From Smile Asymptotics to Market Risk Measures

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Abstract

The left tail of the implied volatility skew, coming from quotes on out-of-the-money put options, can be thought to reflect the market's assessment of the risk of a huge drop in stock prices. We analyze how this market information can be integrated into the theoretical framework of convex monetary measures of risk. In particular, we make use of indifference pricing by dynamic convex risk measures, which are given as solutions of backward stochastic differential equations (BSDEs), to establish a link between these two approaches to risk measurement. We derive a characterization of the implied volatility in terms of the solution of a nonlinear PDE and provide a small time-to-maturity expansion and numerical solutions. This procedure allows to choose convex risk measures in a conveniently parametrized class, distorted entropic dynamic risk measures, which we introduce here, such that the asymptotic volatility skew under indifference pricing can be matched with the market skew.

Keywords dynamic convex risk measures, volatility skew, stochastic volatility models, indifference pricing, backward stochastic differential equations

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JEL subject classification G10

1 Introduction

Risk measurement essentially conveys information about tails of distributions. However, that information is also contained in market prices of insurance securities that are contingent on a large (highly unlikely) downside, if we concede that those prices are mostly reflective of protection buyers' risk aversion. Examples are out-of-the-money put options that provide protection on large stock price drops, or senior tranches of CDOs that protect against the default risk of say 15 – 30% of investment grade US companies over a 5 year period

A central regulatory and internal requirement in recent years, in the wake of a number of financial disasters and corporate scandals, has been that firms report a measure of the risk of their financial positions. The industry-standard risk measure, value-at-risk, is widely criticized for not being convex and thereby penalizing diversification, and a number of natural problems arise:

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1. How to construct risk measures with good properties.

2. Computation of these risk measures under typical financial models.

3. Choice: what is the "right" risk measure?

The first issue has been extensively studied in the static case [ADEH99, FS02] and recent developments in extending to dynamic risk measures with good time-consistency and/or recursive properties are discussed, e.g., in [BK09, KS05, MZ09, FS11]. However, concrete examples of dynamic, time-consistent convex risk measures are scarce, and they typically have to be defined abstractly, for example via the driver of a backward stochastic differential equation (BSDE) or as the limit of discrete time-consistent risk measures [Sta10]. As a result, intuition is lost, and there is at present little understanding what the choice of driver says about the measure of risk. Or, to put it another way, how can the driver be constructed to be consistent with risk aversion reflected in the market?

Let ξ be a bounded random variable representing a financial payoff whose value is known at some future time $T < \infty$. A classical example of a convex risk measure, the entropic risk measure, is related to exponential utility:

$$\varrho(\xi) = \frac{1}{\gamma} \log\left(\mathbb{E}\left[e^{-\gamma\xi}\right]\right), \tag{1}$$

where $\gamma > 0$ is a risk-aversion coefficient. When extending to *dynamic* risk measures $\varrho_t(\cdot)$ adapted to some filtration $(\mathcal{F}_t)$, a desirable property is (strong) time-consistency

$$\varrho_s(-\varrho_t(\xi)) = \varrho_s(\xi), \qquad 0 \le s \le t \le T.$$

This flow property is important if ϱ_t is used as a basis for a pricing system. The static entropic risk measure (1) generalizes to

$$\varrho_t(\xi) = \frac{1}{\gamma} \log\left(\mathbb{E}\left[e^{-\gamma\xi} \mid \mathcal{F}_t\right]\right). \tag{2}$$

The flow property follows simply from the tower property of conditional expectations. However, finding other directly-defined examples is not easy, and to have a reasonable class of choices, we need to resort to more abstract constructions.

In a Brownian-based model, time-consistent dynamic risk measures can be built through BSDEs, as shown in [BK09, KS05], extending the work of Peng [Pen04]. That is, on a probability space with a d-dimensional Brownian motion W that generates a filtration $(\mathcal{F}_t)$, the risk measure of the $\mathcal{F}_T$-measurable random variable ξ ($\in \mathbb{R}$ for simplicity) is computed from the solution $(R_t, Z_t) \in \mathbb{R} \times \mathbb{R}^d$ of the BSDE

$$\begin{aligned} -dR_t &= g(t, Z_t)\, dt - Z_t^*\, dW_t \\ R_T &= -\xi, \end{aligned}$$

where $*$ denotes transpose. Here the driver g, which defines the risk measure, is Lipschitz and convex in z and satisfies $g(t, 0) = 0$. The solution is a process $R \in \mathbb{R}$ that matches the *terminal* condition $-\xi$ on date T (when ξ is revealed and the risk is known), and a process $Z \in \mathbb{R}^d$ that, roughly speaking, keeps the solution non-anticipating. Then $\varrho_t(\xi) := R_t$ defines a time-consistent dynamic convex risk measure. However, the possibility to offset risk by dynamically hedging in the market needs to be accounted for. Setting aside technicalities for the moment, this operation leads to a modification of the driver.

The left tail of the implied volatility skew observed in equity markets is a reflection of the premium charged for out-of-the-money put options. The bulk of the skew reveals the heavy left tail in the risk-neutral density of the stock price S_T at expiration, but the very far left tail, where investor sentiment and crash-o-phobia takes over, could be interpreted as revealing information about the representative market risk measure

and its driver g, if we assume prices are consistent with this kind of pricing mechanism. The question then is to extract constraints on the driver from the observed tails of the skew, an inverse problem.

To put our analysis into a broader framework, we observe that the underlying structural question is the inference of preference structures from observable data. The idea of using (at least in theory) observable consumption and investment streams to reveal the preference structure of a rational utility maximizing investor dates back to Samuelson in the 1940s and Black in the 1960s - for a recent overview on this "backward approach" to utility theory we refer to [CHO11]. The spirit of our presentation is a similar one, except we deal with dynamic risk measures rather than utility functions, and the observable data are not given as consumption and investment strategies but as readily available market implied volatilities.

The main goal here is to utilize short-time asymptotics for the inverse problem of using the observed skew slope to calibrate the driver g. Berestycki *et al.* [BBF04] presented short-time asymptotics for implied volatilities for *no arbitrage pricing* under stochastic volatility models. Further work in this direction includes, among others, [FFF10, FJ09, FJL10] and references therein. In Section 2, we extend this analysis to the nonlinear PDEs characterizing indifference pricing under dynamic convex risk measures.

We find (Theorem 2.12) that the zero-order term in the short-time approximation is the same as found in no-arbitrage pricing by [BBF04]. The next order term is the solution of an inhomogeneous linear transport equation that sees only a particular slope of the partially Legendre-transformed driver, but is independent of the size of the options position (see equation (25)).

Section 3 illustrates the theoretical findings by focusing on a particular class of drivers, introducing distorted entropic convex dynamic risk measures. First we develop explicit calculations for the small time expansion in the Hull-White stochastic volatility model to illustrate the impact of the distortion parameter on the implied volatility skew. Then we illustrate the results in a numerical study (via the pricing PDE) of arctangent stochastic volatility driven by an Ornstein-Uhlenbeck process. Section 4 contains the conclusions and Section 5 gives the more technical proofs omitted in the exposition.

2 Heuristics and Statement of Results

We consider a model of a financial market consisting of a risk-free bond bearing no interest and some stock following the stochastic volatility model on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$

$$\begin{cases} dS_t &= \mu(Y_t)S_t\,dt + \sigma(Y_t)S_t\,dW_t^1, & S_0 = S; \\ dY_t &= m(Y_t)\,dt + a(Y_t)(\rho\,dW_t^1 + \rho'\,dW_t^2), & Y_0 = y, \end{cases} \tag{3}$$

where W^1, W^2 are two independent Brownian motions generating $(\mathcal{F}_t)$ and $\rho' = \sqrt{1-\rho^2}$.

Assumption 2.1. *We assume that:*

i) σ, $a \in C^{1+\beta}_{loc}(\mathbb{R})$, *where* $C^{1+\beta}_{loc}(\mathbb{R})$ *is the space of differentiable functions with locally Hölder-continuous derivatives with Hölder-exponent* $\beta > 0$*;*

ii) both σ *and* a *are bounded and bounded away from zero:*

$$0 < \underline{\sigma} < \sigma < \overline{\sigma} < \infty, \quad \textit{and} \quad 0 < \underline{a} < a < \overline{a} < \infty;$$

iii) μ, $m \in C^{0+\beta}_{loc}(\mathbb{R})$, *and* $|\mu| < \overline{\mu} < \infty$.

The pricing will done via the indifference pricing mechanism for dynamic convex risk measures, which are introduced in the next subsection.

2.1 Dynamic Convex Risk Measures, Indifference Pricing and BSDEs

Definition 2.2. We call the family $\varrho_t : L^\infty(\Omega, \mathcal{F}_T, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$, $0 \le t \le T$, a convex dynamic risk measure, if it satisfies for all $t \in [0,T]$ and all $\xi, \xi^1, \xi^2 \in L^\infty(\Omega, \mathcal{F}_T, P)$ the following properties.

i) Monotonicity: $\xi^1 \ge \xi^2$ P-a.s implies $\varrho_t(\xi^1) \le \varrho_t(\xi^2)$;
ii) Cash invariance: $\varrho_t(\xi + m_t) = \varrho_t(\xi) - m_t$ for all $m_t \in L^\infty(\Omega, \mathcal{F}_t, P)$;
iii) Convexity: $\varrho_t(\alpha\xi^1 + (1-\alpha)\xi^2) \le \alpha\varrho_t(\xi^1) + (1-\alpha)\varrho_t(\xi^2)$ for all $\alpha \in [0,1]$;
iv) Time-consistency $\varrho_t(\xi^1) = \varrho_t(\xi^2)$ implies $\varrho_s(\xi^1) = \varrho_s(\xi^2)$ for all $0 \le s \le t$.

We note that if the risk measure is additionally normalized, i.e. $\varrho_t(0) = 0$ for all $t \in [0,T]$, then iv) is equivalent to the stronger property $\varrho_s(-\varrho_t(\xi)) = \varrho_s(\xi)$ for all $0 \le s \le t$ [KS05, Lemma 3.5]. The risk measure $\varrho_t(\xi)$ should be understood as the risk associated with the position ξ at time t.

