AGT conjecture and AFLT states: a complete construction

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Abstract

A complete construction of the AFLT states is proposed. With this construction and for all the cases we have checked, the AGT conjecture on the equivalence of Nekrasov Instanton Counting (NIC) to the $Vir \oplus u(1)$ conformal block has been verified to be true.

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1 Introduction

Conformal blocks, which are defined on the (punctured) Riemann surfaces, holomorphic in each z_i coordinate except when they meet each other, play an essential role in building correlation functions in two dimensional (Euclidean) conformal field theories[1]. They can be best understood as sewing together chiral vertex operators[2–4], which by definition, are not local objects, but the correlation functions are. The later combine both holomorphic and anti-holomorphic conformal blocks in a consistent way to make modular covariant objects. On the sphere, the n-point conformal block is represented graphically as in fig.1, where h_i is the conformal dimension of the primary field inserted at coordinate z_i, and $\tilde{h}_i$ labels the contribution arising from the conformal family descending from a primary field with the conformal dimension $\tilde{h}_i$. The global conformal invariance is $SL(2)\times SL(2)$, which may be used to fix three coordinates $z_1 = 0$, $z_{n-1} = 1$ and $z_n = \infty$. So the independent variables are z_i, $i = 2, ..., n-2$, with the degrees of freedom $n-3$ for the n-point conformal blocks on the sphere.

The calculation of conformal blocks is based on the conformal Ward-identities,

$$[L_n, V_h(z)] = (z^{n+1}\partial_z + (n+1)hz^n)V_h(z).$$

and carried out perturbatively level by level [1, 5, 6]. In some special cases, the decoupling of the Virasoro null vectors can be implemented as differential equations for the conformal blocks. For the general case, recursion relations have been proposed by Zamolodchikov[5, 6] on the meromorphic structures of the conformal blocks either in complex c-plane or h-plane. However, in general, the global perspective of the sewing procedure for the conformal blocks was still not fully understood until recently when the AGT duality [13] had been proposed.

AGT conjecture relates 2d Liouville conformal field theories to 4d $N = 2$ supersymmetric gauge theories of the A_1 type. The main idea is coupling to the Liouville field a $u(1)$ field[4], then this system is dual to a $U(2) = SU(2)\times U(1)$ superconformal 4d theory. In this case, the partition function by Nekrasov instanton counting(NIC)[15, 16] of the 4d $U(2)$ theory is to be identified with the conformal blocks of the $u(1)$ coupled Liouville type. The Liouville CFT is characterized by a 2d one boson theory with center charge $c \geq 25$. Finally, one can decouple the $U(1)$ factor and obtain the instanton partition function of the $SU(2)$ theory which duals to Liouville conformal blocks. Liouville interaction breaks down the charge conservation explicitly and leads to the introduction of the screening charges. Because of the existence of the screening charges, the conformal blocks of the Liouville type is much more complicated than its counterpart of the $u(1)$ free boson theory. However, the AGT conjecture, if proven true, means that there exists an orthogonal basis upon which the $Liouville \times u(1)$ conformal blocks are built. From the above reasoning, there exists a tree-like structure which describes the duality in coupling space of the $N = 2$ 4d superconformal linear quiver gauge theory. The primary objects for this tree-like structure is the "bifundmental" matter coupling, which, if translated correctly, should be represented by the inner products of the bra and ket descendant fields in 2d conformal families sandwiched by a "primary" vertex operator at position, say, z. Such kind of pants-like diagram can be sewed together to form a linear quiver diagram, which, on the 2d CFT side, is just the n-point functions on the sphere for our consideration. Of course, in the present context, we mean the $Vir \oplus u(1)$ 2d CFT.

At first sight, it seems that such duality does not bring in any conveniences. However, the Nekrasov instanton counting on the 4d field theory shows a rather compact form for the summands which are completely factorized in "momentum" P. And the summation is well organized into the combinatorial enumeration of the Young tableuax. This simple structure implies Liouville theory, in particular, the

[4]In fact, the zero mode of the $u(1)$ field is a gauge symmetry and can be fixed to any desired value.

