A Model connecting Quantum, Diffusion, Soliton, and Periodic Localized States under Brownian motion

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Abstract

We propose new equations of motion under the theory of the Brownian motion to connect the states of quantum, diffusion, soliton, and periodic localization. The new equations are nothing but the classical equations of motion with two additional terms and the one of them can be regarded as the the quantum potential. By choosing a parameter space, various important states are obtained. Further, the equations contain other interesting phenomena such as general dynamics of diffusion process, collapse of the soliton, the nonlinear extension of the Schrdinger equation, and the dynamics of phase transition.

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1 Introduction

The description of microscopic world governed by quantum mechanics is quite different from the one of microscopic world with classical mechanics [1, 2]. Whereas, it is an important issue to develop a framework that includes both of classical and quantum physics for the purpose to study mixed dynamics [4, 3, 5]. Formally, one needs to handle large number of quantum particles to organize macroscopic world. The method treating quantum many-body system is accurate but time-consuming to search a numerical solution unless one can find a good approximation for individual circumstances. On the other hand, extension of classical equations of motion by adding quantum potential which yields the Schrödinger equation is much simpler to involve classical-quantum dynamics [6, 7, 8]. One natural motivation of the quantum potential approach is to explain macroscopic quantum phenomena [9, 10]. For the macroscopic level, classical equations of motion may be applicable even for quantum phenomena. In this paper, we propose more general equations under the theory of the Brownian motion as the origin of quantum potential to connect some famous equations of classical and quantum physics. As a result of this connection, quantum, diffusion, soliton, and periodic localized states can be derived by solving the general equations of motion.

Basic concept is to generalize the idea of the Nelson's stochastic equations [11]. These are based on the Langevin equations and the Fokker-Planck equations to describe non-differentiable process. They can lead the Schrödinger equation by assuming some conditions such as the universal relation and the specific form of the mean acceleration. We generalize the Langevin equations so that both of the Nelson equations and the Schrödinger equation are extended. Including the nonlinear Schrödinger equation (NLS), we provide several new equations that contain interesting solutions.

The extension of the Nelson equations was already attempted in the literature. In the paper [12], Davidson generalized the equations to remedy the universal relation: original relation is $\hbar = 2mD$ and modified one is $\hbar = \sqrt{1-2s}mD$, where m is a particle mass, D is the diffusion coefficient, and s is a free parameter. This modification not only reconstruct a hypothesis of the relation but also it is possible to make physical continuation to the diffusion equation by assuming $1 - 2s < 0$. Moreover, as is known that the limit $\hbar \to 0$ recovers classical physics, choices of the parameter as $s = 1/2 - \hbar^2/2m^2D^2$, $s = 1/2$, and $s > 1/2$ represent quantum, classical, and diffusion equations, respectively. Meanwhile, there is another extension of the Nelson equations with a different motivation [13]. We found a way to put them together in a framework of generalized equations of motion.

The purpose of our generalization is to seek a proof of quantum potential in a similar way of the work [14]. With the modified Nelson equations, new static solutions are easily obtained. For instance, the soliton solution in our model is one of a simple prediction because it can be distinguished from other soliton solutions (see e.g. [15]) by comparing the density function. Besides, we also predict new dynamics such as a generalized diffusion process by inventing new diffusion equations. In several points, the existence of quantum potential in the origin of the Brownian motion can be examined.

This paper is organized as follows. In section 2, we present generalized equations of motion with the generalization of Nelson's equations including a brief review. In section 3, the Schrödinger equation is derived from the equations of motion with the extension to a

Hilbert space. Section 4 deals with the diffusion equation by choosing a parameter space of the equations proposed in section 2. Section 5 leads the soliton equation to exhibit new type of soliton solution. In section 6, we obtain the periodic localized solution that exists in other parameter space of generalized equations of motion. Section 7 is devoted to the summary.

