

RANDOM KÄHLER METRICS

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ABSTRACT. The purpose of this article is to propose a new method to define and calculate path integrals over metrics on a Kähler manifold. The main idea is to use finite dimensional spaces of Bergman metrics, as an approximation to the full space of Kähler metrics. We use the theory of large deviations to decide when a sequence of probability measures on the spaces of Bergman metrics tends to a limit measure on the space of all Kähler metrics. Several examples are considered.

1. INTRODUCTION

1.1. **The aim of the paper and the basic setup.** This article is part of a series (cf. [22, 23]) devoted to a new approach to random metrics on a manifold M . It is assumed that M is a Kähler manifold (M, J, ω_0) , of any complex dimension n , with an integral Kähler form $\omega_0 \in H^2(M, \mathbb{Z})$ and a compatible complex structure J . Motivation to study random Kähler metrics arises in large part from the theory of two dimensional quantum gravity [37], because in this case any riemannian metric is Kähler for some complex structure. More generally, it is also of interest in both physics and mathematics as a theory of random surfaces and stochastic geometry in any complex dimension. In this article we introduce the general probabilistic and geometric foundations. Applications relevant to two dimensional quantum gravity are discussed in details in the companion articles [22, 23].

The novel approach to defining and studying random metrics is based on some relatively new ideas in Kähler geometry involved in the Yau-Tian-Donaldson program of relating existence of constant scalar curvature Kähler metrics in the class of ω_0 to stability in the sense of geometric invariant theory. The main idea is to approximate the infinite dimensional space $\mathcal{K}_{[\omega_0]}$ of Kähler metrics in the cohomology class $[\omega_0]$ of ω_0 by finite dimensional spaces \mathcal{B}_k of “Bergman metrics,” which are pullbacks of the Fubini-Study metric on complex projective space by Kodaira type holomorphic embeddings defined by using holomorphic sections of powers L^k of the line bundle L with Chern class $c_1(L) = [\omega_0]$. Much current work in Kähler geometry involves the approximation of the infinite dimensional geometry of $\mathcal{K}_{[\omega_0]}$ by the finite dimensional geometry of \mathcal{B}_k . The idea of this article is to use this approximation to regularize path integrals over $\mathcal{K}_{[\omega_0]}$ by well-defined integrals over \mathcal{B}_k .

The space $\mathcal{K}_{[\omega_0]}$ can be parametrized by the Kähler potentials ϕ of Kähler metrics ω_ϕ in the cohomology class of ω_0 ,

$$\mathcal{K}_{[\omega_0]} = \{\phi \in C^\infty(M)/\mathbb{R} : \omega_\phi := \omega_0 + i\partial\bar{\partial}\phi > 0\} \quad (1)$$

with the equivalence relation that $\phi \equiv \phi'$ if and only if $\phi - \phi'$ is a constant. Since the riemannian metric associated to the Kähler form is given by $ig_{a\bar{b}} = \omega_{a\bar{b}}$, we use the same term 'metric' both for g and corresponding Kähler form ω . We assume that the background Kähler form ω_0 is integral, i.e. $\omega_0 \in H^{1,1}(M, 2\pi\mathbb{Z})$. In particular, if there exists a constant scalar curvature metric in the class $[\omega_0]$, it is natural to choose ω_0 to be that metric. In the case of a Riemann surface, one usually parametrizes riemannian metrics in a conformal class $\text{Conf}(g_0)$ by a Weyl factor σ , with $g = e^{2\sigma}g_0$ for a background metric g_0 . The cohomology class of ω_0 is entirely determined by the area, and parametrizing the metrics by the Kähler potential ϕ instead of σ is a simple change of variables given by $e^{2\sigma} = 1 - \frac{1}{2}\Delta_0\phi$, where Δ_0 is the positive laplacian for the metric g_0 . We refer to [32] for a detailed comparison of the relations between $\text{Conf}(g_0)$ and $\mathcal{K}_{[\omega_0]}$.

Our aim is to define probability measures on $\mathcal{K}_{[\omega_0]}$ and to study the geometry and physics of the resulting random Kähler manifolds. The natural action functionals in this context are those arising in Kähler geometry. For example, the Aubin-Yau and Mabuchi actions, which are local in the Kähler potential ϕ , are singled out, even though in complex dimension one they are non-local in the Liouville field σ . In [22, 23] we discuss in detail an important rôle these new actions play in the problem of the coupling of non-conformal quantum field theories to gravity.

Our approach relies on some recent developments in Kähler geometry. The space $\mathcal{K}_{[\omega_0]}$ is formally an infinite dimensional symmetric space of non-positive curvature, when equipped with a riemannian metric introduced by Mabuchi [31], Semmes [39] and Donaldson [16]. In a very strong sense, $\mathcal{K}_{[\omega_0]}$ is the infinite dimensional limit of the finite dimensional symmetric spaces \mathcal{B}_k of Bergman metrics. To be precise, \mathcal{B}_k can be identified with positive-definite hermitian matrices of rank N_k and determinant one, i.e. with the coset $SL(N_k, \mathbb{C})/SU(N_k)$. Here N_k is the dimension of the space $H^0(M, L^k)$ of holomorphic sections of L^k . The approximation of the differential geometry of $\mathcal{K}_{[\omega_0]}$ by that of \mathcal{B}_k is now a fast developing theory, see e.g. [17, 18, 33, 34, 35, 36, 40, 38, 24]. We take this theory in a new direction by using integrals over \mathcal{B}_k to regularize, or rigorously define, integrals over $\mathcal{K}_{[\omega_0]}$. At the same time, the identification $\mathcal{B}_k \simeq SL(N_k, \mathbb{C})/SU(N_k)$ makes an interesting direct link between random metrics and random matrix models. Let us emphasize that these matrix models are of a novel type and are *not* directly related to the matrix models used to study two-dimensional gravity in [4, 20, 25, 1].

1.2. Defining path integrals. In physics, path integrals must be defined by suitably regularizing and renormalizing formal infinite dimensional measures $e^{-S(g)}\mathcal{D}g$. The main idea of this article is to define and compute such path integrals over metrics by finding a sequence of measures $\mu_k = e^{-S_k(g)}\mathcal{D}_k g$ on the finite dimensional space \mathcal{B}_k which tends to $e^{-S(g)}\mathcal{D}g$ in the sense that

$$\int_{\mathcal{K}_{[\omega_0]}} F(g)e^{-S(g)}\mathcal{D}g := \lim_{k \rightarrow \infty} \int_{\mathcal{B}_k} F_k(g)e^{-S_k(g)}\mathcal{D}_k g \quad (2)$$

for suitable functionals F on $\mathcal{K}_{[\omega_0]}$. The parameter k , or equivalently N_k , thus plays the rôle of the UV regulator.

Although we do not put the emphasis on applications to quantum gravity in the present work, let us mention that in this case a fundamental constraint on the measure $e^{-S(g)}\mathcal{D}g$ is background independence, i.e. independence with respect to the choice of ω_0 in our context. A remarkable feature of the regularization scheme (2) is that the spaces \mathcal{B}_k possess manifestly background independent measures $\mathcal{D}_k g$ for finite k , which converge to an infinite-dimensional background independent measure $\mathcal{D}\phi$. This is used in the companion papers [22, 23].

There are two natural ways to obtain measures on the space $\mathcal{K}_{[\omega]}$ of Kähler metrics. In the top-down approach, we start with an action functional S of the space $\mathcal{K}_{[\omega_0]}$, which is suggested by a particular problem. For example, we show in [22, 23] that the Mabuchi action enters naturally in the quantization of a non-conformal matter/gravity system. We then need to determine the approximating (regularized) actions S_k . The simplest approach is just to restrict S to the Bergman metrics $\mathcal{B}_k \subset \mathcal{K}_{[\omega_0]}$. But there may be simpler and more natural approximations. For example, in §3.3, we propose an approximation scheme based on the so-called *contraction principle* in the theory of large deviations. Let us note however that in general the finite dimensional approximations can be quite complicated and the corresponding large k limit can be hard to study.

Therefore, it is also fruitful to consider a bottom-up approach where we choose sequences of measures $\mu_k = e^{-S_k(g)}\mathcal{D}_k g$ which are simple from the point of view of our formulation in terms of Bergman metrics, and study their large k limits. It is sometimes possible to determine the asymptotics of correlation functions of these sequences, so they provide computable toy models for the general framework.

However, in general, it is very difficult to determine asymptotics of correlation functions of the μ_k . One can then focus on a simpler problem: does the sequence μ_k have an asymptotic action, or in mathematical terms, does the sequence μ_k satisfy a ‘large deviations principle’ (LDP)? The rate function of the LDP is the asymptotic action. When we construct the sequence μ_k from the bottom-up approach, we usually do not know in advance what infinite dimensional theory it tends to. When the sequence satisfies an LDP with a rate function S , we know that it tends in weak sense to a theory with action S .