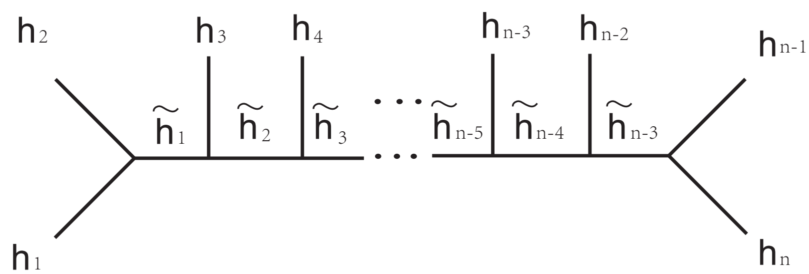

Figure 1: n-point conformal block on S^2

evaluation of the Liouville conformal blocks, could be resolved by embedding it into a bigger system. So one may expect a new construction for the Liouville conformal blocks from the corresponding NIC.

As pointed out by Nakajima[23–25], the instanton counting for $N = 2$ gauge theory is equivalent to the Hilbert scheme of points on the corresponding Seiberg-Witten curve (blow-up Riemann surface)[11, 12, 21]. This can be translated into a topological string description from physicists' point of view. By invoking the D4-D0 brane setup[19, 20] for ADHM construction[18] of the instanton moduli space and the resolving process for ALE singularities[22], these indicate that the instanton counting is a counting for D0 branes in a toric Calabi-Yau 3-fold. Actually, there are two kinds of D0 branes in the Calabi-Yau 3-fold, one is the regular D0 brane, which is in regular representation of Γ, the center of the corresponding ADE group. It carries no flux and can move freely on the Riemann surface. The other is the fractional D0 brane, which is a D2 brane wrapping on a zero-sized two sphere. It is always attached to the ALE singularity since it has a nontrivial monodromy while moving around the singularity. It is these fractional D0 branes that resolve the ALE singularity, and leave fluxes on the blow-up Riemann surface. This property ensures that one can identify these fractional instantons as "anyons" on the Riemann surface. On the other hand, the regular ones are "electric charged" particles on the Riemann surface. So the total counting is equvalent to solving the problem of "electron gas" system with insertions of anyons at the blow-up singularity on the Riemann surface. This point of view is partialy included in Dijkgraaf and Vafa's article[17]. For each pants of the pants decomposition for the (punctured) Riemann surface, one can guess that the instanton partition function can be rewritten as summation over all the intermediate states passing through the sewn holes[4]. For the interests of the present paper, we concern ourselves only with the special pants diagram that one of the tubes is replaced by the blow-up singularity. Then the summand in the instanton partition function represents itself as an inner product of the bra and ket states, sandwiched by the anyonic vertex operator. These bra and ket states should come from the interacting "electronic"[5] particles. A candidate description of the "electronic gas" system is the integrable system of multiple Calogero-Sutherland model, each living on a cycle. The whole (punctured) Riemann surface, can be obtained by sewing together these pants on nonintersecting cycles.

There are many efforts on relating the conformal blocks to the NIC[14, 26–35] from various points of views, and these works confirm the validity of the AGT duality. However, the explicit construction for the Liouville conformal blocks has remained largely unclear until the recent work [7] by Alba, Fateev, Litvinov and Tarnopolsky. In [7], they have put forward the AGT duality in a more explicit

[5]For each simple root of an ADE group, one should introduce a kind of "electronic" field.

form

$$\frac{{}_{\vec{Y}'}\langle P'|V_\alpha|P\rangle_{\vec{Y}}}{\langle P'|V_\alpha|P\rangle} \;=\; Z_{bif}(\alpha|P',\vec{Y}';P,\vec{Y})\ , \tag{1}$$

here specifically for a free field realization,

$$V_\alpha(z) = e^{2i(Q-\alpha)\tilde{\varphi}_-(z)} e^{-2i\alpha\tilde{\varphi}_+(z)} S^n : e^{2i\alpha\varphi(z)} : ,$$