If ϱ_t is normalized, this is nothing else than the minimal capital requirement at time t to make the position riskless since $\varrho_t(\xi + \varrho_t(\xi)) = 0$. In this static setting, the certainty equivalent price of a buyer of a derivative $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ at time t is just the cash amount for which buying the derivative has equal risk to not buying it.

In fact we are much more interested in the case where the buyer of the security is allowed to trade in the stock market to hedge her risk. In describing admissible strategies we follow the setting of continuous time arbitrage theory in the spirit of Delbaen-Schachermayer (for an overview, we refer to the monograph [DS06]). Denote therefore by Θ_t the set of all admissible hedging strategies from time t onwards, i.e. all progressive processes such that $\theta_t = 0$ and $\int_t^u \theta_s(\mu(Y_s)\,ds + \sigma(Y_s)\,dW_s^1)$ exists for all $u \in]t,T]$ and is uniformly bounded from below, and set

$$\mathcal{K}_t := \left\{ \int_t^T \theta_s(\mu(Y_s)\,ds + \sigma(Y_s)\,dW_s^1) \;:\; \theta \in \Theta_t \right\}.$$

The set of all superhedgeable payoffs is then given by $C_t := (\mathcal{K}_t - L^0_+) \cap L^\infty$, where L^0_+ denotes the set of all almost surely non-negative random variables.

The residual risk at time t of the derivative $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ after hedging is given by

$$\hat{\varrho}_t(\xi) := \operatorname*{ess\,inf}_{h \in C_t} \varrho_t(\xi + h). \tag{4}$$

Thus, assuming that the buyer's wealth at time t is x, her dynamic indifference price P_t, which can be viewed as the certainty equivalent after optimal hedging in the underlying market, is given via $\hat{\varrho}_t(x + \xi - P_t) = \hat{\varrho}_t(x)$, whence, using cash invariance,

$$P_t = \hat{\varrho}_t(0) - \hat{\varrho}_t(\xi). \tag{5}$$

We note, while restricting ourselves to the buyer's indifference price, all our considerations are easily adaptable to the seller's indifference price by a simple change of signs of ξ and P_t in (5).

A convenient class of dynamic convex risk measures to which we will stick throughout this paper are defined from solutions of BSDEs. Assume that $g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^2)$-measurable function which is continuous, convex, and quadratic (i.e. bounded in modulus by a quadratic function) in the $\mathbb{R}^2$-component. (In what follows, we will not explicitly denote the dependence of the driver g on $\omega \in \Omega$.) Next, let $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ be a given bounded financial position. Then the BSDE

$$R_t = -\xi + \int_t^T g(Z_s^1, Z_s^2)\,ds - \int_t^T Z_s^1\,dW_s^1 - \int_t^T Z_s^2\,dW_s^2 \tag{6}$$

admits a unique $\mathcal{F}_t$-adapted solution (R_t, Z_t^1, Z_t^2), which defines a dynamic convex risk measure via $\varrho_t(X) := R_t$ [BK09, Theorem 3.21].

The existence of a solution of the BSDE (6) in this quadratic setting was first proved by Kobylanski [Kob00, Theorem 2.3], with some corrections to their arguments given by Nutz [Nut07, Theorem 3.6], whereas the uniqueness follows from the convexity of the driver as shown in [BH08, Corollary 6]. From a financial perspective, the components Z^1, Z^2 of the "auxiliary" process Z can be interpreted as risk sources, describing the risk stemming from the traded asset and the volatility process respectively.

2.2 Transformed BSDE under Hedging

To assure the solvability of the BSDEs and PDEs that arise in our setting, we have to restrict slightly the class of admissible drivers. Throughout, subscripts of functions indicate in the PDE context partial derivatives with the respect to the respective components.

Definition 2.3. We call a $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^2)$-measurable function $g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ an admissible driver (normalized admissible driver) if it satisfies the following conditions i)-iii) (resp. o)-iii)):

o) $g(0,0) = 0 \quad P$-a.s.;
i) $g \in C^{2,1}(\mathbb{R}^2) \quad P$-a.s.;
ii) $g_{z_1 z_1}(z_1, z_2) > 0 \quad P$-a.s. for all $(z_1, z_2) \in \mathbb{R}^2$;
iii) there exist constants $c_1,\ c_2 > 0$ such that

$$c_1\Big(\tfrac{z_1^2}{4c_1^2} - (1 + z_2^2)\Big) \le g(z_1, z_2) \le c_2(1 + z_1^2 + z_2^2) \quad P\text{-a.s. for all } (z_1, z_2) \in \mathbb{R}^2.$$

The normalization of the driver (condition o)) corresponds to the normalization of the risk measure.

Remark 2.4. *To ease the presentation, we only work with drivers that do not depend explicitly on time. While any dependence on R_t would destroy the cash invariance, it is not difficult to add an additional dependence of g on time. The higher order expansions in Section 2.5 will then also depend on some partial derivatives with respect to t at $t = T$.*

In passing from the principal risk measure defined by g to the residual risk measure after hedging, as in (4), we will need the Fenchel-Legendre transform of g in its first component, namely

$$\hat{g}(\zeta, z_2) := \sup_{z_1 \in \mathbb{R}} \Big(\zeta z_1 - g(z_1, z_2)\Big), \qquad \zeta \in \mathbb{R}. \tag{7}$$

Lemma 2.5. *Given that g is a (normalized) admissible driver, then the risk-adjusted driver $\hat{g}$ defined in* (7) *is also a (normalized) admissible driver.*

Proof. To show $\hat{g}$ satisfies condition iii) of Definition 2.3, we fix z_2 and treat the function as classical Fenchel-Legendre transform in one variable. Therefore it holds for proper, continuous convex functions f, g, that $f \le g$ implies $\hat{f} \ge \hat{g}$ and $\hat{\hat{f}} = f$, [HUL01b, Proposition E.1.3.1 and Corollary E.1.3.6]. So the statement is proved by noting that

$$\sup_{z_1}\Big(\xi z_1 - c(1 + z_1^2 + z_2^2)\Big) = c\Big(\frac{z_1^2}{4c^2} - (1 + z_2^2)\Big)$$

for any positive constant c.

To show *i*) and *ii*) in Definition 2.3, we note that condition *iii*) implies that g is 1-coercive in z_1, i.e. $g(z_1, z_2)/|z_1| \to \infty$ as $z_1 \to \pm\infty$ for fixed z_2. Now we can use the fact, that the Fenchel-Legendre transform of any 1-coercive, twice differentiable function with positive second derivative is itself 1-coercive and twice differentiable with positive second derivative, cf. [HUL01a, Corollary X.4.2.10]. Thus it remains only to prove the differentiability of $\hat{g}$ with respect to z_2 which is a consequence of the differentiability properties of g: writing down the difference quotient and noting that the maximizer is differentiable, the positive second derivative with respect to the first component yields the existence of a finite limit. Finally $\hat{g}(0,0) = 0$ follows from the definition if $g(0,0) = 0$. □

In other words, the class of admissible drivers is invariant under the convex conjugation in the first component and the class of normalized admissible drivers is an invariant subclass thereof.

Remark 2.6. *We note that it is important in our setting to stick to the theory of quadratic drivers, since if g would be a Lipschitz driver, $\hat{g}$ would be no more a proper function. This fact is easily seen, since from the Lipschitz condition it follows that*

$$g(z_1, z_2) \leq L(1 + \sqrt{z_1^2 + z_2^2}) \leq \sqrt{2}L(1 + |z_1| + |z_2|)$$

for some constant L and hence

$$\begin{aligned}\hat{g}(\zeta, z_2) = \sup_{z_1}\Big(\zeta z_1 - g(z_1, z_2)\Big) &\geq \sup_{z_1}\Big(\zeta z_1 - \sqrt{2}L(1 + |z_1| + |z_2|)\Big) \\ &= \begin{cases} \infty & \text{if } |\zeta| > \sqrt{2}L \\ -\sqrt{2}L(1 + |z_2|) & \text{if } |\zeta| \leq \sqrt{2}L. \end{cases}\end{aligned}$$

From now on we will assume that g is convex as a function on $\mathbb{R}^2$ and an admissible driver. Our next step is to describe the dynamic hedging risk in terms of BSDEs. These results are in essence due to Toussaint, [Tou07, Section 4.4.1]. Since his thesis is not easily available, we will nevertheless state the proofs here. It is convenient to introduce a notation for the Sharpe ratio:

$$\lambda(y) := \frac{\mu(y)}{\sigma(y)}. \tag{8}$$

Proposition 2.7. *The risk of the financial position $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$ under hedging is $\hat{\varrho}_t(\xi) = \hat{R}_t^{(\xi)}$ where $\hat{R}_t^{(\xi)}$ is given via the unique solution of the BSDE*

$$\hat{R}_t^{(\xi)} = -\xi - \int_t^T \hat{Z}_s^1 \lambda(Y_s) + \hat{g}(-\lambda(Y_s), \hat{Z}_s^2)\, ds - \int_t^T \hat{Z}_s^1\, dW_s^1 - \int_t^T \hat{Z}_s^2\, dW_s^2. \tag{9}$$

Moreover, $\hat{\varrho}_t$ is itself a dynamic convex risk measure.

Proof. It follows from the work of Klöppel and Schweizer [KS05, Theorem 7.17] that the risk is given via the BSDE

$$\hat{R}_t^{(\xi)} = -\xi + \int_t^T \tilde{g}(Z_s^1, Z_s^2)\, ds - \int_t^T \hat{Z}_s^1\, dW_s^1 - \int_t^T \hat{Z}_s^2\, dW_s^2,$$

where $\tilde{g}$ is given by the infimal convolution

$$\tilde{g}(z^1, z^2) := \inf_{v \in \mathbb{R}}\Big(g(z^1 + \sigma(Y_t)v, z^2) + \mu(Y_t)v\Big). \tag{10}$$

To be precise, besides the differences in sign between our convex risk measures and their monetary concave utility functionals, our L^∞ framework is in line with the main part of their paper where they work in L^∞ . However, the result [KS05, Theorem 7.17] is stated in the framework of L^2-BSDEs with Lipschitz drivers. Their detour to L^2 was due to their consideration that this is the natural framework for BSDEs. We have motivated that we have to work with quadratic drivers, for which there is yet no L^2-theory, but it is straightforward (though tedious) to check that their result (10) adapts to our setting due to the regularity enforced by the admissibility conditions in Definition 2.3.