The simplest sequences μ_k are those which are invariant under the unitary group $SU(N_k)$ which acts on $\mathcal{B}_k \simeq SL(N_k, \mathbb{C})/SU(N_k)$ on the left. Such measures can be reduced to standard integrals over eigenvalues. As explained in §4.1, the $U(N)$ -invariance of such measures and the reduction to eigenvalue integrals correspond to an infinite dimensional symmetry of $\mathcal{K}_{[\omega_0]}$ and the quotient by the symmetry is the space $\mathcal{M}(\mathbb{R})$ of probability measures on \mathbb{R} . This reduction is a new observation in Kähler geometry and is related to the study of geodesics of $\mathcal{K}_{[\omega_0]}$ [38]. In §4, we discuss two sequences μ_k of this type. Note that in all cases, $SU(N_k)$ -invariance leads to background-dependent limits, typically to measures which are concentrated at or around the background metric.

1.3. A few examples of interesting questions. Once a notion of ‘random metric’ is defined, one would ideally like to sample the ensemble of random metrics and graph the typical random manifold. Some natural geometric questions are: what is the ‘shape’ of

a random manifold? How smooth is it? What is the expected volume of a given subset $U \subset M$? What is the expected distance between points or the expected diameter of the manifold?

These questions are often related to the computation of correlation functions of natural observables on \mathcal{B}_k , such as values of the Kähler potential $\phi(z)$ or its derivatives at several points (see §3). Hence we would like to find non-trivial models for which these correlation functions are tractable.

A related circle of problems surrounds scaling relations of random metrics and the KPZ formula [28], recently proved rigorously in [21], see also the treatment in [10]. In the Liouville theory the KPZ relation can be derived by looking at the scaling of the path integral with the area A of the surface [9, 15]. A similar problem in our context is to understand how the Mabuchi path integral

$$Z_k(A, \gamma) = \int_{\mathcal{B}_k(A)} e^{-\gamma^2 S_M(\omega_0, \phi)} \mathcal{D}_k \phi \quad (3)$$

behaves in the parameters k, A and γ , where A is the area of the surface and $\mathcal{B}_k(A)$ is the space of Bergman metrics of area A . Convergence of this integral is related to the “stability” of $(M, L, [\omega_0], J)$, i.e. to the growth of the Mabuchi action $S_M(\omega_0, \phi)$ along geodesics of $\mathcal{K}_{[\omega_0]}$.

We plan to study these questions with the formalism introduced in the present paper in the future.

1.4. Plan of the paper. The paper is organized as follows. In §2 we recall the geometric background. In §3, we explain our approach in detail. In subsequent sections, we consider different types of examples where the methods apply. We define probability measures on $\mathcal{K}_{[\omega_0]}$ using sequences $\{\mu_k\}$ of probability measures on \mathcal{B}_k . We construct many interesting sequences μ_k which ‘converge’ to an infinite dimensional limit in the sense that they satisfy a large deviations principle (LDP). Starting from a desired action functional S , we explain how to construct sequences μ_k satisfying an LDP with rate function S , by applying the contraction principle. In §4 we study examples of sequences μ_k arising naturally from the identification of \mathcal{B}_k as a coset space. In the final section we build finite-dimensional approximations to geometric functionals using the contraction principle and pointwise convergence formulas.

2. GEOMETRIC BACKGROUND

2.1. On the geometry of the space $\mathcal{K}_{[\omega_0]}$. There is a natural riemannian metric on the space $\mathcal{K}_{[\omega_0]}$ due to Donaldson, Semmes and Mabuchi [16, 39, 31] defined by

$$G_\phi(\delta\phi_1, \delta\phi_2) = \int_M \delta\phi_1(z) \delta\phi_2(z) \omega_\phi^n, \quad \delta\phi_1, \delta\phi_2 \in T_\phi \mathcal{H} \simeq C^\infty(M). \quad (4)$$

It is shown in [31, 39, 16] that G_ϕ is of constant sectional curvature, i.e. that $(\mathcal{K}_{[\omega_0]}, g_\phi)$ is a symmetric space. It is formally given by $\mathcal{K}_{[\omega_0]} = G_{\mathbb{C}}/G$ where G is the group of Hamiltonian symplectic diffeomorphisms of (M, ω_0) and where $G_{\mathbb{C}}$ is the complexification of G . The associated formal riemannian volume measure $\mathcal{D}\phi$ of $\mathcal{K}_{[\omega_0]}$ is thus the infinite dimensional $G_{\mathbb{C}}$ -invariant Haar measure.

The infinite dimensional geometry is quite formal and many features of finite dimensional symmetric spaces do not exist for $\mathcal{K}_{[\omega_0]}$. For instance, $G_{\mathbb{C}}$ does not exist as a group [16], although it is formally the group $\text{Diff}^{1,1}$ of diffeomorphisms f of M so that $f^*\omega_0$ is of type $(1, 1)$ with respect to the fixed complex structure J . Moreover, the exponential map of $\mathcal{K}_{[\omega_0]}$ is not well-defined [38], and in directions where it is defined it is geodesically incomplete. However, the situation is saved to a large extent by the existence of the finite dimensional approximations \mathcal{B}_k to $\mathcal{K}_{[\omega_0]}$ that we describe in details later in this section.

Let us note that in the case of surfaces, where it is also natural to use the conformal Weyl factor σ instead of ϕ to parametrize the metrics, one can consider the metric

$$G_{\sigma}(\delta\sigma_1, \delta\sigma_2) = \int_M \delta\sigma_1(z)\delta\sigma_2(z)e^{2\sigma}\omega_0. \quad (5)$$

This metric and its associated formal integration volume form $\mathcal{D}\sigma$ is induced from the well-known manifestly diffeomorphism invariant DeWitt-Ebin metric on the space of all Riemannian metrics [12, 13]. In terms of the Kähler potential, (5) becomes the Calabi metric

$$G_{\sigma}(\delta\phi_1, \delta\phi_2) = \int_M (\Delta_{\phi}\delta\phi_1(z))(\Delta_{\phi}\delta\phi_2(z))\omega_{\phi}, \quad (6)$$

where Δ_{ϕ} is the laplacian in the Kähler metric ω_{ϕ} . Since it requires a Green's function to recover $\delta\phi$ from $\Delta_{\phi}(\delta\phi)$, the two metrics (4) and (5) have quite different geometric properties. In particular, when endowed with the metric (4) the space $\mathcal{K}_{[\omega_0]}$ is negatively curved [16, 39] and when endowed with the metric (6) it is positively curved [5, 8].

2.2. Random metrics and random matrices. Crucial to our approach is the existence of finite dimensional approximations \mathcal{B}_k to $\mathcal{K}_{[\omega_0]}$. The properties of these approximations arise from geometric Kähler quantization. Under the integrality assumption $[\omega_0] \in H^2(M, 2\pi\mathbb{Z})$, there exists a holomorphic line bundle $L \rightarrow M$ with $c_1(L) = [\omega_0]$. The space $\mathcal{K}_{[\omega_0]}$ is thus the same as the space \mathcal{H} of hermitian metrics h on L with curvature $(1, 1)$ form $\omega_h = -i\partial\bar{\partial}\log h \in \mathcal{K}_{[\omega_0]}$. and we may identify Kähler metric $\omega_{\phi} \in \mathcal{K}_{[\omega_0]}$ with the hermitian metric $h = e^{-\phi}h_0$ or equivalently with the potential ϕ . The Hilbert spaces ‘quantizing’ (M, ω) are defined in the standard way to be the spaces $H^0(M, L^k)$ of holomorphic sections s_{α} , $\alpha = 1, \dots, N_k$ of $L^k = L \otimes \dots \otimes L$. We set the number of independent sections to be $N_k = \dim H^0(M, L^k)$.

The Kodaira embedding theorem uses a basis of these holomorphic sections to embed M into a high dimensional projective space

$$z \rightarrow [s_1(z), \dots, s_{N_k}(z)] : M \rightarrow \mathbb{C}\mathbb{P}^{N_k-1}.$$

The projective space has a natural Fubini-Study metric on it. Its pull-back to M is called a Bergman metric on M , and it depends on the choice of basis of sections we started with. Any two bases can be related by a $GL(N_k, \mathbb{C})$ transformation and the space \mathcal{B}_k can be identified with the space of all possible bases of sections which lead to inequivalent Bergman metrics. Since the Fubini-Study metric is invariant under $U(N_k)$ transformations of sections and since the overall normalization of the basis vectors is irrelevant since it does not change the associated Bergman metric, we have

$$\mathcal{B}_k \simeq SL(N_k, \mathbb{C})/SU(N_k). \quad (7)$$

The coset on the right side is the space of positive-definite hermitian matrices of determinant one.