with $P + P' + \alpha + nb = 0$, and $S = \oint e^{2ib\varphi(z)}\mathrm{dz}$ is the screening charge in the Virasoro sector. The l.h.s. of eq(1) is the pants-like (with one of the tubes labeled by α shrinks to a line) conformal block. The r.h.s. of (1) reproduces Z_{bif} for the instanton counting, which is given by

$$Z_{bif}(\alpha|P',\vec{Y}';P,\vec{Y}) = \prod_{i,j=1}^{2}\prod_{s\in Y_i}\Big(Q - E_{Y_i,Y'_j}(P_i - P'_j|s) - \alpha\Big)\prod_{t\in Y'_j}\Big(E_{Y'_j,Y_i}(P'_j - P_i|t) - \alpha\Big)\ , \tag{2}$$

where $\vec{P} = (P, -P)$, $\vec{P'} = (P', -P)$ and

$$E_{Y,Y'}(P|s) \equiv P + b^{-1}(a_Y(s) + 1) - b l_{Y'}(s)\,. \tag{3}$$

Here $a_Y(s)$ and $l_Y(s)$ resp. are the arm length and the leg length resp. of the box s in the Young tableau Y, defined as

$$a_Y(s)|_{s=(i,j)} := \lambda_i - j, \quad l_Y(s)|_{s=(i,j)} := \lambda^t_j - i\,,$$

λ_i and λ^t_j resp. are the i-th part of the partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, $\lambda_i \geq \lambda_{i+1}$ and the j-th part of the transpose partition λ^t respectively .

(1) means that the matrix elements of a special “chiral vertex operator” V_α in a suitably chosen basis, can be translated into a 4d theory as an instanton contribution for a special bifundamental contribution of the NIC. By sewing together pants-like diagrams one gets any desired duality diagrams in the coupling space of the linear quiver gauge theory. So, the checking of the AGT duality reduces to the construction of the states $|P\rangle_{\vec{Y}}$, which we shall call the AFLT states[7], with $\vec{Y} \equiv (Y_1, Y_2)$ the Young tableaux. Here the Y’s, the partitions of natural numbers, or equivalently represented by Young tableaux, are labels for the orthogonal basis for the descendant fields (Verma modules) in a $Vir \oplus u(1)$ conformal family from the 2d CFT point of view. By definition, the AFLT states form a complete set of states for the family members in a given $Vir \oplus u(1)$ conformal family and the inner products between them, sandwiched by a vertex operator of the particular form, $V_\alpha(z)$, at position,say, $z = 1$, is factorized exactly as the NIC Z_{bif} presented on the r.h.s. of (1). The explicit formula, (1), puts strong constraints on the possible forms of the AFLT states and make a systematic construction of them unaccessible at first glance. In [7], only the explicit form of the state $|P\rangle_{Y,\varnothing}$ has been found,

$$|P\rangle_{Y,\varnothing} = J^+_{-Y}|P\rangle\Omega_Y(P),$$

with J^+_{-Y} the creator $(-ib)^{-1}a^+_{-n}$’s valued Jack symmetric function, and $\Omega_Y(P)$ the normalization constant.

In our opinion, the AGT conjecture, written in the form of (1), strongly suggests that the $Vir \oplus u(1)$ conformal family is a Hamiltonian system with $|P\rangle_{\vec{Y}}$ the Hamiltonian eigenstates. So the construction of the AFLT states becomes a quantum mechanical problem of solving the Schrodinger equation. Put things in this way, we propose a possible form of the Hamiltonian H and construct its eigenstates explicitly. We shall identify those eigenstates as the AFLT states desired. For in all the cases we have checked, (1) is verified to be true, using the AFLT states we have constructed. We shall present now as the main results of our present paper the explicit form of the Hamiltonian H along with the complete construction of the AFLT states, $|P\rangle_{\vec{Y}}$. More elaborated exposition will come in the subsequent sections.

$$\begin{aligned} H &= H_0 + H_I \\ |P\rangle_{\vec{Y}} &= \frac{1}{1-\frac{1}{E_{\vec{Y}}(P)-H_0}H_I} J_{-\vec{Y}}|P\rangle\Omega_{\vec{Y}}(P). \end{aligned} \tag{4}$$

