Lie-algebraic structure of 2D harmonic oscillator with non-separable complex coupling

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Abstract

Using SU(2)×SU(2) Lie-group structure we obtain the algebraization of 2D harmonic oscillator model with complex quadratic coupling. It is shown that the original time-independent Schrödinger equation in Cartesian coordinates, when mapped to a curved manifold (in general) of arbitrary metric, is expressible as a quadratic combination of group generators modulo a gauge freedom. We propose an improvisation of the usual Lie-algebraic scheme for two critical values of the coupling parameter which makes the problem non-diagonalizable and non-separable. Recently reported results about this interesting non-Hermitian Hamiltonian are confirmed by solving the corresponding spectral problem in a purely algebraic procedure.

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1 Introduction

Finding algebraic structure of a system (classical or quantum mechanical) is always advantageous as such an inherent dynamical symmetry gives us opportunity of classifying the concerned systems in association with different groups. Searching for a Lie-algebraic description becomes even more important and essential for a quantum system due to the increasing complexity arisen for switching from classical observable to a quantum-mechanical operator. As soon as a system is found to possess an explicit or hidden algebraic structure, one can extract all the information about the system encoded in purely algebraic quantities.

Representation space of a Lie group is the arena where we talk in algebraic language. The dimension of representation space may be finite or infinite. However since the spectacular discovery of partial algebraization [1–3] in late 80s, the investigation over the years (see [4] for a lucid discussion covering broad variation of quantal systems) along this direction remains confined within finite-dimensional representation space. This is because in this approach everything, whether it is vector components of group-generators, coefficients of derivatives in the Lie-algebraic realization of Hamiltonian or the wave functions, is expressed in terms of certain polynomials, the best known mathematical entity to deal with.

Over the last two decades the extensive research reports help us to understand that a 1D stationary Schrödinger equation in Hermitian as well as in non-Hermitian platform with a variety

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of quantum potentials [5–9] indeed possesses sl(2) Lie-algebraic structure. This is true for both of exactly solvable (ES) and quasi-exactly solvable (QES) Hamiltonian. The reason for such an apparently surprising phenomena is the exceptional combination of group generators for an ES Hamiltonian making it free from quantum number/s of the Lie group, a fact explained long ago in Ref [4] and recently reinforced by proposing a unified platform [10] through a modification via an additional transformation function. Thus our understanding about the algebraic structure for 1D models is moderately clear. But unfortunately this is not true for multidimensional quantum mechanics (QM).

It should be mentioned that going from 1D to d-dimensional system [11] one encounters several new issues which are not present in one dimension. For instance, apart from solvability the question arises whether the system is integrable or even superintegrable, a lot of literature exists in this field both in classical and quantum level [12–21] These issues for nontrivial higher dimensional models such as non-separable and non-diagonalizable complex quantum Hamiltonian are yet to be properly understood. It is only recently [19] that a novel relation between maximal superintegrability and exact solvability is established for 2D models.

An important recent generalization in QM is due to Bender et al [22] who pointed out that a class of PT-symmetric potentials possesses real eigenvalues (see, for up-to-date survey, [23, 24]). This breakthrough discovery started a new era of non-Hermitian QM thanks to the leading contributions [5, 25–37] from science community. Along this field also 1D models are first reported and then attention goes to the multidimensional models.

Expertise in 1D supersymmetric QM [38] is employed for multidimensional Hermitian and non-Hermitian QM. Series of articles [39–43] contribute in the development of multidimensional Hamiltonian using point-canonical transformation or higher-dimensional intertwining relations and shape-invariance formalism including the possible modifications in the supercharges to accommodate non-Hermitian systems.

However, the Lie-algebraic structures of most of the multidimensional models are not known. It is absolutely necessary to start a systematic investigation on this field. The purpose of present communication is to uncover the hidden algebra of an interesting 2D model namely harmonic oscillator with complex quadratic coupling in real Cartesian plane. Except for two critical values of the coupling parameter (pointed out recently [43]) the Hamiltonian has been shown to be separable [42] in some complex plane. The novelty of our finding is that the algebraization is found not only for separable case but also for non-diagonalizable and non-separable coupling.