2 Modified Nelson equations

The Langevin equation can describe the Brownian motion with a random movement. Normally, this equation computes a motion of differentiable process. Suppose $\boldsymbol{x}(t)$ is a stochastic process, mean forward derivative is given by $\mathcal{D}_+\boldsymbol{x}(t) = \lim_{\Delta t \to +0}\langle(\boldsymbol{x}(t+\Delta t) - \boldsymbol{x}(t))/\Delta t\rangle$ and mean backward derivative is $\mathcal{D}_-\boldsymbol{x}(t) = \lim_{\Delta t \to +0}\langle(\boldsymbol{x}(t) - \boldsymbol{x}(t-\Delta t))/\Delta t\rangle$, where $\langle\rangle$ represents mean value. For differentiable process, mean derivatives are equal $\mathcal{D}_+\boldsymbol{x}(t) = \mathcal{D}_-\boldsymbol{x}(t)$. However, there are many important processes which are not differentiable. To extend the equation to apply non-differentiable processes, independent equation is set for mean forward and backward velocities individually. Basic procedure is formalized by Nelson in [11].

Let us consider kinematics of Markoff process. The position $\boldsymbol{x}(t)$ of the Brownian particle satisfies

$$d_+\boldsymbol{x} = \boldsymbol{b}_+(t, \boldsymbol{x}(t))dt + d\boldsymbol{w}, \quad d_-\boldsymbol{x} = \boldsymbol{b}_-(t, \boldsymbol{x}(t))dt + d\boldsymbol{w}, \tag{1}$$

where $\boldsymbol{b}_+$ ($\boldsymbol{b}_-$) is the mean forward (backward) velocity and $\boldsymbol{w}$ describes the Wiener process which is Gaussian with mean zero. For the Brownian motion, it is usually set $\langle dw_i dw_j\rangle = 2D\delta_{ij}dt$. The Langevin equations provide the space-time evolution of $\boldsymbol{b}_\pm$. Here we assume modified differential equations:

$$\begin{aligned} d_+\boldsymbol{b}_- &= s_1(d_+ - d_-)\boldsymbol{b}_- - s_2 d_+ d_+\boldsymbol{b}_- - \beta\boldsymbol{v}dt + \boldsymbol{K}(\boldsymbol{x}(t))dt + d\boldsymbol{B}(t), \\ d_-\boldsymbol{b}_+ &= -s_1(d_+ - d_-)\boldsymbol{b}_+ + s_2 d_- d_-\boldsymbol{b}_+ + \beta\boldsymbol{v}dt + \boldsymbol{K}(\boldsymbol{x}(t))dt + d\boldsymbol{B}(t), \end{aligned} \tag{2}$$

where $m\beta\boldsymbol{v}$ is the force of friction, $m\boldsymbol{K}$ is a body force ($m\boldsymbol{K} = -\nabla V$), $\boldsymbol{B}$ is a Wiener process representing the residual random impacts, and $d\boldsymbol{B}$ is Gaussian with mean zero. With free parameters s_1 and s_2, the Nelson equations are extended.

Suppose $\boldsymbol{b}_\pm(t, \boldsymbol{x})$ can be expanded in Taylor series, we get

$$\begin{aligned} d_+\boldsymbol{b}_- &= dt\partial_t\boldsymbol{b}_- + ((\boldsymbol{b}_+dt + d\boldsymbol{w})\cdot\nabla)\boldsymbol{b}_- + \frac{1}{2}(d\boldsymbol{w})_i(d\boldsymbol{w})_j\partial_i\partial_j\boldsymbol{b}_-, \\ d_-\boldsymbol{b}_+ &= dt\partial_t\boldsymbol{b}_+ + ((\boldsymbol{b}_-dt + d\boldsymbol{w})\cdot\nabla)\boldsymbol{b}_+ - \frac{1}{2}(d\boldsymbol{w})_i(d\boldsymbol{w})_j\partial_i\partial_j\boldsymbol{b}_+, \\ d_+d_+\boldsymbol{b}_- &= (d\boldsymbol{w})_i(d\boldsymbol{w})_j\partial_i\partial_j\boldsymbol{b}_-, \quad d_-d_-\boldsymbol{b}_+ = (d\boldsymbol{w})_i(d\boldsymbol{w})_j\partial_i\partial_j\boldsymbol{b}_+, \end{aligned} \tag{3}$$

where order $(dt)^2$ is neglected. Taking the average of Eqs. (2), our generalized Langevin equations read