Here, $J^{\pm}_{\pm Y}$ are the Jack states constructed in terms of the oscillators $a^{\pm}_n$'s or $a^{\pm}_{-n}$'s ($n > 0$) solely, $H^{\pm}$ the corresponding Hamiltonian for the Jack symmetric functions, $H_0 \equiv H^+ + H^-$. Thus the eigenstate of H_0 is just $J_{-\vec{Y}}|P\rangle \equiv J^+_{-Y_1}J^-_{-Y_2}|P\rangle$ with the eigenvalue $E_{\vec{Y}}(P)$. $H^{\pm}$ in our formalism is defined to include zero modes $a^{\pm}_0$ also, $-ia^{\pm}_0|P\rangle = \pm P|P\rangle$. It is important that H_I is strictly triangular with respect to the basis vectors of the H_0 eigenstates. By "strictly triangular" we mean the (upper or lower) triangular matrix with zero diagonal entries. It is easy to see that if the interaction term H_I is strictly triangular, then the eigenvalue spectrum of H_0 remains unperturbed and $|P\rangle_{\vec{Y}}$ in (4) well defined for non-degenerate H_0 spectrum descending from a mother state $J_{-\vec{Y}}|P\rangle$ for generic values of P's. Putting things all together, we have

$$\begin{aligned}
& H = H_0 + H_I, \quad H_0 = H^+ + H^-, \quad H_I = \sum_{n=1}^{\infty} 2Qna^+_{-n}a^-_n, \\
& H^{\pm} = \frac{-i}{3}\oint (z\partial_z\varphi^{\pm})^3 \frac{dz}{2\pi iz} + \sum_{n=1}^{\infty} Qna^{\pm}_{-n}a^{\pm}_n, \\
& E_{\vec{Y}}(P) = E_{Y_1} + E_{Y_2} + 2P(|Y_1| - |Y_2|), \quad E_Y = \sum_i (y_i^2 b^{-1} + (2i-1)y_i b), \\
& \Omega_{\vec{Y}}(P) = (-)^{|Y_1|} b^{|Y_1|+|Y_2|} \prod_{Y_1}\Big(2P + (a_{Y_1}+1)b^{-1} - l_{Y_2}b\Big) \prod_{Y_2}\Big(2P - a_{Y_2}b^{-1} + (l_{Y_1}+1)b\Big), \\
& |P\rangle_{\vec{Y}} = \frac{1}{1-\frac{1}{E_{\vec{Y}}(P)-H_0}H_I} J_{-\vec{Y}}|P\rangle\Omega_{\vec{Y}}(P), \\
& H_0 J_{-\vec{Y}}|P\rangle = E_{\vec{Y}}(P) J_{-\vec{Y}}|P\rangle, \quad H|P\rangle_{\vec{Y}} = E_{\vec{Y}}(P)|P\rangle_{\vec{Y}}, \quad -ia^{\pm}_0|P\rangle = \pm P|P\rangle
\end{aligned} \tag{5}$$

Notice that

1) $|P\rangle_{Y,\varnothing}$ constructed in [7] are included in our construction as subcases.

2) The Hamiltonian H constructed by us, albeit in a disguised form, turns out to coincide up to some trivial factor with I_3, one of the integrals of motion found in a different context in appendix C of [7]. I_3 in [7], written in the form of $Vir \oplus u(1)$, makes the Virasoro symmetry manifest, but is not suitable for solving a perturbation theory with perturbation parameter $Q = b + b^{-1}$. The Hamiltonian H written in terms of the interacting bi-Jack polynomial system as in (5), shows Virasoro symmetry only implicitly, but makes the perturbation theory exactly solvable as we shall see soon after.