Next, we rewrite the infimum in (10) to get

$$\tilde{g}(z_1, z_2) = \inf_{u \in \mathbb{R}}(g(u, z^2) - (z^1 - u)\lambda(Y_t)) = -z^1\lambda(Y_t) - \hat{g}(-\lambda(Y_t), z^2).$$

Finally, the uniqueness of the solution of the BSDE (9) follows again from [BH08, Corollary 6] using the convexity of the driver, which is implied by the fact that $\hat{g}$ is concave in the second component. Moreover, this entails also that $\hat{\varrho}$ is a dynamic convex risk measure. □

Remark 2.8. *We remark only in passing that in view of equation* (9) *of the above proposition, the notion of admissibility could be slightly extended: it is possible to replace the lower bound in Definition 2.3, iii) by*

$$c_1(f(z_1) - (1 + z_2^2)) \leq g(z_1, z_2)$$

for an arbitrary real-valued, convex and 1*-coercive function* f*. This is enough to get existence and uniqueness of equation* (9)*, however it would clearly destroy the nice invariance property of Lemma 2.5 and we do not adopt it in the following.*

2.3 Indifference Valuation of European Claims

From the formula (5) for the indifference price P_t and Proposition 2.7, we have that

$$P_t = \hat{R}_t^{(0)} - \hat{R}_t^{(\xi)}. \tag{11}$$

From now on we will restrict ourselves to particular bounded payoffs, namely European put options with strike price K and maturity date T: $\xi = (K - S_T)^+$. Moreover, for the further treatment the substitutions

$$x := \log(S/K), \qquad \tau := T - t, \tag{12}$$

will be convenient and we introduce the following notation. Denote by L_T the layer $[0, T] \times \mathbb{R}^2$ and by $Q_{\tau_0,r}$ the open cylinder above the disk $B(m, r)$ with midpoint m, radius r and height $0 < \tau_0 < T$: $Q_{\tau_0,r} :=]0, \tau_0[\times B(m, r)$. Since the location of the midpoint (once fixed) will play no further role, we skip it in the notation.

The following theorem characterizes the indifference price of a European put with respect to the dynamic convex risk measure with driver g under the stochastic volatility model (3).

Theorem 2.9. *The buyer's indifference price of the European put option is given as*

$$P(\tau, x, y) = \tilde{u}(\tau, x, y) - u(\tau, y) \tag{13}$$

where $u \in C^{1+\beta/2,2+\beta}(Q_{T,r}) \cap C(L_T)$ *for every cylinder* $Q_{T,r}$ *is the solution of the semi-linear parabolic PDE*

$$\begin{cases} -u_\tau + Lu = & \frac{1}{K}\hat{g}(-\lambda(y), \rho' K a(y) u_y); \\ u(0, x, y) = & -(1 - e^x)^+, \end{cases} \tag{14}$$

with operator L *given by*

$$Lu = \frac{1}{2} Tr\left[\begin{pmatrix} \sigma^2(y) & \rho\sigma(y)a(y) \\ \rho\sigma(y)a(y) & a^2(y) \end{pmatrix}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\right] + \begin{pmatrix} -\frac{1}{2}\sigma^2(y) \\ m(y) \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix} - \rho a(y)\lambda(y)u_y,$$

and $\tilde{u}$ *denotes the (*x*-independent) solution of* (14)*, with altered initial condition* $\tilde{u}(0, y) = 0$*. Moreover the solution of the Cauchy problem* (14) *(as well as that with the altered initial condition) is the unique classical solution that is bounded in* L_T *together with its derivatives.*

Proof. By Ladyzhenskaya *et al.* [LSU67, Theorem V.8.1 and Remark V.8.1], there exists a unique solution $v \in C^{1+\beta/2,2+\beta}(Q_{T,r}) \cap C(L_T)$ to the semilinear parabolic PDE

$$v_t + \frac{1}{2}\sigma^2(y)S^2 v_{SS} + \frac{1}{2}a(y)^2 v_{yy} + \rho\sigma(y)a(y)S v_{Sy} + (m(y) - \rho a(y)\lambda(y))v_y = \hat{g}(-\lambda(y), \rho' a(y)v_y), \tag{15}$$

with terminal condition $v(T,S,y) = -(K-S)^+$. Applying Itô's formula to $v(t,S_t,Y_t)$ and defining

$$\begin{aligned}\bar{Z}_t^1 &:= \sigma(Y_t)S_t v_s(t,S_t,Y_t) + \rho a(Y_t)v_y(t,S_t,Y_t)\\ \bar{Z}_t^2 &:= \rho' a(Y_t)v_y(t,S_t,Y_t),\\ \bar{R}_t &:= v(t,S_t,Y_t)\end{aligned}$$

shows that $(\bar{R}_t, \bar{Z}_t^1, \bar{Z}_t^2)$ solves the BSDE (9) for $(\hat{R}_t^{(\xi)}, \hat{Z}_t^1, \hat{Z}_t^2)$ with $\xi = (K-S_T)^+$, and therefore we identify $\hat{R}_t^{(\xi)} = v(t,S_t,Y_t)$. The transformation (12), together with $u(\tau,x,y) = v(t,s,y)/K$ leads to the Cauchy problem (14) for u. Finally, taking zero terminal condition for the PDE (15), and calling the solution $\tilde{v}(t,y)$ leads to $\hat{R}_t^{(0)} = \tilde{v}(t,Y_t)$. Therefore the indifference price in (11) is given by $P_t = \tilde{v}(t,Y_t) - v(t,S_t,Y_t)$, which, in transformed notation, leads to (13). □

Here u is the value function of the holder of the put option, and $\tilde{u}$ is related to the investment (or Merton) problem with trading only in the underlying stock and money market account. The nonlinearity in the PDE (14) is in its "Greek" u_y, that is, the Vega, and enters through the Legendre transform of the driver g in its first variable. For the familiar entropic risk measure, $g(z_1,z_2) = \gamma(z_1^2+z_2^2)/2$, where $\gamma > 0$ is a risk-aversion parameter, we have $\hat{g}(\zeta,z_2) = (\zeta^2/\gamma - \gamma z_2^2)/2$. In this case, the nonlinearity is as u_y^2 (see for example [BK05, SZ05]).

Before we derive a PDE for the implied volatility, we give some a priori bounds on indifference prices and their associated implied volatilities.

Proposition 2.10. *Denote by $P^{BS}(\tau,x;\sigma)$ the Black-Scholes price of the put calculated with constant volatility σ. Then*

$$P^{BS}(\tau,x;\underline{\sigma}) \le P(\tau,x,y) \le P^{BS}(\tau,x;\overline{\sigma}) \tag{16}$$

and

$$\underline{\sigma} \le I(\tau,x,y) \le \overline{\sigma}, \tag{17}$$

where $\underline{\sigma}$ and $\overline{\sigma}$ are the volatility bounds in Assumption 2.1.

The proof is given in Section 5.

2.4 Implied Volatility PDE

Our main goal is to establish an asymptotic expansion of the indifference price implied volatility in the limit of short time-to-maturity. To do so, we now adapt the approach of Berestycki et al. [BBF04] to establish a PDE satisfied by the Black-Scholes volatility $I(\tau,x,y)$ implied by the indifference pricing. Therefore we note that in the Black-Scholes model with unit volatility, the no arbitrage pricing PDE is given by

$$\begin{cases} -U_\tau + \frac{1}{2}(U_{xx} - U_x) &= 0;\\ U(0,x) &= (1-e^x)^+, \end{cases}$$

which can be represented explicitly as

$$U(\tau,x) = \Phi\Big(-\frac{x}{\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\Big) - e^x\Phi\Big(-\frac{x}{\sqrt{\tau}} - \frac{\sqrt{\tau}}{2}\Big),$$

where Φ is the cumulative density function of the standard normal distribution. Using the scaling properties of the Black-Scholes put price, the indifference pricing implied volatility $I(\tau,x,y)$ is hence given by the equation

$$P(\tau,x,y) = U(I^2(\tau,x,y)\tau,x).$$

To derive the PDE for the implied volatility, we plug this into the equation (13) and get after some calculations that the implied volatility I is subject to the nonlinear degenerate parabolic PDE

$$\begin{aligned} &-(\tau I^2)_\tau + H(\tau,x,y,I,I_x,I_y) + \tau\mathcal{L}(y,I,I_{xx},I_{xy},I_{yy}) - 2\tau\rho a(y)\lambda(y)II_y \\ =&2\tau II_y \frac{\hat{g}(-\lambda(y),\rho' Ka(y)\tilde{u}_y) - \hat{g}(-\lambda(y),\rho' Ka(y)u_y)}{K(\tilde{u}_y - u_y)}, \end{aligned} \tag{18}$$

with

$$\begin{aligned} H(\tau,x,y,I,I_x,I_y) &= Tr\left[\begin{pmatrix}\sigma^2(y) & \rho\sigma(y)a(y)\\ \rho\sigma(y)a(y) & a^2(y)\end{pmatrix} I^2 \left(\begin{pmatrix}(\frac{x}{I})_x^2 & (\frac{x}{I})_x(\frac{x}{I})_y\\ (\frac{x}{I})_x(\frac{x}{I})_y & (\frac{x}{I})_y^2\end{pmatrix} - \frac{1}{4}\tau^2\begin{pmatrix}I_x^2 & I_xI_y\\ I_xI_y & I_y^2\end{pmatrix}\right)\right] \\ &\quad +\tau(\rho\sigma(y)a(y) + 2m(y))II_y, \end{aligned} \tag{19}$$

and

$$\mathcal{L}(y,I,I_{xx},I_{xy},I_{yy}) = I\,Tr\left[\begin{pmatrix}\sigma^2(y) & \rho\sigma(y)a(y)\\ \rho\sigma(y)a(y) & a^2(y)\end{pmatrix}\begin{pmatrix}I_{xx} & I_{xy}\\ I_{xy} & I_{yy}\end{pmatrix}\right]. \tag{20}$$