$$\begin{aligned} (\partial_t + \boldsymbol{b}_+\cdot\nabla + D\nabla^2 - s_1(\boldsymbol{b}_+ - \boldsymbol{b}_-)\cdot\nabla - 2D(s_1 - s_2)\nabla^2)\boldsymbol{b}_-dt &= -\beta\boldsymbol{v}dt + \boldsymbol{K}(\boldsymbol{x}(t))dt, \\ (\partial_t + \boldsymbol{b}_-\cdot\nabla - D\nabla^2 + s_1(\boldsymbol{b}_+ - \boldsymbol{b}_-)\cdot\nabla + 2D(s_1 - s_2)\nabla^2)\boldsymbol{b}_+dt &= \beta\boldsymbol{v}dt + \boldsymbol{K}(\boldsymbol{x}(t))dt. \end{aligned} \tag{4}$$

Dividing by dt and summing the two equations, we obtain

$$\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - D\nabla^2\boldsymbol{u} + 2s_1\boldsymbol{u}\cdot\nabla\boldsymbol{u} + 2D(s_1 - s_2)\nabla^2\boldsymbol{u} = \boldsymbol{K}, \tag{5}$$

where $\boldsymbol{v} = (\boldsymbol{b}_+ + \boldsymbol{b}_-)/2$ is the current velocity and $\boldsymbol{u} = (\boldsymbol{b}_+ - \boldsymbol{b}_-)/2$ is the osmotic velocity. Further, we assume the velocities satisfy the Fokker-Planck equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \quad \boldsymbol{u} - D\nabla \ln \rho = 0. \tag{6}$$

Now, we get equations that describe time dependence of $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} &= -D\nabla^2 \boldsymbol{v} - \nabla(\boldsymbol{v} \cdot \boldsymbol{u}), \\ \frac{\partial \boldsymbol{v}}{\partial t} &= -\frac{1}{2}\nabla \boldsymbol{v}^2 + \frac{1-2s_1}{2}\nabla \boldsymbol{u}^2 + (1 - 2s_1 + 2s_2)D\nabla^2 \boldsymbol{u} - \frac{1}{m}\nabla V, \end{aligned} \tag{7}$$

here and hereafter, we assume the current velocity has no rotation. For a case $s_1 = s_2 = 0$, they coincide with the Nelson's equations, for $s_2 = 0$, they become the Davidson's equations [12], and for $s_1 = s_2 = 1/2$, they become the equations we proposed in previous paper [13]. The second equation is derived by the sum of two equations (4), then there is another independent equation. The subtraction of Eqs. (4) yields

$$2(s_1 - 1)(\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + D(2s_1 - 2s_2 - 1)\nabla^2 \boldsymbol{v} = \beta \boldsymbol{v}. \tag{8}$$

This equation will determine the magnitude of friction coefficient $m\beta$. When $\beta = 0$ and $\boldsymbol{v}$ is constant, the equation becomes trivial. In general, right-hand side (RHS) may not be proportional to the current velocity, then, we shall need an extension $\beta v_i \to \beta_{ij} v_j$ by modifying Eqs. (2).

3 Quantum process

The original concept of Nelson's idea is to lead the Schrödinger equation from Eqs. (7). For convenience, we first consider the Lagrangian and the Hamiltonian from the equations of motion. According to [16], they are given by

$$\begin{aligned} L &= \frac{1}{2}m\boldsymbol{v}^2 + \frac{mD^2}{2}(1-2s_1)(\nabla \ln \rho)^2 + mD^2(1 - 2s_1 + 2s_2)\nabla^2 \ln \rho - V, \\ H &= \frac{1}{2}m\boldsymbol{v}^2 - \frac{mD^2}{2}(1-2s_1)(\nabla \ln \rho)^2 - mD^2(1 - 2s_1 + 2s_2)\nabla^2 \ln \rho + V. \end{aligned} \tag{9}$$