The procedure is outlined as follows. On the 2d CFT side, the $Vir \oplus u(1)$ theory can be represented as a theory of two independent scalars $\tilde{\varphi}(z)$ and $\varphi(z)$. $\tilde{\varphi}(z)$ part is essentially a free theory of timelike oscillators, while the scalar field $-i\varphi(z)$ is spacelike but engaged in a Liouville type interaction. The two scalars can be linearly combined to form the "light-cone" scalars. $\varphi^+(z)$ and $\varphi^-(z)$. The labeling $\vec{Y}$ of the basis vectors strongly suggests that there exist a bi-Jack polynomial structure, plus possibly some interactions between these two sectors. That is, the "free" $H^\pm$ spectrums should be described by $J^+_{Y_1}$ and $J^-_{Y_2}$ respectively, here J_Y denotes Jack states related to Young tableau Y. First we construct the "unperturbed" energy operator H_0 which just sums up the "energies" in $J^\pm_Y$ sectors, $H_0 = H^+ + H^-$. The next thing is to specify the interaction between these two sectors. Strictly speaking, H_0 does not describe a free theory, since it also contains the interaction terms proportional to Q. But the new interaction term H_I further mixes the $J^\pm_Y$'s and the coupling is also a first order in Q. It is good to see that H_I is strictly triangular with respect to the basis vectors of H_0 eigenstates. Our method can be easily generalized to wider classes of integrable models, in which the interacting Hamiltonian splits into two parts, $H^{//}$ and $H^\perp$, representing respectively the shift of energies and the rotations (mixings) of states. The later keeps the eigenvalue spectrum untouched[37].

Besides being triangular, the form of the interaction term is however much restricted, also by the Virasoro symmetry. Since the total Hamiltonian is of the form $Vir \oplus u(1)$, an "interaction energy operator" H_I is needed to make the "full Energy operator" $H = H_0 + H_I$ the combination of a_n's and L_n's only. Once the Hamiltonian structure is determined, then the construction of the Hamiltonian eigenstate $|P\rangle_{\vec{Y}}$ is just a quantum mechanical problem. H_0 and H share the same eigenvalue spectrum, but only the eigenstates of H, represented by $|P\rangle_{\vec{Y}}$'s, form a complete set of basis vectors for the $Vir \oplus u(1)$ conformal family.

We have checked by examples the corresponding AGT duality formula, (1) up to level 4, and have found that indeed Nekrasov instanton counting can be reproduced with this construction, (5). In fact, we have also checked more general cases and all get positive answers. But those more general results will appear elsewhere due to lacking of space to include them in this paper.

The insertions of the screening charges play an important role in checking the AGT duality. However, in the present work we concern ourselves only with the cases in which the screening charges can be detached away from the vertex operator V_α and moved on to act on the AFLT states (similar to the Felder's BRST operators)[8, 9]. The more general cases in which screening charges can not be moved away from V_α will be under our future studies.

It is well known that it is possible to map the Liouville theory to the analytic continuation of the Calogero-Sutherland(CS) model, which was originally and in most cases considered to be a theory with the parameter $\beta > 0$, while in the Liouville case $\beta < 0$ is required. Some explanation is given in [36]. The physical space of the CS model are created by Jack polynomials, which are symmetric functions studied in great detail in mathematics and physics literatures[38, 39]. The integrability of the CS model may be derived in different ways, e.g., from the knowledge of the hidden $W_{1+\infty}$ symmetry of the model. A recursion relation related to the Virasoro singular vectors and an integral representation based on it has appeared recently in [36], in which more references can be found on the subjects of the CS model and the Jack symmetric functions. It should be stressed again that for $\beta > 0$, there is no null vectors in the CS model. So the "null" vectors are not the true null vectors of the CS model, since the Virasoro algebra based on which the null vectors are constructed is not the true conformal algebra of the CS model in that case. But for $\beta < 0$, yes, there are null vectors in the CS model. It is possible to describe the $Liouville \times u(1)$ theory in terms of the Jack polynomials considered as analytic continuation from $\beta > 0$ to $\beta < 0$.

There is another hint that the $Liouville \times u(1)$ theory has something to do with $\beta < 0$ CS model. It can be found from the Nekrasov partition function, in which each term in the summation can be written in the form of the Carlsson-Okounkov formula[10], for the special cases when no screening charges are inserted. Carlsson-Okounkov formula is a formula for the inner products between the bra Jack states and the ket Jack states sandwiched with a modified vertex operator. This extraordinary formula is of great help in checking the AGT duality with our construction for the orthogonal basis vectors $|P\rangle_{\vec{Y}}$'s defined in (1).