To motivate an initial condition for the Cauchy problem, we send formally τ to zero in (18). As the "Vega" $\mathcal{V} = \tilde{u}_y - u_y$ tends also to zero as $\tau \downarrow 0$ (this is shown in Lemma 5.2), we observe that the quotient on the right side of (18)

$$\frac{\hat{g}(-\lambda(y),\rho' Ka(y)\tilde{u}_y) - \hat{g}(-\lambda(y),\rho' Ka(y)u_y)}{K(\tilde{u}_y - u_y)} \to \rho' a(y)\hat{g}_{z_2}(-\lambda(y),0),$$

which is bounded by the definition of admissible driver and Lemma 2.5. Dividing by I^2, this leads to the formal limit equation

$$Tr\left[\begin{pmatrix}\sigma^2(y) & \rho\sigma(y)a(y)\\ \rho\sigma(y)a(y) & a^2(y)\end{pmatrix}\begin{pmatrix}(\frac{x}{I(0,x,y)})_x^2 & (\frac{x}{I(0,x,y)})_x(\frac{x}{I(0,x,y)})_y\\ (\frac{x}{I(0,x,y)})_x(\frac{x}{I(0,x,y)})_y & (\frac{x}{I(0,x,y)})_y^2\end{pmatrix}\right] = 1. \tag{21}$$

Remark 2.11. *Our Cauchy problem is similar to that derived for no arbitrage pricing implied volatilities in [BBF04], where they have the same equation* (18)*, but i) without the last term on the left side (which here is due to the change in measure from physical to a risk-neutral one); and ii) without the right side term (which here is due to the dynamic convex risk measure used for indifference pricing). However, our initial condition* (21)*, which does not depend on $\hat{g}$ and the drift of the stochastic volatility model, is exactly the same as theirs.*

Now we turn this heuristic argument into a precise statement.

Theorem 2.12. *The implied volatility function $I(\tau,x,y)$ generated by the indifference pricing mechanism is the unique solution $I \in C^{1+\beta/2,2+\beta}(Q_{T,r}) \cap C(L_T)$ to the following nonlinear parabolic Cauchy problem*

$$\begin{aligned} &-(\tau I^2)_\tau + H(\tau,x,y,I,I_x,I_y) + \tau\mathcal{L}(y,I,I_{xx},I_{xy},I_{yy}) - 2\tau\rho a(y)\lambda(y)II_y \\ =&2\tau II_y \frac{\hat{g}(-\lambda(y),\rho' Ka(y)\tilde{u}_y) - \hat{g}(-\lambda(y),\rho' Ka(y)u_y)}{K(\tilde{u}_y - u_y)} \end{aligned} \tag{22}$$

where H and $\mathcal{L}$ are given by (19) *and* (20)*. The initial condition is given as $I^0(x,y) = x/\psi(x,y)$ where ψ is the unique viscosity solution of the eikonal equation*

$$\begin{cases} Tr\left[\begin{pmatrix}\sigma^2(y) & \rho\sigma(y)a(y)\\ \rho\sigma(y)a(y) & a^2(y)\end{pmatrix}\begin{pmatrix}\psi_x^2 & \psi_x\psi_y\\ \psi_x\psi_y & \psi_y^2\end{pmatrix}\right] = 1; \\ \psi(0,y) = 0; \\ \psi(x,y) > 0 \quad \text{for } x > 0. \end{cases} \tag{23}$$

The proof is given in Section 5.

It is worthwhile to note that indifference prices are not linear. Indeed, buying double the amount of securities will not lead to twice the price. In this way also the volatility implied by indifference prices is quantity-dependent as a consequence of the appearance of u_y and $\tilde{u}_y$ on the right side of equation (22). However, the nonlinearity in quantity is not observed in the zeroth- and first-order small time-to-maturity approximation as we show in the following subsection. Moreover, deriving the PDE for the indifference price implied volatility for buying n put options results in the same Cauchy problem as in Theorem 2.12, where one has only to replace K by nK in every appearance in equation (22).

2.5 Small-Time Expansion

In the short time limit the implied volatility under indifference pricing is equal to the usual one as calculated by Berestycki e.a. [BBF04], as the initial conditions are the same (see Remark 2.11). The subtleties of the indifference pricing appear only away from maturity. Thus we make the *Ansatz* of an asymptotic expansion of the implied volatility:

$$I(\tau, x, y) = I^0(x, y)(1 + \tau I^1(x, y) + O(\tau^2)). \tag{24}$$

As seen above, the term I^0 is given via solution of the eikonal equation (23). To find the PDE for I^1, we plug in the expansion (24) into the equation (22) and compare the first order terms for $\tau \to 0$. This leads to the inhomogenuous linear transport equation

$$2I^1 + \frac{\psi}{2}\mathcal{H}^1(y, I^1_x, I^1_y) = F^1(y) - \rho a(y)\lambda(y)\frac{I^0_y}{I^0} - \rho' a(y)\frac{I^0_y}{I^0}\,\hat{g}_{z_2}(-\lambda(y), 0), \tag{25}$$

with

$$F^1(y) = \frac{1}{2I^0}\left(Tr\left[\begin{pmatrix} \sigma^2(y) & \rho\sigma(y)a(y) \\ \rho\sigma(y)a(y) & a^2(y) \end{pmatrix}\begin{pmatrix} I^0_{xx} & I^0_{xy} \\ I^0_{xy} & I^0_{yy} \end{pmatrix}\right] + 2(\rho\sigma(y)a(y) + 2m(y))I^0_y\right)$$

and operator

$$\mathcal{H}^1(y, I^1_x, I^1_y) = Tr\left[\begin{pmatrix} \sigma^2(y) & \rho\sigma(y)a(y) \\ \rho\sigma(y)a(y) & a^2(y) \end{pmatrix}\begin{pmatrix} 2\psi_x I^1_x & \psi_y I^1_x + \psi_x I^1_y \\ \psi_y I^1_x + \psi_x I^1_y & 2\psi_y I^1_y \end{pmatrix}\right],$$

where ψ is again the solution of the eikonal equation (23). It is important to observe that the dependence of this first order approximation on the risk measure (via its driver g) is given merely by the evaluation at $z_2 = 0$ of the derivative of its Fenchel-Legendre transform $\hat{g}$ with respect to the second component.

Comparing our PDE to the analogous equation in the arbitrage-pricing setting of [BBF04] (who, however, prescribe only the methodology in general and make explicit calculations just in one example), we note the additional presence of the last two terms of the right side of (25): here again the first one is due to the change to a risk-neutral probability measure and the second one a consequence of indifference pricing with a dynamic convex risk measure.

Furthermore we can obtain an interior boundary condition for the PDE at $x = 0$ by sending x formally to zero in (25). Imposing higher regularity on the coefficients and on $\hat{g}$ one can obtain also higher order terms in the expansion of the implied volatility. This is done by using Taylor expansions of $\hat{g}$ (in the second component) and $\tilde{u}_y$ and u_y (in τ).

3 Examples & Computations

In this Section, we introduce a family of dynamic risk measures within which to present the effect of risk aversion on implied volatilities, first using the asymptotic approximation in the Hull-White stochastic volatility model, and later using a numerical solution of the quasilinear option pricing PDE.

3.1 Distorted Entropic Dynamic Convex Risk Measures

To study the impact of the driver on the implied volatility, we will now turn to a nicely parametrized family of risk measures. Therefore we define the following class of drivers, generating distorted forms of the entropic risk measure:

$$g^{\eta,\gamma}(z_1, z_2) := \frac{\gamma}{2}(z_1^2 + z_2^2) + \eta\gamma z_1 z_2 + \frac{\eta^2\gamma}{2} z_2^2 = \frac{\gamma}{2}\Big((z_1 + \eta z_2)^2 + z_2^2\Big). \tag{26}$$

It is clear, that in the case $\eta = 0$ this is the driver connected to the classical entropic risk measure, whereas η can be seen as a parameter which describes in which way volatility risk increases also the risk coming from the tradable asset. As we will see later in Section 3.3, η effectively plays the role of a *volatility risk premium*. In the case of the usual entropic risk measure the driver describes a circle whose radius is governed by the parameter γ. In the distorted case it is now an ellipse where η determines additionally the eccentricity.

Turning to the Fenchel-Legendre transform, we have

$$\hat{g}^{\eta,\gamma}(\zeta, z_2) = \frac{1}{2\gamma}\zeta^2 - \frac{\gamma}{2} z_2^2 - \eta\zeta z_2. \tag{27}$$

Plugging this into (22), we see that the right hand side now reads

$$\tau II_y\Big(2\eta\lambda(y)a(y) - \gamma K {\rho'}^2 a(y)^2(\tilde{u}_y + u_y)\Big)$$

and we remark in particular that γ scales with K and hence also with the number of securities bought (as mentioned at the end of Section 2.4). In particular we see again that the term appearing in the first order approximation of the implied volatility, $\hat{g}^{\eta,\gamma}_{z_2}(-\lambda(y), 0) = \eta\lambda(y)$, is independent of γ.

3.2 Short-Time Asymptotics for the Hull-White Model

In the following we look at an example which is an adaption of Example 6.1/6.3 of [BBF04]. Let the stochastic volatility model be given as the Hull-White model

$$\begin{cases} dS_t &= \mu(Y_t)S_t\,dt + bY_tS_t\,dW_t^1, & S_0 = s; \\ dY_t &= \kappa Y_t\,dW_t^2, & Y_0 = y. \end{cases} \tag{28}$$

for two independent Brownian motions W^1, W^2. Obviously the model does not fall in the class considered above because the volatility $\sigma(Y) = bY$ is a geometric Brownian motion that is not bounded above or away from zero. Nevertheless, by a change of variables we will see that the results hold.