This identification may be viewed in a slightly different way. We can identify \mathcal{B}_k with the space \mathcal{H}_k of all hermitian inner products on $H^0(M, L^k)$. The inner product associated with a given hermitian metric h on L , which induces a hermitian metric h^k on L^k , is given by

$$\text{Hilb}_k(h)_{\alpha\beta} := \frac{1}{V} \int_M \bar{s}_\alpha s_\beta h^k \omega_h^n, \quad (8)$$

where V is the volume, or in two dimensions the area $V = A$, of the manifold M . Having fixed a background metric $\omega_0 \in \mathcal{K}_{[\omega_0]}$, we denote the associated hermitian metric on L by $h_0 \in \mathcal{H}$. This h_0 thus induces a reference inner product $\text{Hilb}_k(h_0)$, which we can fix, without loss of generality, to be equal to the identity matrix. Any other inner product may then be identified with the positive hermitian matrix representing it relative to the background inner product. Since for any metric $h = h_0 e^{-\phi}$ in \mathcal{H} there is corresponding metric $\omega_\phi \in \mathcal{K}_{[\omega_0]}$, we thus obtain a map

$$\text{Hilb}_k(\phi) : \mathcal{K}_{[\omega_0]} \rightarrow \mathcal{H}_k := \{\text{Inner products on } H^0(M, L^k)\}. \quad (9)$$

We can also introduce a map FS_k which converts an inner product $G \in \mathcal{H}_k$ to a Bergman metric in $\mathcal{B}_k \in \mathcal{K}_{[\omega_0]}$ as follows. Given an inner product G , one can form a G -orthonormal basis $\{s_\alpha\}$ and an associated Kodaira holomorphic embedding. If e denotes a local frame for $L \rightarrow U \subset M$, then any $s \in H^0(M, L^k)$ may be expressed locally as $s = f e^{\otimes k}$. The pull back of the Fubini-Study hermitian metric on $\mathcal{O}(1) \rightarrow \mathbb{C}\mathbb{P}^{N_k-1}$ is given by

$$FS_k : \mathcal{H}_k \rightarrow \mathcal{K}_{[\omega_0]}, \quad FS_k(G) = \frac{1}{k} \log \sum_{j=1}^{N_k} |f_j(z)|^2 \in \mathcal{B}_k \subset \mathcal{K}_{[\omega_0]}. \quad (10)$$

Given the background inner product $\text{Hilb}_k(h_0)$, we let $s^0 : M \rightarrow \mathbb{C}\mathbb{P}^{N_k-1}$ denote the associated Kodaira embedding. Let $A \in GL(N_k, \mathbb{C})$ denote the change of basis matrix from a $\text{Hilb}_k(h_0)$ -orthonormal basis to a G -orthonormal basis s . Then the positive matrix P corresponding to G is $P = A^\dagger A$ and the Kähler potential for the corresponding Bergman metric is given by

$$\phi_P(z) = FS_k(P)(z) := \frac{1}{k} \log |As^0(z)|^2. \quad (11)$$

This formula is obviously invariant under the $U(N_k)$ action on A on the left. Since also the space of potentials is defined modulo constant shifts, we thus have an identification

$$FS_k : SL(N_k, \mathbb{C})/SU(N_k) \rightarrow \mathcal{B}_k. \quad (12)$$

This identification can be viewed as a correspondence between random positive ermitian matrices and random Bergman metrics. It allows us to reduce the asymptotic calculation of integrals over metrics to matrix integrals. Thus, under the change of variables $P \in SL(N_k, \mathbb{C})/SU(N_k) \rightarrow \phi_P \in \mathcal{B}_k$, we may rewrite integrals over metrics as integrals over matrices

$$\int_{\mathcal{B}_k} e^{-S_k(\phi)} F_k(\phi) \mathcal{D}_k \phi = \int_{SL(N_k, \mathbb{C})/SU(N_k)} e^{-S_k(P)} F_k(P) dP. \quad (13)$$

Different choices of the discretized action $S_k(P) = S_k(\phi_P)$, the operators $F(P) = F_k(\phi_P)$ and the measure $dP = \mathcal{D}_k \phi_P$ will be considered later in the text.

Remark: We now have three ways to view the space \mathcal{B}_k :

- (1) As a subspace $\mathcal{B}_k \subset \mathcal{K}_{[\omega_0]}$ of Bergman metrics in the space of Kähler metrics;
- (2) As the space \mathcal{H}_k of inner products on $H^0(M, L^k)$;
- (3) As the coset space $SL(N_k, \mathbb{C})/SU(N_k)$.

The identification of (3) with (2) is just the choice of a background inner product, while the identification of (1) and (2) uses the FS_k map (12).

Any Kähler metric on M can be approximated by a Bergman metric with increasing accuracy as $k \rightarrow \infty$. This theorem follows from the large k limit of the asymptotics of Bergman or Szegő kernels [41, 6, 43]. A key statement is that for any $\phi \in \mathcal{K}_{[\omega_0]}$,

$$\lim_{k \rightarrow \infty} FS_k \circ \text{Hilb}_k(\phi) = \phi. \quad (14)$$

This limit formula may be differentiated any number of times. There is even a complete asymptotic expansion for the left side in terms of curvature invariants of the right side, which follows from the following result. If $\{s_\alpha\}$ is an orthonormal basis with respect to the inner product (8), then

$$\rho_k = \sum_{\alpha=1}^{N_k} \bar{s}_\alpha(z) s_\alpha(z) h^k = k^n + \frac{1}{2} k^{n-1} R(\omega_h) + \mathcal{O}(k^{n-2}), \quad (15)$$

where for large k there is a complete asymptotic expansion on the right, depending on various curvature invariants [43, 6, 29], see also [19]. The left side of (15) is the contraction of the diagonal of the Szegő projector

$$\Pi_{h^k}(z, w) = \sum_{\alpha} s_\alpha(z) \otimes s_\alpha(w)^*, \quad (16)$$

i.e. the kernel of the orthogonal projection from all square integrable sections of $L^k \rightarrow M$ to the space $H^0(M, L^k)$ of holomorphic sections, with respect to the inner product (8). The pointwise asymptotics (15) provides an important tool in the problem of approximating the formal integrals over $\mathcal{K}_{[\omega_0]}$ by the integrals (13) over \mathcal{B}_k . However, as well as in random matrix theory, it requires more than just pointwise limits of integrands to obtain the $k \rightarrow \infty$ limits of matrix integrals.

2.3. Invariant metric and the Mabuchi-Semmes-Donaldson metric. Recall some facts on the group $GL(N, \mathbb{C})$ and its subgroup $SL(N, \mathbb{C})$. The group action on the hermitian matrices is defined as

$$A \cdot P = A^\dagger P A.$$

Any matrix $g \in GL(N, \mathbb{C})$ may be expressed uniquely as the product $g = kan$ of a matrix $k \in U(N)$, $a \in D(N)$, the subgroup of diagonal matrices and $n \in T(N)$, the group of upper triangular matrices with ones on the diagonal. This factorization is known as the Iwasawa

decomposition of $GL(N, \mathbb{C})$. We denote the Lie algebras of $D(N)$ and $T(N)$ by \mathfrak{d} and \mathfrak{t} respectively. An important point is that the exponential map

$$\exp_I : \mathfrak{d} \oplus \mathfrak{t} \rightarrow GL(N, \mathbb{C})/U(N) \quad (17)$$

is a diffeomorphism since $GL(N, \mathbb{C})/U(N)$ has non-positive curvature. Hence, we may define the inverse map $\log P$ of $P \in GL(N, \mathbb{C})/U(N)$. In the case of positive-definite hermitian matrices, it is the usual logarithm.

The $GL(N, \mathbb{C})$ -invariant metric on the space $GL(N, \mathbb{C})/U(N)$ can be written as $\text{tr}(P^{-1}\delta P)^2$. With a suitable normalization, the corresponding Haar measure is

$$\mu_N(P) = \frac{1}{(\det P)^N} dP_{11} \cdots dP_{NN} \prod_{1 \leq j < k \leq N} d\text{Re } P_{jk} d\text{Im } P_{jk}. \quad (18)$$

Here we note that the space of positive-definite hermitian matrices is open in the space of all hermitian matrices and has N real diagonal coordinates $P_{11} \cdots P_{NN}$ and $N(N-1)$ off-diagonal coordinates $\text{Re } P_{jk}, \text{Im } P_{jk}$, for $1 \leq j < k \leq N$, so that the real dimension is N^2 . The metric (18) is easily pulled-back to the metric on $SL(N, \mathbb{C})/SU(N)$ by imposing the constraint $\det P = 1$.