2 2D harmonic oscillator with complex coupling & SU(2)×SU(2)

Let us consider following 2D stationary Schrödinger equation in real Cartesian coordinates (x, y) [chosen atomic units: $\hbar = m = 1$]

$$H(x,y)\psi(x,y) \equiv \left[-\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) + V(x,y)\right]\psi(x,y) = E\psi(x,y) \tag{1}$$

for the non-Hermitian potential

$$8\,V(x,y) = \omega^2 x^2 + \tilde{\omega}^2 y^2 + 2\,\mathrm{i}\,\mu\,xy \qquad (\omega, \tilde{\omega}, \mu \in \mathbb{R})\,. \tag{2}$$

Note that for $\mu = 0$, the potential simply becomes sum of two 1D (real) oscillators. Thus in our model $\mu \neq 0$ and also the frequencies $\omega, \tilde{\omega}$ are positive. Clearly PT-symmetry is not preserved for this model. Several significant properties of this system such as its pseudo-Hermiticity, relation between real spectra and the potential parameters etc. are already known. We are interested about its hidden algebraic structure, which is unknown.

In the first step, we are to choose suitable coordinates for the representation space of the Lie group. This space must be preserved by every defined operators including the Hamiltonian. Hence

it is not expected that the group coordinates would coincide with the physical coordinates (x, y) for a general potential interaction. Suppose (ξ, η) denotes the coordinates on the orbit of SU(2)×SU(2) group. Then its representation space $R_{n\tilde{n}}$ is of dimension $(n+1)(\tilde{n}+1)$ with the polynomial basis functions

$$\{\xi^n\eta^{\tilde{n}}, \xi^{n-1}\eta^{\tilde{n}}, \cdots, \eta^{\tilde{n}}; \xi^n\eta^{\tilde{n}-1}, \xi^{n-1}\eta^{\tilde{n}-1}, \cdots, \eta^{\tilde{n}-1}; \cdots \xi^n\eta, \xi^{n-1}\eta, \cdots \eta; \xi^n, \xi^{n-1}, \cdots, 1\}\,.$$

After fixing the coordinates, we can write down six generators of the group

$$\left.\begin{array}{lcl|lcl} J_1 &=& \xi^2\partial_\xi - n\xi & J_4 &=& \eta^2\partial_\eta - \tilde{n}\,\eta \\ J_2 &=& \xi\partial_\xi - n/2 & J_5 &=& \eta\partial_\eta - \tilde{n}/2 \\ J_3 &=& \partial_\xi & J_6 &=& \partial_\eta \end{array}\right\}\,, \tag{3}$$

where $n, \tilde{n}$ are quantum numbers associated with the group assuming (independently of each other) non-negative integer values. The cooresponding Lie algebra is closed with respect to the commutation relations

$$\left.\begin{array}{lll} [J_1, J_2] = -J_1\,, & [J_2, J_3] = -J_3\,, & [J_3, J_1] = 2J_2 \\ [J_4, J_5] = -J_4\,, & [J_5, J_6] = -J_6\,, & [J_6, J_4] = 2J_5 \end{array}\right\}\,. \tag{4}$$

Note that we have two degenerate relations

$$(2J_2)^2 - 2\{J_1, J_3\} = n(n+2)\,, \quad (2J_5)^2 - 2\{J_4, J_6\} = \tilde{n}(\tilde{n}+2)\,. \tag{5}$$

In the next section particular transformation function will be fixed.

3 Linear complex transformation and gauge rotation

In the 2nd step of algebraization, we are to choose the transformation rules between physical coordinates (x, y) and the group coordinates (ξ, η). This is crucial for the success of the program. The observation is as follows. It is inevitable that every non-derivative quantity in transformed coordinate system must be certain polynomial. Now the given potential is already quadratic polynomial in x, y and so must also be a polynomial (not necessarily of same degree) in ξ, η. Thus only polynomial relation between two systems are allowed. We will proceed with the simplest choice of linear transformations as this corresponds to constant metric of the transformed space.