Writing down the pricing PDE in the case of the distorted entropic risk measure (26), we get

$$-u_\tau + \frac{1}{2}(b^2y^2u_{xx} + \kappa^2y^2u_{yy}) - \frac{1}{2}b^2y^2u_x = \frac{1}{2\gamma Kb^2}\frac{\mu(y)^2}{y^2} - \frac{\gamma K\kappa^2}{2}y^2u_y^2 + \frac{\eta\kappa}{b}\mu(y)u_y,$$

and one sees that by the time change $\tau \mapsto \tau y^2$ (the boundary $y = 0$ is not hit when we start with $y_0 > 0$ given that Y_t is a geometric Brownian motion) and setting $\tilde{\mu}(y) := \mu(y)/y^2$ the equation becomes

$$-u_\tau + \frac{1}{2}(b^2u_{xx} + \kappa^2u_{yy}) - \frac{1}{2}b^2u_x = \frac{1}{2\gamma Kb^2}\tilde{\mu}(y)^2 - \frac{\gamma K\kappa^2}{2}u_y^2 + \frac{\eta\kappa}{b}\tilde{\mu}(y)u_y.$$

This equation has a solution (again by [LSU67, Theorem V.8.1 and Remark V.8.1]), at least in the case that $\tilde{\mu}$ is locally β-Hölder continuous (which in turn implies that $\mu(y) = O(y^2)$ as $y \to 0$) .

In absence of global bounds on the volatility we are not able to derive results about existence and uniqueness of solutions for the PDE (22) for the implied volatility. Nevertheless we can postulate the small-time expansion to get

$$\begin{cases} b\psi_x^2 + \kappa\psi_y^2 &= 1/y^2 \\ \psi(0,y) &= 0 \\ \psi(x,y) &> 0 \quad \text{for } x > 0 \end{cases} \tag{29}$$

as the PDE characterizing the zeroth oder term and

$$\begin{cases} 0 &= 2I^1 + b^2y^2\psi\psi_x I_x^1 + \kappa^2y^2\psi\psi_y I_y^1 - \frac{1}{2}b^2y^2\frac{I_{xx}^0}{I^0} - \frac{1}{2}\kappa^2y^2\frac{I_{yy}^0}{I^0} + \eta\mu(y)\frac{I_y^0}{I^0} \\ I^1(0,y) &= \frac{\kappa^2}{12} + \eta\frac{\mu(y)}{2y} \end{cases} \tag{30}$$

for the first. As derived in [BBF04], the zeroth order term of the expansion is given via the solution of (29),

$$\psi(x,y) = \frac{1}{\kappa}\ln\Big(\frac{\kappa x}{by} + \sqrt{1 + \frac{\kappa^2x^2}{b^2y^2}}\Big).$$

as $I^0(x,y) = x/\psi(x,y)$, whereas for (30) we can guarantee only a solution in the case where $\mu(y) = O(y^3)$ since $I_y^0/I^0 = -\psi_y/\psi \sim 1/y$ as $x \to 0$. Obviously this means practically that we need an extreme drift in the Hull-White model to compensate the very volatile volatility process. However, setting e.g. $\mu(y) = \mu y^3$ for some constant μ, we are able to solve the PDE (30) explicitly by the method of characteristics to get

$$I^1(x,y) = \frac{1}{\psi^2(x,y)}\Big(\ln\Big(\frac{by}{x}\psi(x,y)\Big(1 + \frac{\kappa^2x^2}{b^2y^2}\Big)^{\frac{1}{4}}\Big) + \eta\frac{\mu x^2}{2b^2}\Big).$$

In the following graphic we rely on the parameter set

$$\mu = 6; \qquad \kappa = 7; \qquad b = 1; \qquad y_0 = 0.3; \qquad \tau = 0.1.$$

Whereas the parameter γ does not appear in the first order approximation, the distortion parameter η has a double effect. On the one hand side it shifts the smile at the money a small amount, on the other hand it changes more significantly the wing behavior of the smile, adding to the asymptotics the term $\eta\frac{\mu\kappa^2}{2b^2}\frac{x^2}{(\ln|x|)^2}$ (since $x^2/\psi^2 \sim k^2x^2/(\ln|x|)^2$ as $x \to \pm\infty$). This changes the whole wing behavior, since $I^0 \sim \kappa\frac{|x|}{\ln|x|}$ and $I^1 \to 0$ for $\eta = 0$ as $x \to \pm\infty$. Of course $\eta = 0$ corresponds to the first order term of martingale pricing as [BBF04]. Positive η (hence a positive impact of the volatility risk on the risk of the traded asset) increases the implied volatility and steepens the wings.

3.3 Numerical Study

We consider the buyer's indifference price of one European put option with respect to the family of *distorted entropic risk measures* defined by (26). We work within the stochastic volatility model (3) and, for the numerical solution, we return to the primitive variables (t, S, y). Denote by $\mathcal{L}_y$ the generator of the Markov process Y:

$$\mathcal{L}_y = \frac{1}{2}a(y)^2\frac{\partial^2}{\partial y^2} + m(y)\frac{\partial}{\partial y},$$

and by $\mathcal{L}_{S,y}$ the generator of (S, Y):

$$\mathcal{L}_{S,y} = \mathcal{L}_y + \frac{1}{2}\sigma(y)^2S^2\frac{\partial^2}{\partial S^2} + \rho a(y)\sigma(y)S\frac{\partial^2}{\partial S\,\partial y} + \mu(y)\frac{\partial}{\partial S}.$$

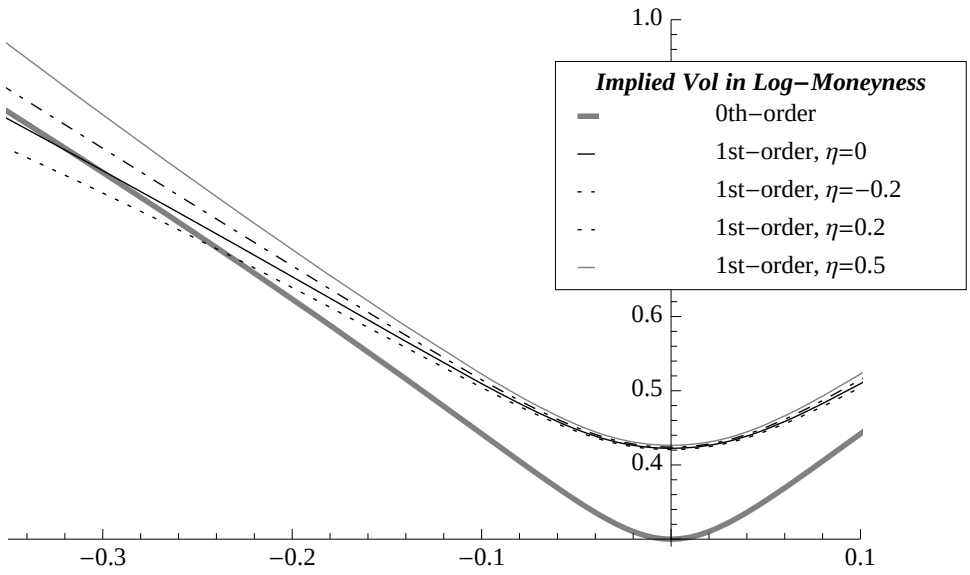

Figure 1: Implied volatility in terms of log-moneyness for the Hull-White model: zeroth and first order approximation in dependence of η.

From (11), the buyer's indifference price of a put option with strike K and expiration date T at time $t < T$ when $S_t = S$ and $Y_t = y$, is given by:

$$P(t,S,y) = \varphi(t,S,y) - \varphi_0(t,y),$$

where φ solves

$$\begin{aligned} \varphi_t + \left(\mathcal{L}_{S,y} - \rho a(y)\lambda(y)\frac{\partial}{\partial y}\right)\varphi &= -\hat{g}(-\lambda(y), -\rho' a(y)\varphi_y), \\ \varphi(T,S,y) &= (K-S)^+, \end{aligned} \tag{31}$$

and $\varphi_0(t,y)$ solves the same PDE without the S-derivatives and with zero terminal condition. Note that $\varphi = -v$, where v was the solution to the PDE problem in (15), and $\varphi_0 = -\tilde{v}$ which was introduced in the proof of Theorem 2.9.

As $\hat{g}$ is given by (27), we can re-write (31) as

$$\varphi_t + \left(\mathcal{L}_{S,y} - (\rho + \eta\rho')a(y)\lambda(y)\frac{\partial}{\partial y}\right)\varphi = -\frac{\lambda^2(y)}{2\gamma} + \frac{1}{2}(1-\rho^2)\gamma a(y)^2\varphi_y^2. \tag{32}$$

This shows that η plays the role of a *volatility risk premium* in that it enters as a drift adjustment for the volatility-driving process Y. However the nonlinearity of the PDE is through a quadratic term in φ_y, as in the case of the entropic risk measure.

