Linear Differential Equations with Fuzzy Boundary Values

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***Abstract*—In this study, we consider a linear differential equation with fuzzy boundary values. We express the solution of the problem in terms of a fuzzy set of crisp real functions. Each real function from the solution set satisfies differential equation, and its boundary values belong to intervals, determined by the corresponding fuzzy numbers. The least possibility among possibilities of boundary values in corresponding fuzzy sets is defined as the possibility of the real function in the fuzzy solution.**

In order to find the fuzzy solution we propose a method based on the properties of linear transformations. We show that, if the corresponding crisp problem has a unique solution then the fuzzy problem has unique solution too. We also prove that if the boundary values are triangular fuzzy numbers, then the value of the solution at any time is also a triangular fuzzy number.

We find that the fuzzy solution determined by our method is the same as the one that is obtained from solution of crisp problem by the application of the extension principle.

We present two examples describing the proposed method.

Keywords: fuzzy boundary value problem, fuzzy set, linear transformation.

I. Introduction

Approaches to fuzzy boundary value problems can be of two types. The first approach assumes that, even if only the boundary values are fuzzy in the handling problem, the solution is fuzzy function, consequently, the derivative in the differential equation can be considered as a derivative of fuzzy function. This derivative can be Hukuhara derivative, or a derivative in generalized sense. Bede [1] has demonstrated that a large class of boundary value problems have not a solution, if Hukuhara derivative is used. To overcome this difficulty, in [2] and [3] the concept of generalized derivative is developed and fuzzy differential equations have been investigated using this concept (see also [4], [5], [6]). Recently, Khastan and Nieto [7] have found solutions for a large enough class of boundary value problems with the generalized derivative. However as it is seen from the examples in mentioned article, these solutions are difficult to interpret because four different problems, obtained by using the generalized second derivatives, often does not reflect the nature of the problem.

The second approach is based on generating the fuzzy solution from the crisp solution. In particular case, for the fuzzy initial value problem this approach can be of three ways. The first one uses the extension principle. In this way, the initial value is taken as a real constant, and the resulting crisp problem is solved. Then the real constant in the solution is replaced with the initial fuzzy value. In the final solution, arithmetic operations are considered to be operations on fuzzy numbers ([8], [9]). The second way, offered by Hüllemerier [10], uses the concept of differential inclusion. In this way, by taking an alpha-cut of initial value, the given differential equation is converted to a differential inclusion and the obtained solution is accepted as the alpha-cut of the fuzzy solution. Misukoshi et al [11] have proved that, under certain conditions, the two main ways of the approach are equivalent for the initial value problem. The third way is offered by Gasilov et al [12]. In this way the fuzzy problem is considered to be a set of crisp problems.

In this study, we investigate a differential equation with fuzzy boundary values. We interpret the problem as a set of crisp problems. For linear equations, we propose a method based on the properties of linear transformations. We show that, if the solution of the corresponding crisp problem exists and is unique, the fuzzy problem also has unique solution. Moreover, we prove that if the boundary values are triangular fuzzy numbers, then the value of the solution is a triangular fuzzy number at each time. We explain the proposed method on examples.

II. Fuzzy Boundary Value Problem

Below, we use the notation $\widetilde{u} = (u_L(r), u_R(r))$, $(0 \leq r \leq 1)$, to indicate a fuzzy number in parametric form. We denote $\underline{u} = u_L(0)$ and $\overline{u} = u_R(0)$ to indicate the left and the right limits of $\widetilde{u}$, respectively. We represent a triangular fuzzy number as $\widetilde{u} = (l, m, r)$, for which we have $\underline{u} = l$ and $\overline{u} = r$.

In this paper we consider a fuzzy boundary value problem (FBVP) with crisp linear differential equation but with fuzzy boundary values. For clarity we consider second order differential equation:

$$\begin{cases} x'' + a_1(t)x' + a_2(t)x = f(t) \\ x(0) = \widetilde{A} \\ x(T) = \widetilde{B} \end{cases} \tag{1}$$

We note that the coefficients of the differential equation are not necessary constant.

Let us represent the boundary values as $\widetilde{A} = a_{cr} + \widetilde{a}$ and $\widetilde{B} = b_{cr} + \widetilde{b}$, where a_{cr} and b_{cr} are vectors with possibility 1 and denote the crisp parts (the vertices) of $\widetilde{A}$ and $\widetilde{B}$; $\widetilde{a}$ and $\widetilde{b}$ denote the uncertain parts with vertices at the origin.