Using the map (9) one can show [7] that the invariant metric on $\mathcal{B}_k = SL(N_k, \mathbb{C})/SU(N_k)$ converges to the Mabuchi-Semmes-Donaldson metric (4) on $\mathcal{K}_{[\omega_0]}$. Indeed, the variation of Hilb_k is given by

$$\delta \text{Hilb}_k(\phi)_{\alpha\beta} = \frac{1}{V} \int_M \left[(-k - \frac{1}{2} \Delta_\phi) \bar{s}_\alpha s_\beta h_0^k e^{-k\phi} \right] \delta\phi \omega_\phi^n. \quad (19)$$

Then the invariant metric on \mathcal{B}_k can be pulled back to a metric on $\mathcal{K}_{[\omega_0]}$ as follows

$$\text{tr}(\text{Hilb}_k^{-1}(\phi) \delta \text{Hilb}_k(\phi))^2 = k^{n+2} \int (\delta\phi)^2 \omega_\phi^n + \mathcal{O}(k^{n+1}), \quad (20)$$

where the expansion on the right follows from the formula (15). Thus, the discretized volume element $\mathcal{D}_k\phi$ in (13) corresponds to the Haar measure (18).

2.4. Geometric action functionals. The full measure on $\mathcal{K}_{[\omega_0]}$ will always be taken of the form $e^{-S} \mathcal{D}\phi$ for some action S . For example, in the case of surfaces, the Liouville action

$$S(g_0, \sigma) = \kappa S_L(g_0, \sigma) + \mu \int_M e^{2\sigma} \omega_0, \quad S_L(g_0, \sigma) = \int_M (\partial\sigma \wedge \bar{\partial}\sigma + \sigma R_0 \omega_0), \quad (21)$$

where R_0 is the Ricci scalar and μ the cosmological constant, is a very natural choice. In applications to two dimensional quantum gravity, the positive coupling constant κ is fixed in terms of the central charge of the matter CFT coupled to gravity. The Liouville action satisfies the cocycle identities,

$$S_L(g_0, g_1) = -S_L(g_1, g_0), \quad S_L(g_0, g_1) + S_L(g_1, g_2) = S_L(g_0, g_2), \quad (22)$$

valid for any metrics g_0, g_1 and g_2 in the same conformal class. These identities are fundamental constraints that any effective gravitational action must satisfy.

In our framework, it is natural to express the action functionals in terms of the Kähler potential, e.g. the Liouville action reads

$$S_L(\omega_0, \phi) = \int_M \left(\frac{1}{4} \partial \log \frac{\omega_\phi}{\omega_0} \wedge \bar{\partial} \log \frac{\omega_\phi}{\omega_0} + \frac{1}{2} R_0 \omega_0 \log \frac{\omega_\phi}{\omega_0} \right). \quad (23)$$

However, and quite remarkably, there are other geometric functionals, satisfying the cocycle conditions, that are local in ϕ (but not in σ in the case of surfaces) that we can naturally consider. The most important one is the so-called Mabuchi action [30] (often called “Mabuchi energy” in the mathematical literature), which reads

$$S_M(\omega_0, \phi) = \int_M \left(\frac{\bar{R}}{2n} \phi \sum_{k=0}^n \omega_\phi^k \wedge \omega_0^{n-k} - \phi \sum_{k=0}^{n-1} \text{Ric}(\omega_0) \wedge \omega_\phi^k \wedge \omega_0^{n-1-k} + \omega_\phi^n \log \frac{\omega_\phi^n}{\omega_0^n} \right), \quad (24)$$

where $\text{Ric}(\omega_0)$ is the Ricci (1, 1)-form associated with ω_0 and \bar{R} the average Ricci scalar given by $\bar{R} = \frac{1}{V} \int_M R \omega_0^n$. The equation of motion of the Mabuchi action yields the condition for the scalar curvature to be constant, a property shared with the Liouville action restricted to a fixed Kähler class in one complex dimension. Another interesting functional, satisfying the cocycle identities, is the so-called Aubin-Yau action,

$$S_{AY}(g_0, \phi) = \int_M \left(- \sum_{k=0}^{n-1} \frac{k+1}{n+1} \partial \phi \wedge \bar{\partial} \phi \wedge \omega_\phi^{n-1-k} \wedge \omega_0^k + \phi \omega_0^n \right). \quad (25)$$

In one complex dimension, the above formulas simplify and yields

$$S_{AY}(\omega_0, \phi) = \int \left(\frac{1}{2} \phi \partial \bar{\partial} \phi + \phi \omega_0 \right), \quad (26)$$

$$S_M(\omega_0, \phi) = \int \left(\frac{1}{2} \bar{R} \phi \partial \bar{\partial} \phi + \phi (\bar{R} \omega_0 - \text{Ric}(\omega_0)) + \omega_\phi \log \frac{\omega_\phi}{\omega_0} \right). \quad (27)$$

Notably, in this case the first two terms of the Mabuchi action are proportional to the Aubin-Yau action.

3. DEFINING PATH INTEGRALS

3.1. Formal definition. The aim of this section is to explain how to define path integrals of the form $\int e^{-S} \mathcal{D}\phi$, where S is a desired action and $\mathcal{D}\phi$ is the volume form of the Mabuchi-Semmes-Donaldson riemannian metric. The infinite dimensional, continuum path integral is only a formal expression. We would like to define path integrals over $\mathcal{K}_{[\omega_0]}$ as a limit $k \rightarrow \infty$ of the finite-dimensional, regularized, path integrals over a sequence of measures $\mu_k = e^{-S_k(\phi)} \mathcal{D}_k \phi$ on \mathcal{B}_k :

$$\int_{\mathcal{K}_{[\omega_0]}} F(\phi) e^{-S(\phi)} \mathcal{D}\phi := \lim_{k \rightarrow \infty} \int_{\mathcal{B}_k} F_k(\phi) e^{-S_k(\phi)} \mathcal{D}_k \phi. \quad (28)$$

To define rigorously the limit on the right side of this equation is clearly a difficult issue, related to the renormalization of the theory, a problem that we will only start to address in the present work.

Let us comment on the choice of the volume forms on \mathcal{B}_k and $\mathcal{K}_{[\omega_0]}$. As we explained in §2, the obvious choice is to take $\mathcal{D}_k\phi$ to be the Haar measure on \mathcal{B}_k (18), i.e. the volume form of the symmetric space riemannian metric, which corresponds in the infinite dimensional limit to the riemannian volume density associated to the metric (4). Another option is to use the fact that the exponential map (17) $\exp_I : T_I\mathcal{B}_k \rightarrow \mathcal{B}_k$ is a diffeomorphism, and that the tangent space is equipped with the natural Euclidean volume form. This volume form converges to the tangent space volume form on $\mathcal{K}_{[\omega_0]}$, which is analogous to the flat volume form used in the Liouville theory [9, 15]. Here we will consider both options.

The main problem is then to construct sequences S_k of finite dimensional actions, so that the right side of Eq. (28) converges to a finite theory with the desired action S on $\mathcal{K}_{[\omega_0]}$. This is the problem of renormalization. Mathematically, we need to explain what it means for a sequence of measures μ_k on \mathcal{B}_k to converge to a measure on $\mathcal{K}_{[\omega_0]}$. Since $\mathcal{B}_k \subset \mathcal{K}_{[\omega_0]}$ we may regard the μ_k as a sequence of measures on $\mathcal{K}_{[\omega_0]}$ which are concentrated on \mathcal{B}_k . Hence we can ask whether the sequence μ_k has a weak limit in the sense of probability theory, i.e. whether the right side of (28) converges for some functions F .

A natural class of correlation functions $\mathcal{K}_k(z_1, \dots, z_n)$ is given by taking F in Eq. (28) to be the product of the Kähler potential field ϕ at several points,

$$\begin{aligned} \mathcal{K}_k(z_1, \dots, z_n) &= \int_{\mathcal{B}_k} \prod_{j=1}^n (\phi(z_j)) e^{-S_k(\phi)} \mathcal{D}_k\phi \\ &= \int_{SL(N_k)/SU(N_k)} \left(\prod_{j=1}^n \log \bar{s}(z_j) P s(z_j) \right) e^{-S_k(P)} \mathcal{D}_k P, \end{aligned} \quad (29)$$

where in the second line we use (11) and (13) to change variables from the Kähler potential ϕ to the positive-definite hermitian matrix P and $\mathcal{D}_k P$ is an appropriate measure. In some simple examples, $e^{-S_k(P)} \mathcal{D}_k P$ splits into the Haar measure on $U(N_k)$ and the measure on the eigenvalues of P , in which case it is often possible to calculate some of these correlation functions at large k .