When the Hamiltonian is constant written by E, the current velocity is zero, $s_2 = 0$, and $f(\boldsymbol{x}) = \sqrt{\rho(\boldsymbol{x})}$, we get the equation

$$E = -2(1-2s_1)mD^2 \frac{\nabla^2 f}{f} + V. \tag{10}$$

It looks like the Schrödinger equation if $2(1-2s_1)mD^2$ is constant and corresponds to $\hbar^2/2m$. If we use ρ instead of f, the first term of RHS corresponds to the quantum potential. Since ρ is real, the function f is also real. It approaches more to the Schrödinger equation if we consider the time dependence of ρ and $\boldsymbol{v}$. Assuming $\boldsymbol{v}$ is gradient of some function, given by $D'\nabla w$ with constant D', and $s_2 = 0$, Eqs. (7) can be rewritten as

$$\begin{aligned} D\frac{\partial(\ln\rho)}{\partial t} &= -DD'\nabla^2 w - DD'\nabla w\cdot\nabla\ln\rho, \\ D'\frac{\partial w}{\partial t} &= -\frac{1}{2}\boldsymbol{v}^2 + \frac{1-2s_1}{2}\boldsymbol{u}^2 + (1-2s_1)D^2\nabla^2(\ln\rho) - \frac{1}{m}V. \end{aligned} \tag{11}$$

Giving a new function $f = e^{\frac{1}{2}\ln\rho + iaw}$, we have

$$\nabla^2 f = (\frac{1}{4}\nabla\ln\rho\cdot\nabla\ln\rho - a^2\nabla w\cdot\nabla w + \frac{1}{2}\nabla^2\ln\rho)f + ia(\nabla\ln\rho\cdot\nabla w + \nabla^2 w)f \tag{12}$$

If we take $a = D'/2D\sqrt{1-2s_1}$, it becomes

$$\nabla^2 f = \frac{1}{2(1-2s_1)D^2}(D'\frac{\partial w}{\partial t} + \frac{V}{m})f - \frac{ia}{D'}\frac{\partial\ln\rho}{\partial t}f. \tag{13}$$

Then, this can be

$$2i\sqrt{1-2s_1}mD\frac{\partial f}{\partial t} = -2(1-2s_1)mD^2\nabla^2 f + Vf, \tag{14}$$

Let $D = \hbar/2\sqrt{1-2s_1}m$ or $s_1 = 1/2 - \hbar^2/8m^2D^2$, it is the same to the Schrödinger equation

$$i\hbar\frac{\partial f}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 f + Vf. \tag{15}$$

Assuming $f(t,\boldsymbol{x}) = f(\boldsymbol{x})e^{-iEt/\hbar}$ (i.e. $aw(t,\boldsymbol{x}) = aw(\boldsymbol{x}) - Et/\hbar$) for the case of constant energy, it is consistent with Eq. (10). A difference from quantum mechanics is that the function f is scalar while the wave function is vector in a Hilbert space. The extension to a Hilbert space is necessary to assign quantum charges and consider multi-particle system. This extension is not difficult if we employ infinite number of probability densities and velocities. Taking stochastic process $\boldsymbol{x}^{(k)}$ for each component of a Hilbert space and giving mean velocities for each process $\boldsymbol{b}_+^{(k)}$ and $\boldsymbol{b}_-^{(k)}$, with k runs from 1 to infinity. For all the velocities, we apply the modified Langevin equations and the Fokker-Planck equations, i.e. Eqs. (14) and Eqs. (6). Then we obtain the Schrödinger like equation for each k to the function $f^{(k)} = e^{\frac{1}{2}\ln\rho^{(k)} + iaw^{(k)}}$. As for the potential $V^{(k)}$, it is not necessary to take independent function in each component. That is, they can be functions of other components $V^{(k)} = V^{(k)}(f^{(1)}, f^{(2)}, \cdots)$. When the potential can be written in linear combinations, we can take $V^{(k)}f_k \to \sum V_{k\ell}f_\ell$. More generally, the potential will be given by operator system with the function of f_k through other components so that we may write $V^{(k)}f_k \to \hat{V}f_k$. Rewriting $f_k = \psi$, the equations turn into

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + \hat{V}\psi. \tag{16}$$

Now we can deal with any potential and any state of quantum mechanics as long as non-relativistic approximation is valid. Compared to the second quantization formalism, it is less convenient to calculate cross section, decay rate, branching ratio, and so on, due to the difficulty of formulating relativistic Brownian motion [17]. However, when one considers a relationship between quantum and classical mechanics, above quantization scheme will be useful. As it is easy to contain classical equations, we can predict intermediate states between classical and quantum solutions to check the method.