We split the problem (1) to following two problems:

1) Associated crisp problem (which is non-homogeneous):

$$\left\{ \begin{array}{c} x'' + a_1(t)x' + a_2(t)x = f(t) \\ x(0) = a_{cr} \\ x(T) = b_{cr} \end{array} \right. \tag{2}$$

2) Homogeneous problem with fuzzy boundary values:

$$\left\{ \begin{array}{c} x'' + a_1(t)x' + a_2(t)x = 0 \\ x(0) = \widetilde{a} \\ x(T) = \widetilde{b} \end{array} \right. \tag{3}$$

It is easy to see that, a solution of the given problem (1) is of the form $x(t) = x_{cr}(t) + x_{un}(t)$ (crisp solution + uncertainty). Here $x_{cr}(t)$ is a solution of the non-homogeneous crisp problem (2); while $x_{un}(t)$ is a solution of the homogeneous problem (3) with fuzzy boundary conditions. $x_{cr}(t)$ can be computed by means of analytical or numerical methods. Hence, (1) is reduced to solving a homogeneous equation with fuzzy boundary conditions (3). Therefore, we will investigate how to solve this problem.

We assume the solution x_{un} of the problem (3) be a fuzzy set $\widetilde{X}$ of real functions such as $x(t)$. Each function $x(t)$ must satisfy the differential equation and must have boundary values a and b from the sets $\widetilde{a}$ and $\widetilde{b}$, respectively. We define the possibility (membership) of the function $x(t)$ to be equal to the least possibility of its boundary values.

Mathematically, the fuzzy solution set can be defined as follows:

$$\widetilde{X} = \{x(t) \mid x'' + a_1(t)x' + a_2(t)x = 0;$$
$$x(0) = a;\ x(T) = b;\ a \in \widetilde{a};\ b \in \widetilde{b}\} \tag{4}$$

with membership function

$$\mu_{\widetilde{X}}(x(t)) = \min\left\{\mu_{\widetilde{a}}(a), \mu_{\widetilde{b}}(b)\right\} \tag{5}$$

The solution $\widetilde{X}$, defined above, can be interpreted as a fuzzy bunch of functions.

One can also interpret that we consider a FBVP as a set of crisp BVPs whose boundary values belong to the fuzzy sets $\widetilde{a}$ and $\widetilde{b}$.

A. A matrix representation of the solution in the crisp case

Here we consider crisp BVP for second order homogeneous linear differential equation:

$$\left\{ \begin{array}{c} x'' + a_1(t)x' + a_2(t)x = 0 \\ x(0) = a \\ x(T) = b \end{array} \right. \tag{6}$$

Let $x_1(t)$ and $x_2(t)$ be linear independent solutions of the differential equation. Then the general solution is $x(t) = c_1x_1(t) + c_2x_2(t)$. For c_1 and c_2 we have the following linear system

$$\left\{ \begin{array}{c} c_1x_1(0) + c_2x_2(0) = a \\ c_1x_1(T) + c_2x_2(T) = b \end{array} \right. \tag{7}$$

Below we obtain a matrix representation for the solution of the BVP. We rewrite the linear system (7) in matrix form:

$$M\ \mathbf{c} = \mathbf{u}$$

where $M = \begin{bmatrix} x_1(0) & x_2(0) \\ x_1(T) & x_2(T) \end{bmatrix}$; $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$; $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$.

The solution of the linear system is

$$\mathbf{c} = M^{-1}\ \mathbf{u} \tag{8}$$

We constitute a vector-function of linear independent solutions $\mathbf{s}(t) = [x_1(t) \quad x_2(t)]$. Then the general solution can be rewritten in matrix form as

$$x(t) = [x_1(t) \quad x_2(t)] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{s}(t)\ \mathbf{c}$$

Using (8) we have $x(t) = \mathbf{s}(t)\ M^{-1}\ \mathbf{u}$, or,

$$x(t) = \mathbf{w}(t)\ \mathbf{u} = w_1(t)\ a + w_2(t)\ b \tag{9}$$

where

$$\mathbf{w}(t) = \mathbf{s}(t)\ M^{-1} \tag{10}$$

B. The solution method for FBVP

Now we show how to find $\widetilde{X}(t)$ (the value of the solution for the problem (3) at a time t).

