s(V_x)$ is zero in the resonance vicinity, and thus the integrand in Eq. (B15) is analytic. Hence, one can take the integral by parts. This yields

$$\epsilon = 1 + \sum_s \frac{\omega_{ps}^2}{k} \int_{-\infty}^{\infty} \frac{f_s'(V_x)}{\omega - kV_x}\,dV_x, \tag{B16}$$

in agreement with Ref. [59].

From, Eq. (B7) the dispersion relation now reads as $\epsilon(\omega, \mathbf{k}) = 0$. In particular, this means $\mathbf{v}_{g0} = -\epsilon_{\mathbf{k}}/\epsilon_\omega$ and

$$\mathcal{I} = \frac{\epsilon_\omega}{16\pi}\,|\tilde{\mathbf{E}}|^2. \tag{B17}$$

Let us show that this expression is consistent with Eq. (51). First, notice that [52]

$$P_x = m_s V_x - \partial_{V_x}\Phi_s, \tag{B18}$$

so $\mathcal{I} = k^{-1}\sum_s n_s\langle\partial_{V_x}\Phi_s\rangle_{f_s}$. From Eq. (B13), one gets

$$k^{-1}\langle\partial_{V_x}\Phi_s\rangle_{f_s} = -\langle\partial_\omega\Phi_s\rangle_{f_s} = -\partial_\omega\langle\Phi_s\rangle_{f_s}, \qquad \text{(B19)}$$

and therefore $\mathcal{I} = -\partial_\omega\sum_s n_s\langle\Phi_s\rangle_{f_s}$, or

$$\mathcal{I} = \frac{|\tilde{\mathbf{E}}|^2}{16\pi}\,\partial_\omega\sum_s 4\pi n_s\langle\alpha_s\rangle_{f_s}, \qquad \text{(B20)}$$

where we substituted Eq. (B1) for Φ_s. Using Eq. (B14), one thereby matches Eq. (B17), as anticipated.

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