Note, as we focus on low energy physics, the system with a single component function is sufficient for the description of dynamics. For the moment, we do not mention about a Hilbert space. In addition, instead of dealing with many particle system with the wave function, we can employ macroscopic physics by reinterpreting ρ as the number density. In that case, Langevin equations should be modified but the same equations can be used with some simplification [18].

4 Diffusion process

It can be inferred from Eq. (14) that the flip of the sign of $1 - 2s_1$ reproduces the diffusion equation. Only, it makes a wrong sign compared to the actual equation. To produce proper one, we set $f' = e^{\frac{1}{2}\ln\rho - a'w}$ and $a' = D'/2D\sqrt{2s_1 - 1}$ with assuming $2s_1 - 1 > 0$, then

$$2\sqrt{2s_1 - 1}mD\frac{\partial f'}{\partial t} = 2(2s_1 - 1)mD^2\nabla^2 f' - 2mD^2 s_2(\nabla^2 \ln\rho)f' + Vf', \tag{17}$$

When $s_2 = 0$ and $V = 0$, it becomes

$$\frac{\partial f'}{\partial t} = \sqrt{2s_1 - 1}D\nabla^2 f'. \tag{18}$$

Thus f' obeys the diffusion equation with the diffusion coefficient $D_0 = \sqrt{2s_1 - 1}D$. This function is not directly related to the probability density so that we need an additional equation. To compute the dynamics, we consider the function $f = e^{\frac{1}{2}\ln\rho + a'w}$ which satisfies

$$\frac{\partial f}{\partial t} = -\sqrt{2s_1 - 1}D\nabla^2 f. \tag{19}$$

Giving initial conditions of f and f', the probability density $\rho = ff'$ can be computed. With a famous solution in two dimensions, i.e. $f = f_0 e^{r^2/4D_0(t+t_0)}/4\pi D_0(t + t_0)$ and $f' = f'_0 e^{-r^2/4D_0(t+t'_0)}/4\pi D_0(t + t'_0)$, the probability density becomes

$$\rho = \frac{t_0 - t'_0}{4\pi D_0(t + t_0)(t + t'_0)} e^{-\frac{r^2(t_0 - t'_0)}{4D_0(t+t_0)(t+t'_0)}}. \tag{20}$$

In this way, time evolution differs from the diffusion equation in general. For a special case, when f is constant, the time-dependence becomes the same. The equations with the system of f and f' describe general dynamics of the diffusion equation.

For the case $f =$ const. or $D'w = -\sqrt{2s_1 - 1}D\ln\rho$, the probability density obeys the diffusion equation. This case means $\boldsymbol{v}$ is proportional to $\boldsymbol{u}$, actually $\boldsymbol{v} = -\sqrt{2s_1 - 1}\boldsymbol{u}$. Suppose a solution $\rho = e^{-r^2/4D_0 t}/4\pi D_0 t$, we can calculate the friction from Eq. (8). Since $\boldsymbol{u} = -\boldsymbol{x}/2\sqrt{2s_1 - 1}t$, Eq. (8) reduces to $\beta = -(s_1 - 1)/\sqrt{2s_1 - 1}t$. This is one prediction for a particle obeying the diffusion equation in our system. It can be rewritten by observable parameters as

$$\beta = -2D_0 W(-\pi r^2 \rho) \frac{\boldsymbol{u}^2 - \boldsymbol{v}^2}{r^2 |\boldsymbol{u}||\boldsymbol{v}|}, \tag{21}$$

where $W(x)$ is the Lambert W-function. For large t or small ρ, it can be approximated by $\beta \approx 2\pi D_0 \rho(\boldsymbol{u}^2 - \boldsymbol{v}^2)/|\boldsymbol{u}||\boldsymbol{v}|$. The sign of the friction force depends on magnitudes of two velocities; it is positive when $|\boldsymbol{u}| > |\boldsymbol{v}|$ and negative when $|\boldsymbol{u}| < |\boldsymbol{v}|$. For general s_2, the modified Nelson equations do not reduce to single equation in the above scheme. It becomes much harder to solve the equations.