Let linear independent solutions of the crisp equation (3), $x_1(t)$ and $x_2(t)$, be known. Then we can constitute the vector $\mathbf{w}$ (see, formula (10)). According (4) and (9) we have:

$$\widetilde{X} = \left\{x(t) = \mathbf{w}(t)\ \mathbf{u} \mid \mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix};\ a \in \widetilde{a};\ b \in \widetilde{b}\right\} \tag{11}$$

Consider a fixed time t. Put $\mathbf{v} = \mathbf{w}(t)$. Then from (11) we have:

$$\widetilde{X}(t) = \left\{\mathbf{v}\ \mathbf{u} \mid \mathbf{u} = [a \quad b]^T;\ a \in \widetilde{a};\ b \in \widetilde{b}\right\} \tag{12}$$

To determine how is the set $\widetilde{X}(t)$ we consider the transformation $T(\mathbf{u}) = \mathbf{v}\ \mathbf{u}$ (here $\mathbf{v}$ is a fixed vector). One can see that $T : R^2 \rightarrow R^1$ is a linear transformation. Therefore, $\widetilde{X}(t)$ is the image of the set $\widetilde{B} = \left\{\mathbf{u} = [a \quad b]^T \mid a \in \widetilde{a};\ b \in \widetilde{b}\right\} = (\widetilde{a},\ \widetilde{b})$ under the linear transformation $T(\mathbf{u})$.

We remember some properties of linear transformations [13]:

1. A linear transformation maps the origin (zero vector) to the origin (zero vector).

2. Under a linear transformation the images of a pair of similar figures are also similar.

3. Under a linear transformation the images of nested figures are also nested.

In addition, we shall reference a property of fuzzy number vectors.

4. The fuzzy set $\widetilde{B} = (\widetilde{a},\ \widetilde{b})$ forms a fuzzy region in the ab-coordinate plane, vertex of which is located at the origin

and boundary of which is a rectangle. Furthermore, the α-cuts of the region are rectangles nested within one another.

The facts 1-4 allow us to derive the following conclusion. The vector $\widetilde{B}$, components of which are the boundary values $\widetilde{a}$ and $\widetilde{b}$, form a fuzzy rectangle in the ab-coordinate plane. The linear transformation $T(\mathbf{u})$ maps this fuzzy rectangle to a fuzzy interval in R^1. Therefore, the solution at any time forms a fuzzy number.

C. Particular case when boundary values are triangular fuzzy numbers

In particular, if $\widetilde{a}$ and $\widetilde{b}$ triangular fuzzy numbers, the α-cuts of the region $\widetilde{B} = (\widetilde{a},\ \widetilde{b})$ are nested rectangles, furthermore, they are similar. According to the discussion above, their images are intervals that also are nested and similar, consequently, form a triangular fuzzy number $\widetilde{X}(t)$. Therefore, $\widetilde{X}(t)$ can be represented in the form $\widetilde{X}(t) = (\underline{x}(t), 0, \overline{x}(t))$. Now we investigate how to calculate $\underline{x}(t)$ and $\overline{x}(t)$.

Let $\widetilde{a} = (\underline{a}, 0, \overline{a})$, $\widetilde{b} = (\underline{b}, 0, \overline{b})$ and $\mathbf{w}(t) = (w_1(t), w_2(t))$. Since $\overline{x}(t)$ is the maximum value among all products $\mathbf{w}(t){\cdot}\mathbf{u} = aw_1(t) + bw_2(t)$, we have:

$$\overline{x}(t) = \max\left\{\underline{a}w_1(t), \overline{a}w_1(t)\right\} + \max\left\{\underline{b}w_2(t), \overline{b}w_2(t)\right\}$$
$$\underline{x}(t) = \min\left\{\underline{a}w_1(t), \overline{a}w_1(t)\right\} + \min\left\{\underline{b}w_2(t), \overline{b}w_2(t)\right\}$$

Note that an α-cut of $\widetilde{X}(t)$ can be determined by similarity:

$$X_\alpha(t) = \left[\underline{x_\alpha}(t), \overline{x_\alpha}(t)\right] = (1-\alpha)\left[\underline{x}(t), \overline{x}(t)\right] = (1-\alpha)X_0(t)$$

Formulas for $\underline{x}(t)$ and $\overline{x}(t)$ allow us to represent the solution in a new way:

$$\widetilde{X}(t) = w_1(t)\ \widetilde{a} + w_2(t)\ \widetilde{b}$$

where the operations assumed to be multiplication of real number with fuzzy one, and addition of fuzzy numbers.