Moreover, introducing the transformation

$$\varphi_0(t,y) = -\frac{1}{\gamma(1-\rho^2)}\log f(t,y),$$

leads to the *linear* PDE problem for f:

$$f_t + \left(\mathcal{L}_y - (\rho + \eta\rho')a(y)\lambda(y)\frac{\partial}{\partial y}\right) f - \frac{1}{2}\lambda^2(y)(1-\rho^2) f = 0, \tag{33}$$

$$f(T,y) = 1. \tag{34}$$

Therefore the indifference price is given by

$$P(t,S,y) = \varphi(t,S,y) + \frac{1}{\gamma(1-\rho^2)} \log f(t,y).$$

In the numerical solutions, we take the volatility-driving process (Y_t) to be an Ornstein-Uhlenbeck process with the dynamics:

$$dY_t = \alpha(m - Y_t)\,dt + \nu\sqrt{2\alpha}\,(\rho\,dW_t^1 + \rho'\,dW_t^2),$$

and we choose a function $\sigma(Y)$ that gives realistic volatility characteristics. For the OU process (Y_t), the rate of mean-reversion is α, the long-run mean-level is m and the long-run variance is ν^2. For the computations, we will take $\alpha = 5$, $m = 0$, $\nu^2 = 1$, $\rho = -0.2$ and

$$\sigma(y) = \frac{0.7}{\pi}(\arctan(y-1) + \pi/2) + 0.03,$$

so that $\sigma(m) = 0.2050$. The parameter ν measures approximately the standard deviation of volatility fluctuations. The values are chosen such that the one standard deviation confidence interval for Y is $(-1, 1)$ and this translates roughly to the confidence interval $(0.13, 0.38)$ for volatility σ. The two standard deviation interval for volatility is approximately $(0.10, 0.56)$.

We first solve the quasilinear PDE (32) for φ using implicit finite-differences on the linear part, and explicit on the nonlinear part. Then we solve the linear PDE problem (33) for f. We do this for fixed current stock price $S_0 = 100$ and $\sigma(Y_0) = 0.223$. and for various strikes $K \in [70, 110]$ and for a range of distortion parameters η and risk-aversions $\gamma > 0$. The put option is three months from expiration. Figure 2 reveals a more complex picture regarding the effect of η away from the short maturity asymptotic approximation. We see, as in Figure 1 from the asymptotics, increasing η increases the skew slope; however it also shifts down the levels of implied volatility around the money (as opposed to the opposite effect we saw in Figure 1).

Figure 3 shows, as we would expect, that increasing risk aversion γ decreases the implied volatility skew which comes from the indifference price of the buyer who is willing to pay less for the risk of the option position. It also has a secondary effect of flattening the skew out of the money.

4 Conclusion

We have derived a nonlinear PDE for the implied volatility from indifference pricing with respect to dynamic convex risk measures defined by BSDEs under diffusion stochastic volatility models. Our asymptotic analysis has highlighted the principal effect of the risk measure on option implied volatility at short maturities, namely through the appearance of $\hat{g}_{z_2}$ in the first order correction solving (25).

In the example of Section 3.2, this translates explicitly to a steepening effect on the implied volatility smile from the distortion parameter η. Numerical computations confirm this away from short maturity too, as well as quantifying the effect of risk aversion on the level of implied volatilities.

In principle, the analysis could be used to infer some information about the driver, for example η and γ in the family (26), from market implied volatilities; calibration is beyond the scope of the current paper.

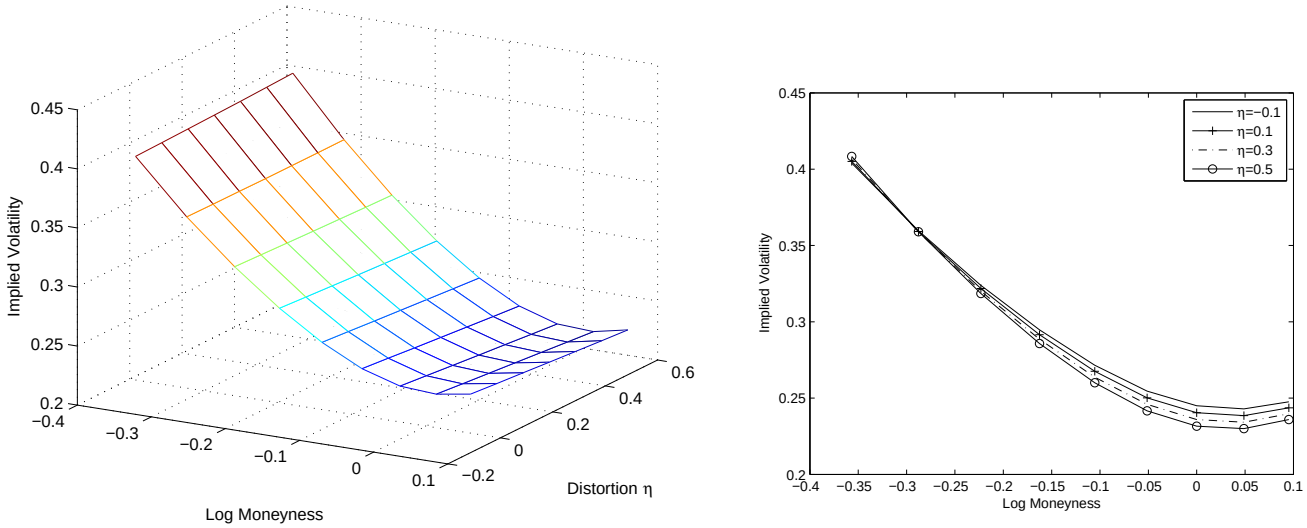

Figure 2: Implied volatility from the arctangent stochastic volatility model in terms of log moneyness $\log(K/S_0)$ and η with fixed $\gamma = 0.5$.

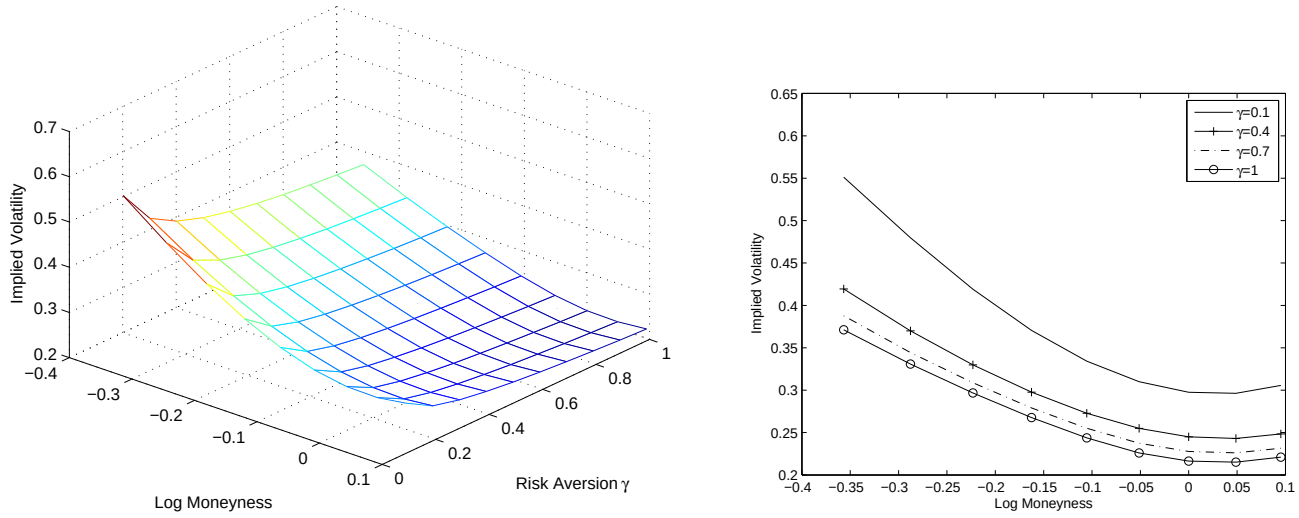

Figure 3: Implied volatility from the arctangent stochastic volatility model in terms of log moneyness $\log(K/S_0)$ and γ with fixed $\eta = 0.2$.

5 Proofs

Price- and Volatility Bounds

Lemma 5.1 (First Price Comparison). *Suppose that* $\underline{P} = \tilde{\underline{u}} - \underline{u}$ *and* $\overline{P} = \tilde{\overline{u}} - \overline{u}$ *for some* $\underline{u}$, $\tilde{\underline{u}}$, $\overline{u}$, $\tilde{\overline{u}} \in C^{1,2,}(Q_{\tau_0,r}) \cap C(\overline{Q_{\tau_0,r}})$, $0 < \tau_0 < T$, *satisfy*

$$\overline{P}_\tau \geq \tilde{L}\overline{P} - \frac{1}{K}\Big(\hat{g}(-\lambda(y), \rho' Ka(y)\tilde{\overline{u}}_y) - \hat{g}(-\lambda(y), \rho' Ka(y)\overline{u}_y)\Big), \quad \text{in } Q_{\tau_0,r} \tag{35}$$

$$\underline{P}_\tau \leq \tilde{L}\underline{P} - \frac{1}{K}\Big(\hat{g}(-\lambda(y), \rho' Ka(y)\tilde{\underline{u}}_y) - \hat{g}(-\lambda(y), \rho' Ka(y)\underline{u}_y)\Big), \quad \text{in } Q_{\tau_0,r} \tag{36}$$

as well as

$$\overline{P} \geq \underline{P} \quad \text{on} \quad \{0\} \times \mathbb{R}^2 \cap \overline{Q_{\tau_0,r}} \quad \text{and} \quad]0, \tau_0] \times \mathbb{R}^2 \cap \partial Q_{\tau_0,r}. \tag{37}$$

Then $\overline{P} \geq \underline{P}$ *on* $Q_{\tau_0,r}$.

Proof. Even if the form of this comparison principle for sub- and superprices seems to be quite unusual, the proof follows directly along the lines of Friedman [Fri64, Theorem 2.16] since the functions $\hat{g}$ contains no second derivatives. To be precise: this argument leads a version where the inequalities in (35), (37) and the conclusion are strict. But setting $\overline{P}^\varepsilon = \overline{P} + \varepsilon(1+\tau)$ one gets a strict superprice and sending ε to zero yields the stated version. □

Proof of Proposition 2.10:

Proof. To prove (16), we intend to invoke the above comparison principle for the price process given in Theorem 2.9 since it is clear that the Black-Scholes pricing functions are sub- resp. supersolutions of the PDE. Unfortunately we have the indifference price only as solution of a Dirichlet problem which does not give rise to directly comparable lateral boundary conditions. thus we have to alter the argument a bit.