5 Soliton

We have analyzed $s_1 > 1/2$ and $s_1 < 1/2$, and this section deals with $s_1 = 1/2$. The case $s_1 = 1/2$ and $s_2 = 0$ illustrate just classical physics. This is natural because this limit corresponds to $\hbar \to 0$. Under the words of stochastic process, these assumptions lead $\frac{1}{2}(d_+ + d_-)\boldsymbol{b}_\pm = \pm\beta\boldsymbol{v} + \boldsymbol{K} + d\boldsymbol{B}$. Then we get $(d_+ + d_-)\boldsymbol{v} = \boldsymbol{K} + d\boldsymbol{B}$ and $\frac{1}{2}(d_+ + d_-)\boldsymbol{u} = \beta\boldsymbol{v}$. Since the average of $(d_+ + d_-)/dt$ corresponds to the Euler-Lagrange derivative, ordinary equations of motion realize. Precisely, they can be

$$(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{K}, \quad (\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla)\boldsymbol{u} = \beta\boldsymbol{v}. \tag{22}$$

The first equation can lead Newton's equation of motion or it is related to the Navier-Stokes equations (to derive them we need to treat fluid density properly, see [16, 19]). The second equation is not trivial so it is a constrained system. Using the Fokker-Planck equations, it can be $(\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + \beta\boldsymbol{v} = 0$.

For the Hamiltonian with zero current-velocity, the condition $s_1 = 1/2$ makes

$$H = -2s_2 m D^2 \nabla^2 \ln \rho + V. \tag{23}$$

When the total energy and the potential are constant with written by $E - V = \Delta E$, it can be

$$\Delta E = -2s_2 m D^2 \nabla^2 \ln \rho. \tag{24}$$

In this section, we assume $s_2 > 0$ and $\Delta E > 0$. For one spatial dimension, the probability density becomes Gaussian. For two dimensions with circular symmetry, the solution is $\rho = (c_1 + c_2 r)e^{-\Delta E r^2/4s_2 m D^2}$, where c_1 and c_2 are constant. For three dimensions with spherical symmetry, it is solved as

$$\rho = c_1 e^{-\frac{\Delta E r^2}{12 s_2 m D^2} - c_2/r}. \tag{25}$$

When $c_2 > 0$, it can be normalized by

$$c_1^{-1} = \frac{1}{4}\sqrt{\pi} c_2^3 G_{0,3}^{3,0}\left(\frac{c_2^2 \Delta E}{48 s_2 m D^2}\middle| -\frac{3}{2}, -1, 0\right), \tag{26}$$

where G is the Meijer G-function. When the current velocity is non-zero constant, we have

$$E - \frac{1}{2}m\boldsymbol{v}^2 - V = -\frac{2s_2 mD^2}{4}\nabla^2 \ln\rho, \quad \frac{\partial \ln\rho}{\partial t} = -\boldsymbol{v}\cdot\nabla\ln\rho. \tag{27}$$

Writing $E - \frac{1}{2}m\boldsymbol{v}^2 - V = \Delta E$, the solution can be written in the same way with $\rho(t, \boldsymbol{x}+\boldsymbol{v}t)$:

$$\rho = c_1 e^{-\frac{\Delta E(\boldsymbol{x}+\boldsymbol{v}t)^2}{12s_2 mD^2} - c_2/\sqrt{(\boldsymbol{x}+\boldsymbol{v}t)^2}}. \tag{28}$$

This solution contains both of Gaussian soliton and bubble soliton. When $c_2 = 0$ it is a Gaussian and otherwise it can be seen as a bubble. The size of a bubble is $r_0 = (6s_2c_2mD^2/\Delta E)^{1/3}$ and the thickness of a film mainly depends on $\Delta r = \sqrt{12s_2mD^2/\Delta E}$. The soliton will be stable as it is a general solution in that environment. By changing s_1 and s_2, the soliton will collapse in several ways.

More generally, when $\boldsymbol{v} = D'\nabla w$, $f = e^{\frac{1}{2}\ln\rho + ibw}$, and $b = D'/\sqrt{8s_2D^2}$, we have

$$\nabla^2 f = (\frac{1}{4}(\nabla\ln f^* f)^2 + \frac{D'}{4s_2D^2}\frac{\partial w}{\partial t} + \frac{V}{4s_2mD^2} - \frac{ib}{D'}\frac{\partial\ln\rho}{\partial t})f, \tag{29}$$

where $*$ denotes complex conjugate. Then we get

$$i\sqrt{8s_2}mD\frac{\partial f}{\partial t} = -4s_2mD^2\nabla^2 f + (s_2mD^2(\nabla\ln f^* f)^2 + V)f. \tag{30}$$

Thus the equations reduce to one differential equation for complex function f as in the same way of the Schrödinger equation. This equation will generalize the above soliton solution.