Without loss of generality, we assume following relations between two coordinate systems

$$\xi = x + ay\,, \quad \eta = x + \tilde{a}y\,, \qquad (a, \tilde{a} \in \mathbb{C}\,, a \neq \tilde{a})\,. \tag{6}$$

Note that the metric corresponding to this transformation is constant: $g = 1/(\tilde{a} - a)^2$. The kinetic part of original Schrödinger Hamiltonian (1) then transforms to the form

$$-2T(x, y) \to -2T(\xi, \eta) = (1 + a^2)\partial_\xi^2 + (1 + \tilde{a}^2)\partial_\eta^2 + 2(1 + a\tilde{a})\partial_\xi\partial_\eta\,. \tag{7}$$

We see that the new coordinate system is non-orthogonal for general values of $a, \tilde{a}$. For simplicity, let us make it orthogonal by choosing digonalized metric. This implies that the two transformation parameters are no longer independent

$$1 + a\tilde{a} = 0\,. \tag{8}$$

The algebraic structure of the model is hidden. To uncover it, let us utilize the gauge freedom

$$\psi(x, y) = \exp[-\Omega(\xi, \eta)]\;\chi(\xi, \eta)|_{\xi = x+ay\,, \eta = x+\tilde{a}y}\,, \qquad (\chi \in R_{n\tilde{n}}) \tag{9}$$

The purpose of this gauge transformation is straightforward. We need the eigenfunctions of group eigenvalue equation $H_G(\xi, \eta)\,\chi = E\,\chi$ as polynomials in ξ, η. Two points worth mentioning. In the first place, the maximum degree of the polynomial $\chi(\xi, \eta)$ must be $n + \tilde{n}$ (degree in $\xi \leq n$ & that in $\eta \leq \tilde{n}$). In the second place, the total number of allowed levels is equal to dim $R_{n\tilde{n}} = (n+1)(\tilde{n}+1)$.

Hence generically the generated potential will belong to QES class unless an exceptional combination of generators makes the Hamiltonian free from $n, \tilde{n}$. Below we will show that this is actually the case for our model qualifying it as an ES member.

The explicit expression for the gauge-transformed Hamiltonian $H_G(\xi,\eta) \equiv \mathrm{e}^{\Omega} H(\xi,\eta)\mathrm{e}^{-\Omega}$ is as follows

$$2H_G(\xi,\eta) = -(1+a^2)\partial_\xi^2 - (1+\tilde{a}^2)\partial_\eta^2 + 2\left\{(1+a^2)\Omega_\xi\partial_\xi + (1+\tilde{a}^2)\Omega_\eta\partial_\eta + V + \triangle V\right\}, \qquad (10)$$

where the suffix denotes partial differentiation with respect to that. The gauge-potential $\triangle V$ is given by the expression

$$2\triangle V \equiv (1+a^2)(\Omega_{\xi\xi} - \Omega_\xi^2) + (1+\tilde{a}^2)(\Omega_{\eta\eta} - \Omega_\eta^2)\,. \qquad (11)$$

In the next section we will show that the gauged Hamiltonian H_G, given by (10), can be expressed in terms of group generators (3). The general relation must be quadratic.

4 Lie-algebraic representation for diagonalizable potential

In this section we will find the algebraization for the Hamiltonian $H_G(\xi,\eta)$ with diagonalized kinetic term [see equation (10)]. A close inspection on the generators (3) reveals that the Hamiltonian (10) should have following Lie-algebraic representation

$$-2(H_G - E) = (1+a^2)J_3^2 + (1+\tilde{a}^2)J_6^2 + k_1 J_2 + \tilde{k}_1 J_5 + 2D_1(n,\tilde{n})\,. \qquad (12)$$

Thus the bilinear terms are fixed to match with second order derivative terms. The linear terms are to be determined to get the desired form (2). The reason for using suffix '1' in the coefficients of linear terms will be clear shortly. The additive constant D_1 is introduced to have the convenient form of the potential and the eigenvalue equation $H_G\,\chi = E\,\chi$.

The issue is to balance the first order derivative terms between (10) and its Lie-algebraic representation (12), which yields two first-order partial differential equations (PDE)

$$-2(1+a^2)\Omega_\xi = k_1\xi\,, \qquad -2(1+\tilde{a}^2)\Omega_\eta = \tilde{k}_1\eta\,. \qquad (13)$$

On integration, we get the form of the phase function

$$-4\Omega(\xi,\eta) = k_1\xi^2/(1+a^2) + \tilde{k}_1\eta^2/(1+\tilde{a}^2)\,. \qquad (14)$$