3.2. Large deviations approach. A simpler criterion for convergence of a sequence of finite dimensional measures $\mu_k = e^{-S_k} \mathcal{D}_k\phi$ to the infinite dimensional measure $e^{-S} \mathcal{D}\phi$ is in terms of large deviations theory. In fact, a sequence may satisfy an LDP without converging in the previous sense, i.e. without all correlators necessarily having finite limits, so the LDP may be viewed as a rough check on whether the sequence is defining an infinite dimensional theory.

As has been already mentioned, we may regard the measures μ_k on \mathcal{B}_k as measures on $\mathcal{K}_{[\omega_0]}$ which are supported on (i.e. are zero away from) the submanifolds $\mathcal{B}_k \in \mathcal{K}_{[\omega_0]}$. One says that the sequence of measures μ_k satisfies an LDP with the speed n_k and a rate function S if, for all balls $B(\omega, \delta) \subset \mathcal{K}_{[\omega_0]}$ of radius δ , one has

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \log \mu_k(B(\omega, \delta)) = - \inf_{\omega_\phi \in B(\omega, \delta)} S(\omega_\phi). \quad (30)$$

Here, the radius of the ball $B(\omega, \delta)$ is computed with the riemannian metric on $\mathcal{K}_{[\omega_0]}$. Thus, a basic criterion for the sequence μ_k to tend to $e^{-S(\phi)} \mathcal{D}\phi$ is that it satisfies an LDP with the

rate functional S . This criterion is rather crude since it only determines the action S and does not take into account the integration measure $\mathcal{D}\phi$. But this is an advantage when the infinite dimensional measure is hard to define and only the corresponding action is known.

It may be hard to compute the rate function by integrating over balls. However, one could use a simpler method, based on Gärtner-Ellis theorem, see [11] for background. Suppose that there exists a limit logarithmic moment generating function

$$W(J) = \lim_{k \rightarrow \infty} \frac{1}{n_k} \log \mathbf{E}_k \{ e^{\langle J, \phi_k \rangle} \},$$

where ϕ_k is our random variable, and $\langle \cdot, \cdot \rangle$ is an appropriate pairing, which is case-dependent. Then the LDP rate functional is determined by the Legendre transform of the generating function

$$S(\phi) = \sup_J \{ \langle J, \phi \rangle - W(J) \}.$$

This method is a finite-dimensional analog of the method of effective action in the quantum field theory, where the generating functional is defined as a Fourier-Laplace transform of the full quantum measure.

3.3. Contraction principle. No general geometric method for constructing simple and tractable approximating actions S_k to a given geometric action S is known. The most straightforward way is to restrict the functional S to \mathcal{B}_k

$$S_k = S(\phi)|_{\mathcal{B}_k}. \quad (31)$$

This is a good approximation in the sense that $S_k \simeq S(\phi)$ as $k \rightarrow \infty$. On the other hand, the restriction does not simplify the action nor does it arise naturally from the point of view of the geometry of \mathcal{B}_k .

In special cases, considered in the last section, such as the Aubin-Yau action or Mabuchi action, the pointwise Bergman kernel expansion (15) can be used to construct alternative finite dimensional approximations. Other techniques related to the geometric invariant theory have been used [44, 35, 36, 34], see also [24]. These approximations are used to construct S_k so that $S_k(T_k(\phi)) \rightarrow S(\phi)$, where

$$T_k = FS_k \circ \text{Hilb}_k : \mathcal{K}_{[\omega_0]} \rightarrow \mathcal{B}_k. \quad (32)$$

It is not clear whether knowing such pointwise approximations is enough to approximate the path integrals.

However, the contraction principle of large deviation theory suggests some new methods for constructing finite dimensional approximations. Namely, we may define

$$S_k(P) = \inf_{\{\phi \in \mathcal{K}_{[\omega_0]}, \text{Hilb}_k(\phi) = P\}} S(\phi). \quad (33)$$

It is clear that the S_k of (33) does tend to S as $k \rightarrow \infty$ since the fiber of the map $\text{Hilb}_k(\phi)$ shrinks to a single point $\{\phi\}$ in this limit. We now explain how (33) is related to the contraction principle.

If we are given a formal infinite dimensional measure μ , we can form approximating measures by pushing μ forward to \mathcal{B}_k , meaning

$$\mu_k = (\text{Hilb}_k)_* \mu, \quad (34)$$

where $(\text{Hilb}_k)_*$ denotes the push-forward under the map (9). It is equivalent but often more natural geometrically to consider instead the pushforward

$$\mu_k = (T_k)_* \mu. \quad (35)$$

We observe that

$$\lim_{k \rightarrow \infty} \mu_k = \mu, \quad (36)$$

so that this method of obtaining finite dimensional approximations is devised to automatically reproduce the original measure in the large k limit. Indeed, the statement (36) is equivalent to

$$\lim_{k \rightarrow \infty} \int_{\mathcal{K}_{[\omega_0]}} f(FS_k \circ \text{Hilb}_k(\phi)) \mu_k = \int_{\mathcal{K}_{[\omega_0]}} f(\phi) \mu$$

for suitable functions f . But this holds by (14) as long as the limit can be taken under the integral sign since

$$\lim_{k \rightarrow \infty} f(FS_k \circ \text{Hilb}_k(\phi)) = f(\phi)$$

for all continuous functions f .

The contraction principle (see e.g. [11, 26]) says that given a sequence of measures μ_k satisfying an LDP with speed n_k and rate function I , and a map $T : X \rightarrow Y$, the pushforward measures $T_* \mu_k$ also satisfy an LDP with speed n_k and the rate function

$$J(y) = \inf_{\{x \in X, T(x)=y\}} I(x). \quad (37)$$

Our proposal (33) does not in fact follow from the contraction principle because we have only one measure μ on $\mathcal{K}_{[\omega_0]}$ rather than a sequence of measures and also because we have a sequence of maps T_k between $X = \mathcal{K}_{[\omega_0]}$ and $Y = \mathcal{B}_k$ rather than one fixed map T , as in (37). Therefore we view (33) as an approximation scheme suggested by the contraction principle, and the formal considerations above only suggest why it is natural.

This procedure is actually reminiscent of the Wilsonian notion of renormalizing path integrals by integrating out higher frequencies, in the sense that if we have $\mu_k \simeq e^{-n_k I(\phi)} \mathcal{D}(\phi)$ then the pushforward measure corresponds to computing the path integral

$$T_* \mu_k \simeq \int_{T^{-1}(\phi)} e^{-n_k I(\phi)} \mathcal{D}(\phi)$$

over all metrics that correspond to the same regularized Bergman metric.

In principle, we can modify our problem so that the contraction principle does apply. For this we need to embed the measure μ and action S into a one parameter family $\mu_\varepsilon, S_\varepsilon$, satisfying an LDP. Then we can apply the map T_k to a sequence $T_k * \mu_\varepsilon$. We now have two parameters k, ε . We can then apply the contraction principle with k fixed to the family in ε .

Since there are many ways to embed μ, S into a family μ_ε satisfying an LDP, we would like to choose one which leads to a good approximation as $k \rightarrow \infty$. For some special μ, S

there may be natural parameters one can use for ε . We now consider two ways to proceed when no such parameters occur.

First, if the measure μ on $\mathcal{K}_{[\omega_0]}$ is well-defined, we can just replace our path integral $\int e^{-S(\phi)} \mathcal{D}\phi$ by $\int e^{-\frac{1}{\varepsilon}S(\phi)} \mathcal{D}\phi$. This new family automatically has an LDP with rate function S and it directly gives (33) as the approximate rate function.