6 Periodic localization

The Schrödinger equation appears by taking $s_2 = 0$ and $\rho = f^2$ in the Hamiltonian system. However, even when $s_2 \neq 0$, we can get the same equation. By reparametrizing the function $\rho = f^{2(1-2s_1+2s_2)/(1-2s_1)}$, the constant Hamiltonian without current velocity becomes

$$E = -\frac{2(1-2s_1+2s_2)^2 mD^2}{1-2s_1}\frac{\nabla^2 f}{f} + V. \tag{31}$$

If $1 - 2s_1 > 0$ and $E - V > 0$, a typical solution is

$$f = c_0\left|\cos[\frac{x_1}{\Delta x_1} + c_1]\right|\left|\cos[\frac{x_2}{\Delta x_2} + c_2]\right|\left|\cos[\frac{x_3}{\Delta x_3} + c_3]\right|. \tag{32}$$

For simplicity, we choose $c_1 = c_2 = c_3 = 0$ and $\Delta x_i = \Delta x$, then

$$\Delta x = \sqrt{\frac{6(1-2s_1+2s_2)^2 mD^2}{(1-2s_1)(E-V)}}. \tag{33}$$

In the view of probability density, it becomes

$$\rho = \rho_0\left|\cos[\frac{x_1}{\Delta x}]\right|^{\frac{1-2s_1}{2(1-2s_1+2s_2)}}\left|\cos[\frac{x_2}{\Delta x}]\right|^{\frac{1-2s_1}{2(1-2s_1+2s_2)}}\left|\cos[\frac{x_3}{\Delta x}]\right|^{\frac{1-2s_1}{2(1-2s_1+2s_2)}}. \tag{34}$$

equation (solution)	s_1	s_2	f
quantum	$s_1 = 1/2 - \hbar^2/8m^2D^2$	$s_2 = 0$	complex
diffusion	$s_1 > 1/2$	$s_2 = 0$	real
classical	$s_1 = 1/2$	$s_2 = 0$	real
soliton	$s_1 = 1/2$	$s_2 > 0$	real
periodic localization	$s_1 < 1/2$	$s_2 \neq 0$	real
NLS	$s_1 = 1/2 - \hbar^2/8m^2D^2$	$s_2 \neq 0$	complex

Table 1: Appearance of interesting equations and solutions are shown, depending on s_1, s_2 and f. If one believes that Planck constant is constant rigorously in quantum phenomena, s_1 should retain $1/2 - \hbar^2/8m^2D^2$. However, if this is not true at some macroscopic scale, it can be any value within $s_1 > 1/2$.

If we take ρ as the number density, $(1-2s_1)/2(1-2s_1+2s_2) \gg 1$, and the Born-Oppenheimer approximation, it can roughly describe nuclei of solid state. Suppose the diameter of a nucleus is 10fm and the size of atom is 100pm, a realistic model can be obtained in the vicinity of $(1-2s_1)/2(1-2s_1+2s_2) \sim 10^8$. If s_1 is exactly equal to s_2, strong localization is occurred by making the absolute values of s_1 and s_2 very large. In other cases, it needs fine-tuning to keep $1-2s_1+2s_2$ very small for the realization of such localization.

Naively thinking, if it is regarded as a solid, phase transition can be performed in the framework of equations of motion. Varying the parameter s_1 to $s_1 > 1/2$, it satisfies the diffusion equation so that nuclei in the solid will diffuse. Considering the diffusion process describe liquid state, this procedure can be interpreted as solid to liquid phase transition. Depending on how s_1 changes, whole dynamics of this phase transition can be predicted.