D. General case when boundary values are parametric fuzzy numbers

In the general case, when $\widetilde{a}$ and $\widetilde{b}$ are arbitrary fuzzy numbers, the solution can be obtained by using α-cuts. Let $a_\alpha = \left[\underline{a_\alpha}, \overline{a_\alpha}\right]$ and $b_\alpha = \left[\underline{b_\alpha}, \overline{b_\alpha}\right]$. Then $B_\alpha = \left[\underline{a_\alpha}, \overline{a_\alpha}\right] \times \left[\underline{b_\alpha}, \overline{b_\alpha}\right]$. By similar argumentation to the preceding case, for the α-cut of the solution we obtain the following formulas:

$$X_\alpha(t) = \left[\underline{x_\alpha}(t), \overline{x_\alpha}(t)\right]$$

where

$$\overline{x_\alpha}(t) = \max\left\{\underline{a_\alpha}w_1(t), \overline{a_\alpha}w_1(t)\right\} + \max\left\{\underline{b_\alpha}w_2(t), \overline{b_\alpha}w_2(t)\right\}$$
$$\underline{x_\alpha}(t) = \min\left\{\underline{a_\alpha}w_1(t), \overline{a_\alpha}w_1(t)\right\} + \min\left\{\underline{b_\alpha}w_2(t), \overline{b_\alpha}w_2(t)\right\}$$

Based on the formulas above we can conclude, that the representation for solution

$$\widetilde{X}(t) = w_1(t)\ \widetilde{a} + w_2(t)\ \widetilde{b} \qquad (13)$$

is valid in general. Thus, the solution, defined by formula (4), becomes the same as the solution obtained from (9) by extension principle.

Remark: The approach is valid also for the general case, when nth-order m-point boundary value problem is considered.

E. Solution algorithm

Based on the arguments above, we propose the following algorithm to solve the problem (1):

1. Represent the boundary values as $\widetilde{A} = a_{cr} + \widetilde{a}$ and $\widetilde{B} = b_{cr} + \widetilde{b}$.
2. Find linear independent solutions $x_1(t)$ and $x_2(t)$ of the crisp differential equation $x'' + a_1(t)x' + a_2(t)x = 0$. Constitute the vector-function $\mathbf{s}(t) = [x_1(t) \quad x_2(t)]$, the matrix M and calculate the vector $\mathbf{w}(t) = (w_1(t), w_2(t))$ by formula (10).
3. Find the solution $x_{cr}(t)$ of the non-homogeneous crisp problem (2).
4. The solution of the given problem (1) is

$$\widetilde{x}(t) = x_{cr}(t) + w_1(t)\ \widetilde{a} + w_2(t)\ \widetilde{b} \qquad (14)$$

III. EXAMPLES

Example 1. Solve the FBVP:

$$\begin{cases} x'' - 3x' + 2x = 4t - 6 \\ x(0) = (1.5,\ 2,\ 3) \\ x(1) = (\ 2,\ \ 3,\ 4) \end{cases} \qquad (15)$$

Solution: We represent the solution as $\widetilde{x}(t) = x_{cr}(t) + \widetilde{x}_{un}(t)$.

1. We solve crisp non-homogeneous problem

$$\begin{cases} x'' - 3x' + 2x = 4t - 6 \\ x(0) = 2 \\ x(1) = 3 \end{cases}$$

and find the crisp solution $x_{cr}(t) = 2t + \frac{1}{e^2-e}\left[2(e^{2+t} - e^{1+2t}) + (e^{2t} - e^t)\right]$ (the thick line in Fig.1).