The approximation (33) can often be difficult to evaluate since it is a constrained variational problem. A drastic simplification is to linearize it around the critical metric φ_c , in the case when the action $S(\varphi)$ has a unique critical point. With no essential loss of generality we may assume $S(\varphi_c) = 0$. We then change the action from S to S_ε depending on a small parameter ε by defining $S_\varepsilon(\varphi) = \frac{1}{\varepsilon}S(\varphi_c + \sqrt{\varepsilon}\eta)$. We expand the right side in powers of ε ,

$$\frac{1}{\varepsilon}S(\varphi_c + \sqrt{\varepsilon}\eta) = \frac{1}{2} \int \eta \frac{\delta^2 S}{\delta \varphi^2} \Big|_{\varphi_c} \eta + \mathcal{O}(\sqrt{\varepsilon}).$$

We then change variables $\phi_0 + \varepsilon\eta \rightarrow \eta$ in the functional integral. We are simply approximating the original measure by a gaussian measure on the tangent space $T_{\varphi_c} \mathcal{K}_{[\omega_0]}$. We ignore the Jacobian factor of powers of ε since we also normalize the measure by dividing by its mass, which has the same power of ε . The resulting integral is then a gaussian integral

$$\int_{T_{\varphi_c} \mathcal{K}_{[\omega_0]}} e^{-\frac{1}{\varepsilon}[\langle \delta^2 S_{\varphi_c} \eta, \eta \rangle + \langle J, \eta \rangle]} \mathcal{D}\eta$$

which satisfies an LDP with rate function $\delta^2 S_{\varphi_c}$. Hence we may apply the contraction principle to this sequence (or family) of measures when we push forward by

$$DHilb_k(\varphi_c) : T_{\varphi_c} \mathcal{K}_{[\omega_0]} \rightarrow T_{Hilb_k(\varphi_c)} \mathcal{B}_k.$$

Here we use that $Hilb_k(\varphi_c + \sqrt{\varepsilon}\eta) = Hilb_k(\varphi_c) + \sqrt{\varepsilon}DHilb_k(\varphi_c)\eta + \mathcal{O}(\varepsilon)$. Since the rate function for the family is the quadratic part $S_2(\eta) = \langle \delta_\varphi^2 S_{\varphi_c} \eta, \eta \rangle$, the finite dimensional approximation to this rate functional is given by

$$S_k(\delta P) = \inf_{\eta: DHilb_k(\varphi_c)\eta = \delta P} I_2(\eta). \quad (38)$$

The result we get is equivalent to the well-known fact that the pushforward of a gaussian measure under a linear map is a new gaussian measure.

This linearizes the problem to the tangent spaces and we would only expect (38) to be a good approximation to S near the critical point φ_c .

4. MEASURES OF THE MATRIX MODEL TYPE ON BERGMAN METRICS

In this section we consider the bottom-up approach, namely we take the finite dimensional integrals (13), that arise naturally from the random matrix side of the identity rather than from the geometric side. In particular we consider $U(N_k)$ -invariant measures. We emphasize that the corresponding actions $S_k(\phi)$ do not necessarily have a good geometric interpretation. In this approach geometric features of Kähler metrics are suppressed and only the spectral theory of random positive-definite hermitian matrices is retained. Hence one should not expect to get a geometric theory of random surfaces arising in the large k limit. Our

purpose is simply to illustrate the LDP's for such measures, and to compute some interesting correlation functions.

4.1. Eigenvalue models. From the point of view of the standard matrix models the simplest class of measures on \mathcal{B}_k has the form

$$\mu(P) = \mathcal{F}(\Lambda)dP, \quad (39)$$

where $\mathcal{F}(\Lambda)$ depends only on the eigenvalues Λ of the matrix $P \in \mathcal{B}_k$, and not on the angular part U in the decomposition $P = U\Lambda U^\dagger$ and the measure dP can be decomposed as $d\Lambda dU$, where dU is the Haar measure on $U(N_k)$. However, we should emphasize that even for the eigenvalue type measures the computation of geometric correlation functions (29) will involve nontrivial integration over angular variables.

In principle, the integration in (13) goes over the matrices $P \in \mathcal{B}_k$ which have unit determinant. In the actual computations it is easier to use an unrestricted positive hermitian matrix $P \in GL(N_k)/U(N_k)$ as integration variable. Since the dilatations $P \rightarrow cP$ for $c > 0$ do not change the associated Bergman metric, they correspond to a 'gauge' freedom that must be mod out. We shall do that in the standard way by inserting the gauge constraint $\det P = 1$ in the path integral with a δ -function, as follows,

$$\delta(\log \det P)\mu(P) = \int_{-i\infty}^{i\infty} da [\det P]^a \mu(P) = \int_{-i\infty}^{i\infty} da \mu_a(P). \quad (40)$$

Obviously, the constrained measures are still of the type of Eq. (39).

4.2. Empirical measures. Recall that, after fixing a background metric ω_0 , any inner product $\text{Hilb}_k(\phi)$ may be represented as a positive hermitian operator $P_k(\phi)$ relative to $\text{Hilb}_k(\omega_0)$. In mathematical terms, we can encode Kähler potentials by the empirical measure of eigenvalues (that is to say, by the density of eigenvalues) of either $\frac{1}{k} \log P_k(\phi)$ or $P_k(\phi)$. In the first case we encode the hermitian operator

$$X_k(\phi) := \frac{1}{k} \log P_k(\phi) \quad (41)$$

by the density

$$\nu_{X_k(\phi)}(x) = \frac{1}{N_k} \sum_{j=1}^{N_k} \delta(x - x_{k,j}), \quad (42)$$

where $\{x_{k,j}\}_{j=1}^{N_k}$ are the eigenvalues of $X_k(\phi)$. This density defines a measure $d\nu_{X_k(\phi)} = \nu_{X_k(\phi)} dx \in \mathcal{M}(\mathbb{R})$, where $\mathcal{M}(\mathbb{R})$ is the convex set of probability measures on \mathbb{R} . In the second case we can do the same for P_k . We can then choose our matrix model such that at large k the density (42) associated with the matrix $\frac{1}{k} \log P_k$, or the density associated with the matrix P_k itself, has a smooth large k limit. Clearly this would correspond to quite different models. But in either case we have an embedding

$$\nu_k \circ \text{Hilb}_k : \mathcal{K}_{[\omega_0]} \rightarrow \mathcal{M}(\mathbb{R}_+). \quad (43)$$

Under this embedding, we may push forward the given probability measure on $\mathcal{K}_{[\omega_0]}$ to obtain a probability measure on $\mathcal{M}(\mathbb{R})$:

$$(\nu_k \circ \text{Hilb}_k)_* \mu = \mathbf{Prob}_k \in \mathcal{P}(\mathcal{M}(\mathbb{R})), \quad (44)$$

where $\mathcal{P}(\mathcal{M}(\mathbb{R}))$ is the metric space of probability measures on $\mathcal{M}(\mathbb{R}_+)$. The rate function for the sequence of measures \mathbf{Prob}_k , if it exists, will then be a functional on $\mathcal{M}(\mathbb{R})$ rather than a geometric functional. This approach is common in the theory of large deviations of random matrices, see e.g. [26], and comes up naturally in our problem.

We also remark that the empirical measure map induces a new family of correlation functions, the spectral correlation functions on \mathbb{R}^n , defined by

$$\mathcal{K}_k(\mu_1, \dots, \mu_n) = \mathbf{E}_k \prod_{j=1}^n \nu_k(\mu_j) \otimes \dots \otimes \nu_k(\mu_n). \quad (45)$$

Here, $\nu_k \otimes \nu_k \cdots \otimes \nu_k$ denotes the product probability measure on \mathbb{R}^n determined by ν_k , i.e. its integral against a function $f(\mu_1, \dots, \mu_n)$ is given by $\int_{\mathbb{R}^n} f(\mu_1, \dots, \mu_n) \nu_k(\mu_1) \cdots \nu_k(\mu_n)$.

4.3. Wishart ensemble. The Wishart measure is defined by

$$\mu_a(P) = e^{-g \operatorname{tr} P} [\det P]^{a+N} \mu_N(P).$$

Here $\mu_N(P)$ is the Haar measure (18). Under the decomposition $P = U\Lambda U^\dagger$ the radial part of the Wishart measure is the eigenvalue distribution

$$\frac{1}{Z_N} e^{-g \sum_{j=1}^N \lambda_j} |\Delta(\lambda)|^2 \prod_{j=1}^N \lambda_j^a d\lambda_j, \quad (46)$$

where $\Delta(\lambda) = \prod_{j < k} (\lambda_j - \lambda_k)$ is the usual Vandermonde determinant. The normalizing constant is $Z_N = \int \mu(P) = \int_{-i\infty}^{i\infty} da g^{-N(N+a)} \prod_{j=1}^N \Gamma(j+1)\Gamma(j+a)$. We have also mod out by the angular integral.