The determination of two constants $k_1, \tilde{k}_1$ will be the final step of algebraization. Comparing the derivative-free terms, we get the following equation

$$V - E = -\triangle V - D + (nk_1 + \tilde{n}\tilde{k}_1)/4\,. \qquad (15)$$

Substituting the phase function $\Omega(\xi,\eta)$ from (14) into (11), the gauge-potential $\triangle V$ can be calculated. Hence the final forms of the generated potential and energy eigenvalues are

$$8V(x,y) = \left(\frac{k_1^2}{1+a^2} + \frac{\tilde{k}_1^2}{1+\tilde{a}^2}\right)x^2 + \left(\frac{a^2k_1^2}{1+a^2} + \frac{\tilde{a}^2\tilde{k}_1^2}{1+\tilde{a}^2}\right)y^2 + 2\left(\frac{ak_1^2}{1+a^2} + \frac{\tilde{a}\tilde{k}_1^2}{1+\tilde{a}^2}\right)xy \qquad (16)$$

$$E_{n\tilde{n}} = -\frac{k_1}{4}(n+1) - \frac{\tilde{k}_1}{4}(\tilde{n}+1) + D\,, \quad n,\tilde{n} = 0,1,2,\ldots \qquad (17)$$

At this point we recognize the potential as a genuine ES candidate as its explicit form (16) is free from quantum numbers $n, \tilde{n}$. Now the two forms (16) and (2) for the potential must be identical.

This gives us following three linear equations for three unknowns $k_1, \tilde{k}_1, a$ [note that $a\tilde{a} = -1$ according to (8)]

$$\frac{k_1^2}{1+a^2} + \frac{\tilde{k}_1^2}{1+\tilde{a}^2} = \omega^2\,, \quad \frac{a^2k_1^2}{1+a^2} + \frac{\tilde{a}^2\tilde{k}_1^2}{1+\tilde{a}^2} = \tilde{\omega}^2\,, \quad \frac{ak_1^2}{1+a^2} + \frac{\tilde{a}\tilde{k}_1^2}{1+\tilde{a}^2} = \mathrm{i}\,\mu\,. \tag{18}$$

Our algebraization will work if the system (18) is consistent. Examining the first two equations, it is not difficult to convince that for consistency $a^2 \neq \tilde{a}^2$ must hold. Now $a \neq \tilde{a}$ is set from the beginning as a requirement of non-vanishing Jacobian. The other restriction, namely $a \neq -\tilde{a}$ must be the automatic choice, for otherwise $a\tilde{a} = -1$ would imply $a = \pm 1, \tilde{a} = \mp 1$, pushing us towards a particular case $\omega = \tilde{\omega}$. We, therefore conclude that our algebraization is successful. Solving first two equations for $k_1, \tilde{k}_1$ and substituting them into the last equation of (18), we have the following expressions

$$k_1^2 = \frac{(1+a^2)(\omega^2\tilde{a}^2 - \tilde{\omega}^2)}{\tilde{a}^2 - a^2}\,, \quad \tilde{k}_1^2 = \frac{(1+\tilde{a}^2)(\tilde{\omega}^2 - a^2\omega^2)}{\tilde{a}^2 - a^2}\,, \quad \omega^2 - \tilde{\omega}^2 + \mathrm{i}\,\mu(a+\tilde{a}) = 0\,. \tag{19}$$

Using the relation $a\tilde{a} = -1$, the last equation gives following complex values for the transformation parameters

$$-2\mu a = \mathrm{i}\left[\left(\tilde{\omega}^2 - \omega^2\right) + \delta\right]\,, \quad -2\mu\tilde{a} = \mathrm{i}\left[\left(\tilde{\omega}^2 - \omega^2\right) - \delta\right]\,, \quad \delta = +\sqrt{(\tilde{\omega}^2 - \omega^2)^2 - 4\mu^2}\,. \tag{20}$$

Substituting back the values (20) for $a, \tilde{a}$ into first two expressions for $k_1, \tilde{k}_1$ in equation (19), one may express their final values after some manipulations

$$k_1\sqrt{2} = -\sqrt{\omega^2 + \tilde{\omega}^2 + \delta}\,, \qquad \tilde{k}_1\sqrt{2} = -\sqrt{\omega^2 + \tilde{\omega}^2 - \delta}\,. \tag{21}$$

Note that the negative square root is taken for $k_1, \tilde{k}_1$ to ensure the normalizability of the wave functions for real case $\mu = 0$ [see equations (9) and (14)].