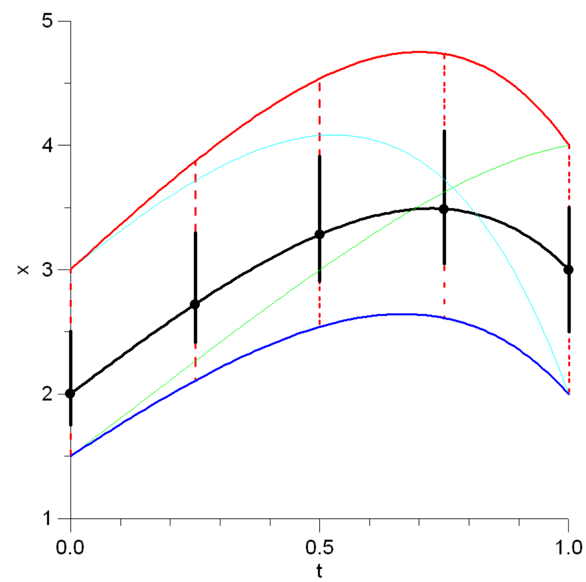

Fig. 1. The fuzzy solution and its $\alpha = 0.5$-cut for Example 1. Dashed and thick bars represent the values of the fuzzy solution and its $\alpha = 0.5$-cut at different times, respectively.

2. We consider fuzzy homogeneous problem

$$\begin{cases} x'' - 3x' + 2x = 0 \\ x(0) = (-0.5,\ 0,\ 1) \\ x(1) = (-1,\ 0,\ 1) \end{cases}$$

$x_1(t) = e^t$ and $x_2(t) = e^{2t}$ are linear independent solutions for the differential equation. Then $\mathbf{s}(t) = \begin{bmatrix} e^t & e^{2t} \end{bmatrix}$,
$M = \begin{bmatrix} 1 & 1 \\ e & e^2 \end{bmatrix}$ and
$\mathbf{w} = \mathbf{s}(t)\ M^{-1} = \frac{1}{e^2-e}\begin{bmatrix} e^{2+t} - e^{1+2t} & e^{2t} - e^t \end{bmatrix}$.
The formula (13) gives the solution of homogeneous problem:

$$\widetilde{x}_{un}(t) = \tfrac{1}{e^2-e}((e^{2+t} - e^{1+2t})\ (-0.5,\ 0,\ 1)+ \\ (e^{2t} - e^t)\ (-1,\ 0,\ 1)) \tag{16}$$

where the arithmetic operations are considered to be fuzzy operations. We add this solution to the crisp solution and get the fuzzy solution of the given FBVP (15):

$$\widetilde{x}(t) = 2t + \tfrac{1}{e^2-e}((e^{2+t} - e^{1+2t})\ (1.5,\ 2,\ 3)+ \\ (e^{2t} - e^t)\ (0,\ 1,\ 2)) \tag{17}$$

The fuzzy solution $\widetilde{x}(t)$ forms a band in the tx-coordinate plane (Fig. 1). Since $w_1(t) > 0$ and $w_2(t) > 0$ for $0 < t < T$, the upper border of the band, $\overline{x}(t)$, becomes the solution of the crisp non-homogeneous problem with the upper boundary values $\overline{A} = 3$ and $\overline{B} = 4$, while the lower border $\underline{x}(t)$ corresponds to $\underline{A} = 1.5$ and $\underline{B} = 2$:

$$\begin{aligned} \overline{x}(t) &= 2t + \frac{1}{e^2-e}((e^{2+t} - e^{1+2t})\cdot\ 3\ + (e^{2t} - e^t)\cdot 2) \\ \underline{x}(t) &= 2t + \frac{1}{e^2-e}((e^{2+t} - e^{1+2t})\cdot 1.5 + (e^{2t} - e^t)\cdot 0) \end{aligned}$$

We can express the solution $\widetilde{x}(t)$ also via α-cuts, which are intervals $x_\alpha(t) = \left[\underline{x_\alpha}(t),\ \overline{x_\alpha}(t)\right]$ at any time t. Since the boundary values are triangular fuzzy numbers, $\widetilde{x}_{un}(t)$ also is a triangular fuzzy number, say $\widetilde{x}_{un}(t) = (\underline{x_{un}}(t),\ 0,\ \overline{x_{un}}(t))$. Consequently, an α-cut of $\widetilde{x}_{un}(t)$ can be determined by similarity with coefficient $(1-\alpha)$, i.e.

$$x_{un,\ \alpha}(t) = (1-\alpha)\left[\underline{x_{un}}(t),\ \overline{x_{un}}(t)\right]$$

Adding the crisp solution gives the α-cut of the solution $\widetilde{x}(t)$:

$$\left[\underline{x_\alpha}(t),\ \overline{x_\alpha}(t)\right] = 2t + (1-\alpha)\cdot\tfrac{1}{e^2-e}((e^{2+t} - e^{1+2t})\ [1.5,\ 3] + \\ (e^{2t} - e^t)\ [0,\ 2])$$

In Fig. 1 we show the fuzzy solution (dashed bars) and its $\alpha = 0.5$-cut (thick bars) at different times.