Denote for $N \in \mathbb{N}$ by $u^{N,\underline{\sigma}}$ the solution of the initial/boundary-value problems

$$\begin{cases} -u^{N,\underline{\sigma}}_\tau + \tilde{L}u^{N,\underline{\sigma}} = & \frac{1}{K}\hat{g}(-\lambda(y), \rho' Ka(y)u^{N,\underline{\sigma}}_y); \\ u^{N,\underline{\sigma}}(0,x,y) = & -(1-e^x)^+; \\ u^{N,\underline{\sigma}}(\tau,x,y)\big|_{\partial B(0,N)} = & -P^{BS}(\tau,x;\underline{\sigma}). \end{cases}$$

By a classical argument [LSU67, Section V.§8], we can extract a subsequence $u^{N_k,\underline{\sigma}}$ of $u^{N,\underline{\sigma}}$ such that $u^{N_k,\underline{\sigma}}$ converges together with its derivatives to u and it's derivatives pointwise in L_T. The same is true for $\tilde{u}^N$ given by

$$\begin{cases} -\tilde{u}^N_\tau + \tilde{L}\tilde{u}^N = & \frac{1}{K}\hat{g}(-\lambda(y), \rho' Ka(y)\tilde{u}^N_y); \\ \tilde{u}^N(0,x,y) = & 0; \\ \tilde{u}^N(\tau,x,y)\big|_{\partial B(0,N)} = & 0. \end{cases}$$

Thus $P^{N,\underline{\sigma}}(\tau,x,y) = u^{N,\underline{\sigma}}(\tau,x,y) - \tilde{u}(\tau,x,y)$ satisfies

$$\begin{cases} -P^{N,\underline{\sigma}}_\tau + \tilde{L}P^{N,\underline{\sigma}} = & \frac{1}{K}\hat{g}(-\lambda(y), \rho' Ka(y)\tilde{u}^N_y) - \frac{1}{K}\hat{g}(-\lambda(y), \rho' Ka(y)u^{N,\underline{\sigma}}_y); \\ P^{N,\underline{\sigma}}(0,x,y) = & (1-e^x)^+; \\ P^{N,\underline{\sigma}}(\tau,x,y)\big|_{\partial B(0,N)} = & P^{BS}(\tau,x;\underline{\sigma}). \end{cases} \tag{38}$$

and $P^{N,\underline{\sigma}} \to P$ along a subsequence.

Noting that $P^{BS}(\tau,x;\underline{\sigma})$ is a subprice on every cylinder $Q_{T,N}$ by writing it in the odd form $P^{BS}(\tau,x;\underline{\sigma}) = 0 - (-P^{BS}(\tau,x;\underline{\sigma}))$ to satisfy the comparison principle of Lemma 5.1, we have $P^{BS}(\tau,x;\underline{\sigma}) \leq P^{N,\underline{\sigma}}$ on $Q_{T,N}$

and hence in the limit $P^{BS}(\tau, x; \underline{\sigma}) \leq P$. The other direction of Theorem 2.9 is proved, of course, by the same argument using $P^{BS}(\tau, x; \overline{\sigma})$ as superprice.

Finally a reformulation of the achieved result reads

$$U(\underline{\sigma}^2\tau, x) \leq U(I(\tau, x, y)^2\tau, x) \leq U(\overline{\sigma}^2\tau, x)$$

and so the monotonicity of U yields (17). □

Vanishing Vega

Lemma 5.2. *It holds that $\mathcal{V} = \tilde{u}_y - u_y \to 0$ uniformly on compacts as $\tau \to 0$.*

Proof. Choose the cylinder $Q_{T,r}$ such that the compact set is contained. Thus $u, \tilde{u} \in C^{1+\beta/2,2+\beta}(Q_{T,r}) \cap C(L_T)$ implies that u_y and $\tilde{u}_y$ are $\beta/2$-Hölder continuous with some Hölder constant c, whence

$$\begin{aligned}|\mathcal{V}(\tau, x, y)| &= |\tilde{u}_y(\tau, x, y) - u_y(\tau, x, y)| \\ &\leq |\tilde{u}_y(\tau, x, y) - \tilde{u}_y(0, x, y)| + |u_y(\tau, x, y) - u_y(0, x, y)| \leq 2c\tau^{\frac{\beta}{2}} \to 0\end{aligned}$$

as $\tau \to 0$ since $\tilde{u}_y(0, x, y)$ and $u_y(0, x, y)$ exist and are equal to zero by the definition of the initial conditions. □

Implied Volatility - Proof of the Main Theorem

Lemma 5.3 (Second Price Comparison). *Recall that u is the solution of the Cauchy problem* (14) *and ũ of the same problem with initial condition equal to zero. Suppose that $\overline{P}$, $\underline{P} \in C^{1,2}(Q_{\tau_0,r}) \cap C(\overline{Q_{\tau_0,r}})$, $0 < \tau_0 < T$, satisfy*

$$\overline{P}_\tau \geq \tilde{L}\overline{P} - \frac{1}{K}\Big(\hat{g}(-\lambda(y), \rho' Ka(y)\tilde{u_y}) - \hat{g}(-\lambda(y), \rho' Ka(y)u_y)\Big), \quad \text{in } Q_{\tau_0,r} \tag{39}$$

$$\underline{P}_\tau \leq \tilde{L}\underline{P} - \frac{1}{K}\Big(\hat{g}(-\lambda(y), \rho' Ka(y)\tilde{u_y}) - \hat{g}(-\lambda(y), \rho' Ka(y)u_y)\Big), \quad \text{in } Q_{\tau_0,r} \tag{40}$$

as well as

$$\overline{P} \geq P \geq \underline{P} \quad \text{on} \quad]0, \tau_0] \times \mathbb{R}^2 \cap \partial Q_{\tau_0,r}, \tag{41}$$

and

$$\overline{P}(0, x, y) = \underline{P}(0, x, y) = P(0, x, y) = (1 - e^x)^+$$

then $\overline{P} \geq P \geq \underline{P}$ on $Q_{\tau_0,r}$.

Proof. We note that inequality (39) implies $(\overline{P} - P)_\tau \geq \tilde{L}(\overline{P} - P)$ which implies together with the lateral bound $(\overline{P} - P) \geq 0$ on $]0, \tau_0] \times \mathbb{R}^2 \cap \partial Q_{\tau_0,r}$ and the initial condition $\overline{P}(0, x, y) - P(0, x, y) = 0$ that $\overline{P} \geq P$ on $Q_{\tau_0,r}$ by the classical comparison principle. The second inequality is proved in the same way. □

Lemma 5.4 (Volatility Comparison). *Suppose that $\overline{I}$, $\underline{I} \in C^{1,2}(Q_{\tau_0,r}) \cap C(\overline{Q_{\tau_0,r}})$, $0 < \tau_0 < T$, satisfy*

$$\begin{aligned}(\tau\underline{I}^2)_\tau \leq& H(\tau, x, y, \underline{I}_x, \underline{I}_y) + \tau\mathcal{L}(y, \underline{I}_{xx}, \underline{I}_{xy}, \underline{I}_{yy}) \\ &- 2\tau\underline{I}\underline{I}_y\left(\rho a(y)\lambda(y) + \frac{\hat{g}(-\lambda(y), \rho' Ka(y)\tilde{u}_y) - \hat{g}(-\lambda(y), \rho' Ka(y)u_y)}{K(\tilde{u}_y - u_y)}\right)\end{aligned}$$

resp.

$$\begin{aligned}(\tau\overline{I}^2)_\tau \geq & H(\tau,x,y,\overline{I}_x,\overline{I}_y)+\tau\mathcal{L}(y,\overline{I}_{xx},\overline{I}_{xy},\overline{I}_{yy})\\ & -2\tau\overline{I}\overline{I}_y\left(\rho a(y)\lambda(y)+\frac{\hat{g}(-\lambda(y),\rho' Ka(y)\tilde{u}_y)-\hat{g}(-\lambda(y),\rho' Ka(y)u_y)}{K(\tilde{u}_y-u_y)}\right)\end{aligned}$$

in $Q_{\tau_0,r}$ together with the lateral comparison

$$\underline{I}(\tau,x,y)\leq I(\tau,x,y)\leq\overline{I}(\tau,x,y)\quad on\quad]0,\tau_0]\times\mathbb{R}^2\cap\partial Q_{\tau_0,r}$$

and the initial growth condition

$$\lim_{\tau\to0}\tau\underline{I}^2(\tau,x,y)=\lim_{\tau\to0}\tau\overline{I}^2(\tau,x,y)=0\quad on\quad \{0\}\times\mathbb{R}^2\cap\overline{Q_{\tau_0,r}}. \tag{42}$$

Then it holds that

$$\underline{I}(\tau,x,y)\leq I(\tau,x,y)\leq\overline{I}(\tau,x,y)\quad in\quad Q_{\tau_0,r}.$$

Proof. Define first $\overline{P}(\tau,x,y):=U(\overline{I}^2(\tau,x,y)\tau,x)$ and $\underline{P}(\tau,x,y):=U(\underline{I}^2(\tau,x,y)\tau,x)$. Then by the same calculation as in the derivation of the PDE (22) of the implied volatility we get by Lemma 5.3

$$\begin{aligned}\overline{P}_\tau &\geq \tilde{L}\overline{P}-\frac{1}{K}\left(\hat{g}(-\lambda(y),\rho' Ka(y)\tilde{u_y})-\hat{g}(-\lambda(y),\rho' Ka(y)u_y)\right),\quad \text{in } Q_{\tau_0,r}\\ \underline{P}_\tau &\leq \tilde{L}\underline{P}-\frac{1}{K}\left(\hat{g}(-\lambda(y),\rho' Ka(y)\tilde{u_y})-\hat{g}(-\lambda(y),\rho' Ka(y)u_y)\right),\quad \text{in } Q_{\tau_0,r}\end{aligned}$$

as well as the lateral boundary condition

$$\underline{P}(\tau,x,y)\leq P(\tau,x,y)\leq\overline{P}(\tau,x,y)\quad \text{in}\quad Q_{\tau_0,r}.$$

Moreover, the growth condition (42) implies by the continuity of U that $\overline{P}(0,x,y)=\underline{P}(0,x,y)=P(0,x,y)=(1-e^x)^+$. Thus we can use Lemma 5.3 to infer $\underline{P}(\tau,x,y)\leq P(\tau,x,y)\leq\overline{P}(\tau,x,y)$ in $Q_{\tau_0,r}$ and the monotonicity of the function U in the first component yields the result. □

Proof of Theorem 2.12:

Proof. Remark first that if there exists a solution to the PDE with some fixed initial condition, it has to be unique by the smoothness and monotonicity of U (and the boundedness of I) since otherwise the solution of the pricing PDE (Theorem 2.9) would not be unique. By the same reasoning we get also $I\in C^{1+\beta/2,2+\beta}(Q_{T,r})\cap C(L_T)$. Moreover, the eikonal equation (23) has a unique viscosity solution as proved in [BBF04, Section 3.2]. To prove the theorem we will hence show that the solution of the eikonal equation is the only possible initial condition, i.e. that any solution of the PDE (22) has the eikonal equation as it's small time limit. More precisely we will show that there exist parametrized families of (time-independent) local super- and subsolutions of (22) which converge locally uniformly to the eikonal equation. This is done quite similar as in [BBF04, Section 3.4.], using an adapted vanishing viscosity method. However, in our setting the bounds on the volatility σ enable us to simplify the proof and circumvent some obscurities in the local volatility argument in [BBF04].