The wave functions and the spectra can be obtained explicitly following purely algebraic procedure. The procedure is to fit a polynomial (in ξ, η of degree $\xi \leq n$ and degree $\eta \leq \tilde{n}$) in the eigenvalue equation $H_G\,\chi = E\,\chi$ for each pair of values of $n, \tilde{n}$, where H_G is given by the Lie-algebraic realization (12). For the ES potential, the resulting PDE usually coincides with some known form. Indeed the substitution $\chi(\xi, \eta) = \chi^{(1)}(\xi)\chi^{(2)}(\eta)$ splits it into two Hermite equations, provided we chose zero value for separation constant and the constant D_1 is taken as

$$-4D_1(n, \tilde{n}) = nk_1 + \tilde{n}\tilde{k}_1\,. \tag{22}$$

The final expressions for the wave functions and energy eigenvalues emerge as

$$\left.\begin{aligned} &\chi^{(1)} \propto \mathrm{H}_n\left(\xi\sqrt{\tfrac{-k_1}{2(1+a^2)}}\right)\,, \quad \chi^{(2)} \propto \mathrm{H}_{\tilde{n}}\left(\eta\sqrt{\tfrac{-\tilde{k}_1}{2(1+\tilde{a}^2)}}\right)\,, \\ &-4E_{n\tilde{n}} = (2n+1)k_1 + (2\tilde{n}+1)\tilde{k}_1\,. \end{aligned}\right\} \tag{23}$$

For a non-Hermitian model, one of the important issue is the existence of real spectra. From the above results it is clear that the quantity δ, given by (20), plays a crucial role in this context. In other words the range of value of the coupling parameter μ controls the nature of the spectra due to the following fact

$$\delta \in \mathbb{R} \text{ or } \in \mathrm{i}\mathbb{R} \text{ according as } 2\mu < (\text{ or } >)|\tilde{\omega}^2 - \omega^2|\,. \tag{24}$$

It is not difficult to show that the reality of δ implies the reality of $k_1, \tilde{k}_1$. On the other hand, when δ is purely imaginary, we have $\tilde{k}_1 = k_1^*$. Consequently, we have the following conclusions, reported first in [42]:

1. For $2\mu < |\tilde{\omega}^2 - \omega^2|$, full spectra is real.

2. For $2\mu > |\tilde{\omega}^2 - \omega^2|$, the spectra is real if and only if $n = \tilde{n}$ and otherwise energy eigenvalues appear in conjugate pair.

Deliberately we didn't include two critical values of the coupling parameter in the above discussion, which are

$$2\mu = |\tilde{\omega}^2 - \omega^2| \,. \tag{25}$$

This is because we have to check carefully whether these two values are allowed in our algebraic scheme. The answer is unfortunately negative. Note that we set $1 + a\tilde{a} = 0$ to get orthogonal coordinate system (ξ, η). This, in turn, means $1 + a^2 \neq 0, 1 + \tilde{a}^2 \neq 0$ must hold for otherwise we would have vanishing metric, which is non-physical. These restrictions imply from the values (20) of $a, \tilde{a}$ that the two critical values (25) are prohibited in this scheme.

Hence we have found the algebraization of the Schrödinger equation (1) with 2D harmonic oscillator with complex coupling (2) for all values of the potential parameters $\omega, \tilde{\omega}, \mu$ except for two critical values of the coupling parameter μ given in (25). It is interesting to note that these are the values for which the model is studied recently [43]. Within the supersymmetric context, the authors show that the model has a shape-invariance property for which it becomes an ES model. The most important conclusion made there is that the Hamiltonian is non-diagonalizable and non-separable for those two critical values of μ. In the next section we will show that this non-separable Hamiltonian also have hidden algebraic structure.

5 Improvised Lie-algebraic approach for non-diagonalizable potential

Our purpose, in this section, is to find algebraic structure of non-diagonalizable and non-separable Hamiltonian (1) and (2) for two critical values (25) of the coupling parameter. For definiteness, we will consider

$$2\mu = \tilde{\omega}^2 - \omega^2 \,. \tag{26}$$

Let us recall that the non-zero restrictions for the diagonal metric components $1 + a^2$, $1 + \tilde{a}^2$ were responsible for excluding two critical values in the previous algebraization. This motivates us to choose an unusual metric induced by

$$1 + a^2 = 0 \,, \quad 1 + \tilde{a}^2 = 0 \,, \quad 1 + a\tilde{a} \neq 0 \,, \tag{27}$$

whose solution is

$$a = \mathrm{i} \,,\ \tilde{a} = -\mathrm{i} \Rightarrow 1 + a\tilde{a} = 2 \,,\ \xi = x + \mathrm{i}y \,,\ \eta = x - \mathrm{i}y = \xi^* \,. \tag{28}$$