Example 2. Solve the FBVP:

$$\begin{cases} x'' + 16x = 47 - 8t^2 \\ x(0) = (\ \ 2,\ \ 3,\ 3.5) \\ x(2) = (0.5,\ 1,\ 1.5) \end{cases} \tag{18}$$

Solution:

Associated crisp non-homogeneous problem

$$\begin{cases} x'' + 16x = 47 - 8t^2 \\ x(0) = 3 \\ x(2) = 1 \end{cases}$$

has the solution $x_{cr}(t) = 3 - 0.5t^2$ (thick line in Fig. 2).

To find the uncertain part of the fuzzy solution, $\widetilde{x}_{un}(t)$, we solve fuzzy homogeneous problem

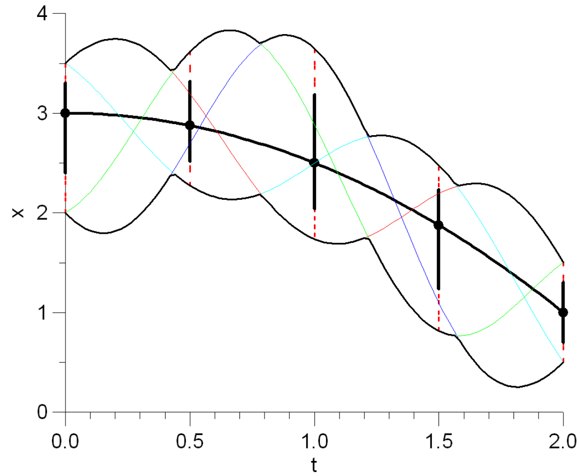

Fig. 2. The fuzzy solution and its $\alpha = 0.6$-cut for Example 2. Dashed and thick bars represent the values of the fuzzy solution and its $\alpha = 0.6$-cut at different times, respectively.

$$\begin{cases} x'' + 16x = 0 \\ x(0) = (\ -1,\ 0,\ 0.5) \\ x(2) = (-0.5,\ 0,\ 0.5) \end{cases}$$

$x_1(t) = \cos 4t$ and $x_2(t) = \sin 4t$ are linear independent solutions for the differential equation. Then
$\mathbf{s}(t) = [\cos 4t \quad \sin 4t]$, $M = \begin{bmatrix} 1 & 0 \\ \cos 8 & \sin 8 \end{bmatrix}$ and
$\mathbf{w} = \mathbf{s}(t)\ M^{-1} = \frac{1}{\sin 8}[\sin(8-4t) \quad \sin 4t]$.

Using the formula (13) we obtain the solution of homogeneous problem and adding the crisp solution we get the solution of the given FBVP (18):

$$\widetilde{x}(t) = 3 - 0.5t^2 + \tfrac{1}{\sin 8}(\sin(8-4t)\ (-1,\ 0,\ 0.5)+ \\ \sin 4t\ (-0.5,\ 0,\ 0.5)) \tag{19}$$

Fuzzy solution generates a band in tx-plane (Fig. 2). Unlike Example 1, the functions $w_1(t)$ and $w_2(t)$ takes both positive and negative values in the interval $0 < t < T$. Because of that, in generation of upper and lower borders of the band, $\overline{a}$ and $\underline{a}$, $\underline{b}$ and $\overline{b}$ take charge in alternately.

IV. Conclusion

In this paper we have investigated the fuzzy boundary value problem as a set of crisp problems. We have proposed a solution method based on the properties of linear transformations. For clarity we have explained the proposed method for second order linear differential equation. We have shown that the fuzzy solution by our method coincides with the solution by extension principle. We are planning to make a comparative analysis between the proposed method and the method including generalized Hukuhara derivative in future.

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