Next, let us consider generalization of above solution with non-zero current velocity. If we take $s_1 = 1/2 - \hbar^2/8m^2D^2$, $s_2 \neq 0$, and $f = e^{\frac{1}{2}\ln\rho + iaw}$, a nonlinear term appears in the Schrödinger equation:

$$i\hbar\frac{\partial f}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 f - \frac{s_2\hbar^2}{2(1-2s_1)m}(\nabla^2 \ln f^* f)f + Vf. \tag{35}$$

When the current velocity is zero and $\hbar$ is free, above periodic localized solution is appeared. The nonlinear term vanishes when the function is the plane wave, otherwise it affects to the energy at the same order of the kinetic term. By extending to a Hilbert space, the nonlinear term exists in each component, i.e.

$$i\hbar\frac{\partial f_k}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 f_k - \frac{s_2\hbar^2}{2(1-2s_1)m}(\nabla^2 \ln f_k^* f_k)f_k + \hat{V} f_k, \tag{36}$$

where k is the index of the Hilbert space. Therefore, our nonlinear extension is quite different from ordinary interactions of quantum mechanics.

Table 1 summarizes the equations and solutions that we have considered. In general, we need to solve nonlinear differential equations (7), but when $1-2s_1 \geq 0$ and $\boldsymbol{v} = D'\nabla w$, they reduce to one complex equation. For the case $1-2s_1 < 0$ and $s_2 = 0$, the equations become two independent diffusion equations about f and f' to give the density $\rho = ff'$. A remaining region ($1-2s_1<0$ and $s_2\neq 0$) is much more difficult to be solved since the equations cannot reduce to single equation.

7 Conclusion

We have proposed a general framework that includes the Schrödinger equation, the diffusion equation, the soliton equation, and the Newton's equations of motion. Based on the modified Langevin equations for non-differentiable processes, the system can describe various motions depending on two parameters s_1 and s_2. Keeping $s_2=0$, the parameter s_1 can connect quantum, classical, and diffusion equations. On the other hand, taking non-zero value for s_2, there exist some interesting solutions like soliton and periodic localization.

Since the framework can provide a lot of predictions, the confirmation of the theory seems possible. For instance, it includes generalized diffusion equations with the system of $\rho = ff'$ and the new type of soliton equation with the potential $\nabla^2 \ln\rho$. Concretely, the potential terms $\boldsymbol{u}^2$ and $\nabla\cdot\boldsymbol{u}$ with arbitrary coefficients are the prediction of the theory. If one finds such potential in soliton, diffusion, and quantum states, these states can be related in our simple model.

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References

[1] G. A. Luna-Acosta, J. A. Méndez-Bermúdez, P. Šeba, K. N. Pichugin, *Phys. Rev. E* **65**, 046605 (2002).

[2] A. A. Neufelda, D. Schwarzer, J. Schroeder, *J. Troe, J. Chem. Phys.* **119**, 2502 (2003).

[3] J. Berges, T. Gasenzer, *Phys. Rev. A* **76**, 033604 (2007).

[4] D. Cohen, *Phys. Rev. Lett.* **78**, 2878 (1997).

[5] S. Nielsen, R. Kapral, G. Ciccotti, *J. Chem. Phys.* **115**, 5805 (2001).

[6] L. D. Site, *Physica B* **349**, 218 (2004).

[7] P. Markowich, T. Paul, C. Sparber, *J. Funct. Anal.* **259**, 1542 (2010).

[8] C. H. Chou, B. L. Hub, T. Yuc, *Physica A* **387**, 432 (2008).

[9] D. M. Appleby, *Found. Phys.* **29**, 1885 (1999).

[10] A. Matzkin, V. Nurock, *Studies in History and Philosophy of Modern Physics* **39**, 17 (2008).

[11] E. Nelson, *Phys. Rev.* **150**, 1079 (1966).

[12] M. Davidson, *Lett. Math. Phys.* **3**, 271 (1979).

[13] H. Isimori, *Int. J. Theor. Phys.* **49**, 1029 (2010).

[14] L. Nottale, *J. Phys. A: Math. Theor.* **42**, 275306 (2009).

[15] V. S. Gerdjikov, arXiv:nlin/0604004.

[16] J. Cresson, S. Darses, *J. Math. Phys.* **48**, 072703 (2007).

[17] J. Dunkel, P. Hänggi, *Phys. Rep.* **471**, 1 (2009).

[18] T. Nakamura, A. Yoshimori, *J. Phys. A, Math. Theor.* **42**, 065001 (2009).

[19] R. J. Harvey, *Phys. Rev.* **152**, 1115 (1966).