The gauged Hamiltonian corresponding to this non-vanishing off-diagonal metric $g = -4$ may be written as

$$H_G\, \chi \equiv [2\,(-\partial_\xi \partial_\eta + \Omega_\eta \partial_\xi + \Omega_\xi \partial_\eta) + V + 2\,(\Omega_{\xi\eta} - \Omega_\xi \Omega_\xi)]\, \chi = E\, \chi \,. \tag{29}$$

Note that the expression for V in (ξ, η) coordinates is given by

$$16V(\xi, \eta) = (\omega^2 + \tilde{\omega}^2)\xi\eta - 2\mu\eta^2 \,, \tag{30}$$

where we have utilized the critical value (26).

Clearly the Hamiltonian (29) together with (30) is not directly expressible in terms of group generators. We propose an improvisation by introducing an arbitrary function $f(\xi, \eta)$. Hence, instead of H_G, we claim that $H_G + f$ can be expressed as a bilinear combination of group generators as follows

$$-\,(H_G + f) = 2J_3 J_6 + k_2 J_2 + \tilde{k}_2 J_5 + D_2(n, \tilde{n}) - E \,. \tag{31}$$

The purpose of using suffix '2' in the coefficients of linear terms should be clear now. Comparing first-order derivative terms between (29) and (31), we get two PDEs

$$-2\Omega_\eta = k_2\xi\,, \qquad -2\Omega_\xi = \tilde{k}_2\eta\,, \tag{32}$$

from which, we conclude, for consistency, the following

$$k_2 = \tilde{k}_2 = -2\ell\ (\ell > 0)\,, \qquad \Omega(\xi,\eta) = \ell\,\xi\eta\,. \tag{33}$$

The constant ℓ is yet to be determined.

Balancing derivative-free term, we get

$$V - E = 2(\ell^2\xi\eta - \ell) - \ell(n+\tilde{n}) - D_2 - f\,, \tag{34}$$

where we substitute the phase function Ω from (32). The above equation may be split as

$$V = 2\ell^2\xi\eta - f\,, \quad E_s = \ell(s+2) + D_2\,, \qquad s = 0,1,2,\ldots \tag{35}$$

It is important to note that we are using index s in place of $n+\tilde{n}$, but this means that a given value of s is achieved for $(s+1)$ different pairs of values of $n,\tilde{n}$. The function f can now be found by substituting (30) for V into first equation of (35). This gives

$$f = \mu\eta^2/8\,, \qquad [\mu \text{ is given by (26)}]\,, \tag{36}$$

for the choice

$$32\ell^2 = \omega^2 + \tilde{\omega}^2\,. \tag{37}$$

The most crucial step is to find the wave functions and spectra by solving the equation $H_G\,\chi = E\,\chi$. Note that introduction of f in Lie-algebraic realization (31) for H_G definitely destroys the algebraic procedure, since now $\chi \notin R_{n\tilde{n}}$. To restore the algebraic setup, we assume that there exists a second gauge transformation

$$\chi = \exp[-\tilde{\Omega}]\tilde{\chi} \tag{38}$$

such that $\tilde{\chi}$ becomes a polynomial in ξ,η. The success of our improvised algebraic scheme depends on the existence of such a phase function $\tilde{\Omega}(\xi,\eta)$.

A straightforward computation shows that the equation $H_G\,\chi = E\,\chi$ converts, after second gauge transformation (38) into following equivalent eigenvalue equation

$$\Big[\partial_\xi\partial_\eta - \left(\ell\xi + \tilde{\Omega}_\eta\right)\partial_\xi - \left(\ell\eta + \tilde{\Omega}_\xi\right)\partial_\eta + \tilde{\Omega}_\xi\tilde{\Omega}_\eta - \tilde{\Omega}_{\xi\eta} \\ +\ell\left(\xi\tilde{\Omega}_\xi + \eta\tilde{\Omega}_\eta\right) + \frac{\mu}{16}\eta^2 + \frac{1}{2}\left(D_2 + \ell s\right)\Big]\tilde{\chi} = 0\,. \tag{39}$$