Now, we would like to perform a large deviation analysis for the sequence of probability measures (46). Since we have $U(N)$ invariance, we only need to study the logarithmic asymptotics of integrals over eigenvalues, which is of a type studied in [27] (see Theorem 2.1). To obtain the large N limit, it is necessary to embed the spaces \mathbb{R}_+^N into one fixed space. In the context of Wishart models, the density of eigenvalues

$$d\nu_\lambda(x) := \frac{1}{N} \sum_{j=1}^N \delta(x - \lambda_j) dx$$

of the matrix P will have a smooth limit. Positive matrices with the same density of eigenvalues are conjugate under $U(N)$. Thus we have mapped the symmetric space \mathcal{P}_N into the space $\mathcal{M}_1^+(\mathbb{R}_+)$ of probability measures on the positive real numbers \mathbb{R}_+ . We then push-forward the measure $d\nu_N$ under this map to define a probability measure P_N on $\mathcal{M}_1^+(\mathbb{R}_+)$. Thus, P_N is supported on the set of measures (46) in terms of $d\nu_\lambda$ as

$$|\Delta(\lambda)|^2 e^{-g \sum_{j=1}^N \lambda_j} \prod_{j=1}^N \lambda_j^a = e^{-gN \int_{\mathbb{R}_+} x d\nu_\lambda + aN \int_{\mathbb{R}_+} \log x d\nu_\lambda + N^2 \Sigma(\nu_\lambda)}, \quad (47)$$

where

$$\Sigma(\nu) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \log |x - y| d\nu(x) d\nu(y)$$

is the standard Coulomb repulsion term between eigenvalues. The coupling constant in (47) scales as $g = g_0 N$, which is the usual 't Hooft's scaling. Then the sequence of measures (46) satisfies a large deviation principle with the rate function [27]

$$S(\nu_\lambda) = -g_0 \int_{\mathbb{R}_+} x d\nu_\lambda + \Sigma(\nu_\lambda). \quad (48)$$

Also, integration over a imposes the constraint

$$\int_{\mathbb{R}_+} \log x d\nu_\lambda = 0. \quad (49)$$

It is not difficult to solve the saddle point equation associated with (48) taking into account the constraint (49). The result is a smooth eigenvalue distribution ν_λ that depends on g and whose support is an interval of \mathbb{R}_+ containing the point $x = 1$. Since the details of this solution will not be used presently, we shall explain it in a separate publication.

It is also possible to explicitly calculate the geometric correlation functions (29). Here we would like to quote the results which will appear elsewhere. We start with the choice of a basis of sections s , which is orthonormal with respect to a choice of background metric h_0^k on the line bundle L^k , with corresponding background Kähler metric $\omega_0(z) = -\frac{1}{k} \partial \bar{\partial} \log h_0^k$. The random metric is then parametrized as $\omega_{\phi(P)}(z) = \frac{1}{k} \partial \bar{\partial} \log \bar{s}(z) P s(z)$. The expectation value of a single metric in the large k limit can then be easily shown to be equal to the background metric

$$\mathbf{E}_k \omega_\phi(z) = \frac{1}{k} \partial \bar{\partial} \log |s(z)|^2 = \omega_0(z) + \mathcal{O}(1/k).$$

In fact, it is not hard to show that the same result holds for an arbitrary measure of the eigenvalue type (39)

$$\int_{\mathcal{B}_k} \omega_{\phi(P)}(z) \mu(P) = \omega_0(z) + \mathcal{O}(1/k)$$

In this scaling limit the two-point function has a similar behavior

$$\mathbf{E}_k \omega_\phi(z) \omega_\phi(y) = \omega_0(z) \omega_0(y) + \mathcal{O}(1/k).$$

Essentially, these calculations indicate that in the large k limit the Wishart measures concentrate at the Dirac measure δ_{ω_0} , i.e. that random metrics concentrate at the background metric.

This result is not contradictory with the fact that the eigenvalue distribution of P has a non-trivial smooth limit as indicated above. Indeed, the eigenvalues of $P_k(\phi)$ for $\phi \neq 0$ grow (or decay) exponentially with k , in such a way that the eigenvalue distribution of X_k (41)-(42) has a smooth limit. This scaling follows from the large k asymptotics (20) and the fact that the size N_k of the matrix grows as k^n .

More precisely, let $\nu_{X_k(\phi)}(x)$ be the density of eigenvalues of $X_k(\phi)$ and let $\nu_\phi(x)$ be its limit when $k \rightarrow \infty$. We call $d\nu_\phi$ the *Szegő limit measure* of ϕ . We conjecture that the limit exists and is given by

$$d\nu_\phi = (\exp_{\phi_0}^{-1}(\phi))_* d\nu_{\phi_0}, \quad (50)$$

where

$$\exp_{\phi_0} : T_{\phi_0} \mathcal{K}_{[\omega_0]} \rightarrow \mathcal{K}_{[\omega_0]}$$

is the exponential map for the Donaldson-Mabuchi-Semmes metric (4). Here,

$$\exp_{\phi_0}^{-1}(\phi) = \dot{\phi}_0,$$

is the initial tangent vector to the unique geodesic with endpoints ϕ_0, ϕ . The conjecture follows formally from the fact that $\frac{1}{k} \log P_k(\phi)$ is the (scaled) inverse of the exponential map of \mathcal{H}_k and that geodesics of \mathcal{H}_k tend to geodesics of $\mathcal{K}_{[\omega_0]}$ [33]. The authors have verified the conjecture in the case of toric varieties.

These considerations motivate the study of the model of the next subsection.

4.4. Models on the tangent space. Let us thus use X_k defined by (41) as the basic variable. In effect, we are inverting the exponential map (17) and defining the probability measure on the tangent space $T_I \mathcal{B}_k$ of the symmetric space of positive matrices.

The simplest such measure is the gaussian measure

$$\mu(X) = e^{-g N \text{tr} X^\dagger X} dX. \quad (51)$$

Here dX is the usual Lebesgue measure on the tangent space. The action $\text{tr} X^\dagger X$ is simply the square of the geodesic distance in the symmetric space from 0 to $\exp_0 X$. Indeed, the exponential map follows the unit speed geodesic from 0 to $\exp_0 X$ and its length is $\|X\|$. Consequently, the action has the natural infinite dimensional limit

$$S(\phi_0, \phi) = d_{\mathcal{K}_{[\omega_0]}}^2(\phi_0, \phi),$$

the square of the geodesic distance of $\mathcal{K}_{[\omega_0]}$ from the background metric (or potential) ϕ_0 to ϕ . If instead of dX we choose the Haar measure as the integration measure, it is known that the distance function $d_{\mathcal{B}_k}$ on \mathcal{B}_k converges to the distance function of $\mathcal{K}_{[\omega_0]}$ [7, 3]. Thus, we formally have

$$\int_{\mathcal{B}_k} F_k(\phi) e^{-d_{\mathcal{B}_k}(\phi_0, \phi)^2} \mathcal{D}_k \phi \rightarrow \int_{\mathcal{K}_{[\omega_0]}} F(\phi) e^{-d_{\mathcal{K}_{[\omega_0]}}^2(\phi_0, \phi)} \mathcal{D} \phi, \quad (52)$$

where $\mathcal{D}_k \phi$ denotes the Haar measure and $\mathcal{D} \phi$ denotes the Mabuchi measure, as usual. In effect, by using $\frac{1}{k} \log P_k$ instead of P_k , the limit measure is spread out from a delta function at the background metric to a bell curve centered at the background metric.

We note that the actions do not depend a priori on derivatives of ϕ , and thus we may expect these random potentials to be very singular. Recall that the corresponding metric is $\omega_0 + i\partial\bar{\partial}\phi$. We could smooth out the singularity by adding a term to the action involving derivatives of ϕ , but for the moment we consider the simple case without derivative terms.

The LDP rate functional for the model (51) was determined in [2] and is given by

$$I(\nu) = g \int x^2 d\nu(x) - \Sigma(\nu) + \text{constant}. \quad (53)$$

Here we would like to discuss the geometric meaning of this functional in the context of Kähler geometry. The pullback of (53) to $\mathcal{K}_{[\omega_0]}$ is given by

$$I(\phi) = g \int_M (\exp_{\phi_0}^{-1} \phi)^2 d\mu_{\phi_0} - \int \int_{M \times M} \log |\exp_{\phi_0}^{-1} \phi(x) - \exp_{\phi_0}^{-1} \phi(y)| d\mu_{\phi_0}(x) d\mu_{\phi_0}(y). \quad (54)$$

This is new action functional in Kähler geometry. The exponential map is not well-defined (see [38]), but $\exp_{\phi_0}^{-1} \phi$ is well-defined as the Dirichlet-to-Neumann operator, taking the

solution of the endpoint problem for geodesics to the initial data of the geodesic. It is known that $\exp_{\phi_0}^{-1} \phi$ is Lipschitz continuous, so the integrals are well-defined. If we write $\dot{\phi} = \exp_{\phi_0}^{-1} \dot{\phi}(x)$ for the initial velocity, then the functional becomes

$$I(\phi_0, \dot{\phi}_0) = g \int_M (\dot{\phi})^2 d\mu_{\phi_0} - \int \int_{M \times M} \log |\dot{\phi}(x) - \dot{\phi}(y)| d\mu_{\phi_0}(x) d\mu_{\phi_0}(y). \quad (55)$$

The first term is the one obtained in the previous section, and is simply the Hamiltonian of the geodesic flow on $T^*\mathcal{K}_{[\omega_0]}$, which is equal to the energy of the geodesic from ϕ_0 to ϕ .