Define $\overline{I}^{\varepsilon,\delta}(x,y)$ for $\varepsilon,\delta>0$ as the solution of

$$\begin{cases} -\delta = & -(\overline{I}^{\varepsilon,\delta})^2+H(0,x,y,\overline{I}^{\varepsilon,\delta},\overline{I}^{\varepsilon,\delta}_x,\overline{I}^{\varepsilon,\delta}_y)+\varepsilon\Delta(\ln(\overline{I}^{\varepsilon,\delta}));\\ \overline{I}^{\varepsilon,\delta}\big|_{\partial B(m,r)} = & \overline{\sigma}.\end{cases} \tag{43}$$

where $B(m,r)$ is an arbitrary disk. We will show that for r, δ, $\varepsilon > 0$ there exists a solution to this equation and for fixed r and δ there exist $\varepsilon_0 > 0$, $\tau_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ this is a supersolution of (22) in $Q_{\tau_0,r}$. Moreover, we show that $\overline{I}^{\varepsilon,\delta} \to I^0$ locally uniformly as we send fist ε and then δ to zero.

First step: Making the change of variables $w := \ln(\overline{I}^{\varepsilon,\delta})$ one gets

$$\begin{aligned} H(0,x,y,\overline{I}_x^{\varepsilon,\delta},\overline{I}_y^{\varepsilon,\delta}) &= Tr\left[\begin{pmatrix}\sigma^2(y) & \rho\sigma(y)a(y)\\ \rho\sigma(y)a(y) & a^2(y)\end{pmatrix} I^2 \begin{pmatrix}(\frac{x}{I})_x^2 & (\frac{x}{I})_x(\frac{x}{I})_y\\ (\frac{x}{I})_x(\frac{x}{I})_y & (\frac{x}{I})_y^2\end{pmatrix}\right] \\ &= Tr\left[\begin{pmatrix}\sigma^2(y) & \rho\sigma(y)a(y)\\ \rho\sigma(y)a(y) & a^2(y)\end{pmatrix}\begin{pmatrix}(1-xw_x)^2 & -x(1-w_x)w_y\\ -x(1-w_x)w_y & x^2w_y^2\end{pmatrix}\right] \\ &=: \tilde{H}(x,y,w_x,w_y) \end{aligned}$$

and the equation (43) becomes

$$\begin{cases} -\delta = & -e^{2w} + \tilde{H}(x,y,w_x,w_y) + \varepsilon\Delta w; \\ w|_{\partial B(m,r)} = & \overline{\sigma}. \end{cases}$$

which admits a solution $w \in C^{2+\beta}(B(m,r))$ [LU68, Theorem 4.8.3] which is unique for sufficiently small r [LU68, Theorem 4.2.1].

Second Step: By the Hölder property of the derivatives of w (as well as that of u_y and $\tilde{u}_y$ and the differentiability of $\hat{g}$) we can conclude that there exists constants c_1-c_5 solely depending on r such that

$$\begin{aligned} |H(\tau,x,y,\overline{I}^{\varepsilon,\delta},\overline{I}_x^{\varepsilon,\delta},\overline{I}_y^{\varepsilon,\delta}) - H(0,x,y,\overline{I}^{\varepsilon,\delta},\overline{I}_x^{\varepsilon,\delta},\overline{I}_y^{\varepsilon,\delta})| &\le c_1(r)\tau + c_2(r)\tau^2 \\ \tau\mathcal{L}(y,\overline{I}^{\varepsilon,\delta},\overline{I}_{xx}^{\varepsilon,\delta},\overline{I}_{xy}^{\varepsilon,\delta},\overline{I}_{yy}^{\varepsilon,\delta}) &\le c_3(r)\tau \\ -2\tau\overline{II}_y\left(\rho a(y)\lambda(y) + \frac{\hat{g}(-\lambda(y),\rho' Ka(y)\tilde{u}_y) - \hat{g}(-\lambda(y),\rho' Ka(y)u_y)}{K(\tilde{u}_y - u_y)}\right) &\le c_4(r)\tau \\ -\varepsilon\Delta(\ln(\overline{I}^{\varepsilon,\delta})) &\le \varepsilon c_5(r) \end{aligned}$$

in $B(m,r)$. We can conclude that

$$\begin{aligned} ((\tau(\overline{I}^{\varepsilon,\delta})^2)_\tau =& (\overline{I}^{\varepsilon,\delta})^2 = \delta + H(0,x,y,\overline{I}^{\varepsilon,\delta},\overline{I}_x^{\varepsilon,\delta},\overline{I}_y^{\varepsilon,\delta}) + \varepsilon\Delta(\ln(\overline{I}^{\varepsilon,\delta})) \\ \ge& H(\tau,x,y,\overline{I}^{\varepsilon,\delta},\overline{I}_x^{\varepsilon,\delta},\overline{I}_y^{\varepsilon,\delta}) + \tau\mathcal{L}(y,\overline{I}^{\varepsilon,\delta},\overline{I}_{xx}^{\varepsilon,\delta},\overline{I}_{xy}^{\varepsilon,\delta},\overline{I}_{yy}^{\varepsilon,\delta}) \\ &- 2\tau\overline{II}_y\left(\rho a(y)\lambda(y) + \frac{\hat{g}(-\lambda(y),\rho' Ka(y)\tilde{u}_y) - \hat{g}(-\lambda(y),\rho' Ka(y)u_y)}{K(\tilde{u}_y - u_y)}\right) \\ &+ \delta - \left(c_1(r)\tau + c_2(r)\tau^2 + c_3(r)\tau + c_4(r)\tau + \varepsilon c_5(r)\right), \end{aligned}$$

for ε, τ small enough - thus for given $\delta > 0$ and $r > 0$ we can find indeed positive bounds on ε_0, τ_0 such that $\overline{I}^{\varepsilon,\delta}$ is a supersolution of (22) for $0 < \varepsilon \le \varepsilon_0$ in $Q_{\tau_0,r}$. In the same way one proves that we get for $\overline{I}^{\varepsilon,\delta}$ of

$$\begin{cases} \delta = & -(\underline{I}^{\varepsilon,\delta})^2 + H(0,x,y,\underline{I}^{\varepsilon,\delta},\underline{I}_x^{\varepsilon,\delta},\underline{I}_y^{\varepsilon,\delta}) + \varepsilon\Delta(\ln(\underline{I}^{\varepsilon,\delta})) - (\underline{I}^{\varepsilon,\delta})^2; \\ \overline{I}^{\varepsilon,\delta}\big|_{\partial B(m,r)} = & \underline{\sigma}. \end{cases}$$

subsolutions.

Third Step: Having now super- and subsolutions, we can invoke now the comparison principle Lemma 5.4 to conclude that

$$\underline{I}^{\varepsilon,\delta}(x,y) \le I(\tau,x,y) \le \overline{I}^{\varepsilon,\delta}(x,y) \quad \text{in} \quad Q_{\tau_0,r}$$

for all $0 < \varepsilon < \varepsilon_0$, ε_0 and τ_0 chosen as above. Thus we have

$$\underline{I}^{\varepsilon,\delta}(x,y) \leq \liminf_{\tau\to 0} I(\tau,x,y) \leq \limsup_{\tau\to 0} I(\tau,x,y) \leq \overline{I}^{\varepsilon,\delta}(x,y).$$

Next we want to send ε to zero. Therefore we note that the families of solutions $\overline{I}^{\varepsilon,\delta}$, $\underline{I}^{\varepsilon,\delta}$ are bounded and equicontinuous in ε (since Hölder-continuous with the same Hölder constants). Thus by the Arzelà-Ascoli theorem $\overline{I}^{\varepsilon,\delta}$ converges along a subsequence uniformly on compacts to some limit function $\overline{I}^{\delta} \in C^{0+\beta}(Q_{\tau_0,r})$. This function is the unique viscosity solution of the PDE

$$\begin{cases} -\delta = & -(\overline{I}^{\delta})^2 + H(0,x,y,\overline{I}^{\delta},\overline{I}^{\delta}_x,\overline{I}^{\delta}_y); \\ \overline{I}^{\varepsilon,\delta}\big|_{\partial B(m,r)} = & \overline{\sigma}, \end{cases}$$

compare [Eva98, Section 10.1]. An analogous result holds true for the subsolutions. now sending $\delta \to 0$, this gives by the same argument a solution of the PDE

$$(I^0)^2 = H(0,x,y,I^0,I^0_x,I^0_y)$$

which satisfies $I(0,y) = 0$. Thus for $\tau \to 0$, $I(\tau,x,y)$ converges locally uniformly to I^0 which is nothing else then the unique viscosity solution of the eikonal equation (23) with $\psi = x/I^0$. □

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