The fact that $\tilde{\chi}$ is some polynomial in ξ,η implies that the phase function $\tilde{\Omega}(\xi,\eta)$ must also be certain polynomial in ξ,η. Thus their generic forms are

$$\tilde{\Omega} = \lambda\xi^a\eta^b\,, \qquad \tilde{\chi} = \xi^c\eta^d\,. \tag{40}$$

Our job remains to find the unknown quantities in (40). The underlying procedure is algebraic: To substitute (40) into (39) and then to equate the coefficients of each independent powers of ξ,η to zero. Clearly seven cases are to be investigated separately:– i) $abcd \neq 0$, ii) $d = 0, ab \neq 0$, iii) $d = b = 0$, iv) $d = a = 0$, v) $c = 0, ab \neq 0$, vi)$c = b = 0$, vii) $c = a = 0$. Without further details, we quote the final result. Only case vii) corresponds to consistent and non-trivial solutions for $\tilde{\Omega}$ and $\tilde{\chi}$:

$$\tilde{\Omega} = -\frac{\mu}{32\ell}\eta^2\,, \quad \tilde{\chi}_s = \eta^{\frac{E_s}{2\ell}-1}\,, \qquad s = 0,1,2,\ldots \tag{41}$$

where we have replaced D_2 in (39) according to second equation of (35). Let us mention that the phase function $\tilde{\Omega}$ in (41) produces the term opposite in sign of the term $f/2$ in (39).

Last step is to find E_s. So far we have not utilized the conjugate relation between the coordinates (ξ, η) namely $\eta = \xi^*$, which will now be used. Note that $\tilde{\chi}_s$ may be expressed in terms of modulus $r = +\sqrt{x^2+y^2}$ and the amplitude ϕ ($\cos\phi = x/r\,, \sin\phi = y/r$):

$$\tilde{\chi}_s(r,\phi) = r^{\frac{E_s}{2\ell}-1} \exp\left[\mathrm{i}\left(\frac{E_s}{2\ell}-1\right)\phi\right]\,. \tag{42}$$

Physical requirement is that the wave function must be single valued which implies 2π-periodicity of $\tilde{\chi}_s$, i. e. $\tilde{\chi}(r,\phi+2\pi) = \tilde{\chi}(r,\phi)$. This is true if and only if

$$\frac{E_s}{2\ell} - 1 = s\,, \text{an integer} \Rightarrow E_s = 2\ell(s+1)\,, \qquad s = 0,1,2,\ldots \tag{43}$$

Then the wave functions in the final form are

$$\psi_s(x,y) = \exp\left(-\ell\xi\eta + \frac{\mu}{32\ell}\eta^2\right)\eta^s\Big|_{\xi=x+iy,\eta=\xi^*}\,, \tag{44}$$

where μ and ℓ are respectively given by (26) and (37).

Hence, we have successfully obtained the algebraization for the critical value (26). The other critical value, namely $2\mu = \omega^2 - \tilde{\omega}^2$, may be included by simply replacing μ by $-\mu$ in the whole procedure. It may be mentioned that both of the expressions (43) and (44) for the spectra and wave functions, derived above, were obtained in Ref. [43] by using shape-invariance method within supersymmetric framework.

6 Conclusion and future problems

In recent times several 2D quantum potentials are reported. Among them the most interesting one is harmonic oscillator with complex quadratic coupling. Although the so-called PT-symmetry is not present, the model is pseudo-Hermitian and possesses either fully real spectra or conjugate pairs of energy eigenvalues. Very recently, in an important work, the authors pointed out that the model becomes non-separable and non-diagonalizable for two critical values of the coupling parameter, but surprisingly remains exactly-solvable with fully real spectra.

Our work explains that the cause behind such an amazing phenomena is the hidden algebraic structure of the model. Within SU(2)×SU(2) group, we have discovered the Lie-algebraic realization of the Hamiltonian. It is shown that for two critical values of the coupling parameter, an improvised Lie-algebraic approach is required to get its distinct algebraic structure. To the best of our knowledge, the algebraization of this model is not reported previously.

Following issues are being investigated presently. Whether s associated functions, needed for completeness of basis for non-separable case, may be constructed algebraically. Interestingly, the index $s = n+\tilde{n}$ means $(s+1)$-fold degeneracy for a given value of s, one of which is the wave function in s-th excited state. The improvisation proposed in this work may find interesting applications such as generating new non-trivial non-Hermitian Hamiltonian which are non-separable and non-diagonalizable.

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