5. APPROXIMATION OF GEOMETRIC ACTIONS

5.1. Aubin-Yau action and the contraction principle. In this section, we consider the top-down approach, where we construct the random sequences which converge to a given geometric functional on $\mathcal{K}_{[\omega_0]}$. We have already shown that the invariant metric on \mathcal{B}_k converges to the Mabuchi metric (4), see (20). Therefore, getting the approximation to actions is the second important ingredient in our construction.

Here we apply the contraction principle, formulated in §3.3, to the Aubin-Yau action (26) and determine the corresponding sequence of finite-dimensional approximations S_k , which converge to S_{AY} in the sense of large deviation theory.

We claim that the minimizing potential for the variational problem

$$S_k(P) = \inf_{\{\phi \in \mathcal{K}_{[\omega_0]}, \text{Hilb}_k(\phi) = P \in \mathcal{B}_k\}} S_{AY}(\phi)$$

is a Bergman metric $\phi(P) \in \mathcal{B}_k$ and that

$$S_k(P) = S_{AY}(\phi(P)) + O(1/k).$$

Thus, to leading order, the finite dimensional approximation to the Aubin-Yau action obtained in this way is the same as its restriction to \mathcal{B}_k .

We set up the constrained variational problem as a Lagrange multiplier problem. We want to find the critical points for

$$S_{AY}(\phi) + \text{tr}(L(\text{Hilb}_k(\phi) - P)),$$

where the Lagrange multiplier L is a hermitian matrix. We then get the following equations determining the critical point

$$\frac{1}{V} \left[k + \frac{1}{2} \Delta_\phi \right] (s_\alpha L_{\alpha\beta} \bar{s}_\beta) h_0^k e^{-k\phi} = 1, \quad \text{Hilb}_k(\phi)_{\alpha\beta} = P_{\alpha\beta}.$$

Since Δ_ϕ is a positive definite laplacian on M , the only solution to the first equation is

$$(s_\alpha L_{\alpha\beta} \bar{s}_\beta) h_0^k e^{-k\phi} = V/k.$$

Therefore we can solve for ϕ in terms of the matrix L ,

$$e^{-k\phi(L)} = \frac{V}{k} \frac{1}{(sL\bar{s})h_0^k}$$

and plug the solution to the second critical point equation. We get

$$\frac{1}{k} \int \frac{\bar{s}_\alpha s_\beta}{(sL\bar{s})} \omega_{\phi(L)}^n = P.$$

The last two equations imply that $\phi(L) = FS_k(L)$, where the matrix L is implicitly determined by the matrix P , according to the last equation. It follows that ϕ_L is a Fubini-Study metric, and that to leading order

$$S_k(P) = S_{AY}(\phi(P)) + O(1/k). \quad (56)$$

Therefore the contraction principle yields the same action as the pull-back of the action under the FS_k map.

5.2. Mabuchi action and Donaldson functional. The Mabuchi action (27) has a minimizer ϕ_c corresponding to the constant scalar curvature metric, which always exists in complex dimension one in each Kähler class. The contraction principle, applied to the Mabuchi action, leads to a constrained variational problem that is very hard to solve.

However, there exists another approximation method, proposed by Donaldson [18], which is based on the pointwise Bergman kernel asymptotics (15). The idea is to consider the variation of the log det Hilb with respect to ϕ ,

$$\delta \log \det \text{Hilb}_k(\phi) = \text{Tr} \text{Hilb}_k^{-1}(\phi) \delta \text{Hilb}_k(\phi) = \frac{1}{V} \int_M (-k\rho_k(\omega_\phi) - \frac{1}{2}\Delta_\phi \rho_k(\omega_\phi)) \delta \phi \omega_\phi^n,$$

where we have used Eq. (19). Using the expansion Eq. (15), one can then integrate the right side and get

$$V \log \det \text{Hilb}_k(\phi) = -kN_k S_{AY}(\omega_0, \phi) + \frac{k}{2} S_M(\omega_0, \phi) + \mathcal{O}(1),$$

provided that we chose the basepoint for the Hilb inner product to be the identity matrix, as we've always done. Therefore the following functional on $\mathcal{K}_{[\omega_0]}$ approximates the Mabuchi action on $\mathcal{K}_{[\omega_0]}$ in the large k limit

$$\mathcal{L}(\omega_0, \phi) = \frac{2V}{k} \log \det \text{Hilb}(\phi) + 2N_k S_{AY}(\omega_0, \phi). \quad (57)$$

Following Donaldson in [18], let us now define the following functional on positive hermitian matrices $P \in GL(N_k, \mathbb{C})/U(N_k)$,

$$Z(P) = \frac{V}{k} \log \det P + N_k S_{AY}(\omega_0, \phi(P)). \quad (58)$$

This functional is obtained by pulling back the first term in (57) from $\mathcal{K}_{[\omega_0]}$ to \mathcal{H}_k using the Hilb map and by pushing forward the second term in (57) in the opposite direction using the FS_k map (12). The difference between the two operations is at least of order $1/k$ on \mathcal{B}_k , as we pointed out in (14). Hence, the functional (58) defines automatically a good approximation to the Mabuchi action on the space \mathcal{B}_k .

We can then consider the following sequence of measures on positive-definite hermitian matrices in $GL(N_k, \mathbb{C})/U(N_k)$,

$$\mu_k(P) = e^{-gZ(P)} d\mu_N(P), \quad (59)$$

where g is a coupling constant and we use the Haar measure (18). Note that, as required, $Z(P)$ is invariant under the multiplication of P by a strictly positive constant and so is the measure (18). Therefore (59) is well-defined on the space \mathcal{B}_k . If we impose explicitly the constraint $\det P = 1$, the functional $Z(P)$ becomes simply proportional to the functional (56), that we obtained by applying the contraction principle to the Aubin-Yau action. The fact that this functional effectively defines the restriction of the Mabuchi functional to the space of Bergman metrics has been also observed in [34].

The study of the $k \rightarrow \infty$ limit of (59), see also (3), is a very interesting open problem that we plan to discuss in the future.

A second approach to regularizing the Mabuchi action is to linearize the problem around the constant scalar curvature metrics by applying the contraction principle to the modified Mabuchi functional,

$$\frac{1}{\varepsilon} S_M(\phi_c + \sqrt{\varepsilon} \eta),$$

see section 3.3. As in the Liouville theory [42], the semiclassical expansion can be developed for the Mabuchi action around the critical point and this reduces the Mabuchi action to its gaussian approximation

$$\frac{1}{\varepsilon} S_M(\phi_c + \sqrt{\varepsilon} \eta) = \frac{1}{8} \int \eta(\Delta_0 - 2\bar{R})\Delta_0\eta\omega_0 + \dots \quad (60)$$

where Δ_0 is the laplacian on the critical metric. The contraction principle leads to the following constrained problem,

$$S_{M,k}(\delta P) = \inf_{D_{\phi_c} \text{Hilb}_k(\phi) = \delta P} \frac{1}{8} \int \eta(\Delta_0 - 2\bar{R})\Delta_0\eta\omega_0.$$

It would be interesting to see how well this approximates our original problem.

6. CONCLUSIONS

Given a Kähler manifold M , we have introduced a new way to integrate over the space of Kähler metrics in a fixed Kähler class on M . The main ideas of the method are:

- To use the Kähler potential ϕ as the basic dynamical variable. The corresponding metric $\omega_0 + i\partial\bar{\partial}\phi$ is local in ϕ .
- To use Kähler quantization to construct finite dimensional approximations \mathcal{B}_k to $\mathcal{K}_{[\omega_0]}$. The spaces \mathcal{B}_k may be identified with symmetric spaces after fixing a background metric, hence there are many natural choices of integration measures and of action functionals on them. There is a long developed theory of approximating geometric functionals on $\mathcal{K}_{[\omega_0]}$ by finite dimensional ones on \mathcal{B}_k .
- To use the theory of large deviations to decide when a sequence μ_k of probability measures on \mathcal{B}_k tends to a limit measure on $\mathcal{K}_{[\omega_0]}$ and to determine the limit rate function.

Several examples were explored in some detail. Our method is very general and we believe it could be applied to and shed a new light on a wide range of problems in physics and mathematics.

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