We notice that the construction we found shares many similarities with the construction of the Jack functions themselves. Namely, we take the state $J^{+}_{-Y_1} J^{-}_{-Y_2} |P\rangle$ as the mother state and its descendants are constructed in such a manner that two partitions are "squeezed" into other pairs. The squeezing does not change the total level of the two partitions, but does make the inner products of the descendants a triangular form.

Although the 4d to 2d duality has just begun to be understood, it has been known for sometime that 2d conformal blocks can be equivalently described as insertions of Wilson lines in 3d pure Chern-Simons topological gauge theory. In fact, we can interpret the n-point conformal block represented by fig.1 as a Wilson line insertion inside a three-ball. The path integral in Chern-Simons-Witten gauge theory thus creates a state living on the boundary of the three ball, which is punctured S^2. So it should not be a too big surprise that 2d conformal field theory has something to do with higher dimensional quantum field theories. Taking into account that Jack symmetric polynomials can be taken as some special limit of the two parameter Macdonald symmetric polynomials, one natural guess is that our construction can be generalized to the case of Macdonald symmetric polynomials. In that case there should be a 5d to 3d duality.

This paper is organized in the following way. Our general formalism on the construction of the AFLT states is presented in the introduction. In section 2, we explore the general structure of the $Vir \oplus u(1)$ conformal family. We found in some cases it is more convenient to work with the bi-Jack function basis. Section 3 contains the major derivation of our construction. Section 4 is the conclusion. And in appendix A the explicit construction of the AFLT states up to level 3 is presented.

2 Exploring the $Vir \oplus u(1)$ Structure

We are dealing with a 4d N=2 $U(2)$ linear quiver gauge theory coupled to special bi-fundmental matter in a superconformal way. According to the standard AGT duality dictionary, the corresponding 2d conformal block is of the $Vir \oplus u(1)$ type, which reproduces the instanton part of the Nekrasov partition function for the $U(2)$ theory. There are two sets of Young diagrams which measure the partitions of the instantons. If one wants to extract the Virasoro basis of the conformal blocks, one need to factor out the $u(1)$ factor.

In this section we shall mainly explore the Hilbert space for the $Vir \oplus u(1)$ theory and find the requirements that the energy operator H should meet. Our procedure depends heavily on the Nekrasov instanton counting formula written more suitably for the construction of the conformal blocks, (1). First, the 2d $u(1)$ conformal block, realized in terms of the oscillators of the scaler field $\tilde{\varphi}$, is essentially of free theory with center charge $c = 1$. The zero modes can be integrated out trivially and does not play any significant role here. The vertex operators for $\tilde{\varphi}$, take a peculiar form

$$e^{2i(Q-\alpha)\tilde{\varphi}_{(-)}(z)} e^{-2i\alpha\tilde{\varphi}_{(+)}(z)}$$

Here, $\tilde{\varphi}_{(\pm)}$ means the positive (negative) mode part of the scalar field $\tilde{\varphi}$. Although the above vertex operator is not the standard one in 2d CFT, its contribution to the conformal block can be easily read off and factored out. Second, the Vir part is a Liouville conformal field theory of the $\varphi(z)$ scalar field and is more complicated because of the existence of the screening charges.

We have the following mode expansion for the scalar fields $\varphi(z)$ and $\tilde{\varphi}(z)$,

$$\begin{aligned}
\varphi(z) &= q + c_0 \log(z) + \sum_{n\in Z, n\neq 0} \frac{c_{-n}}{n} z^n \,, \\
\tilde{\varphi}(z) &= \tilde{q} + a_0 \log(z) + \sum_{n\in Z, n\neq 0} \frac{a_{-n}}{n} z^n \,, \\
[c_n, c_m] &= \frac{n}{2}\delta_{n+m,0} \,, \quad (c_{-n})^\dagger = c_n \,, \quad [c_0, q] = \frac{1}{2} \\
[a_n, a_m] &= \frac{n}{2}\delta_{n+m,0} \,, \quad (a_{-n})^\dagger = -a_n \,, \quad [a_0, \tilde{q}] = \frac{1}{2} \,.
\end{aligned} \tag{6}$$

Virasoro generators in the Vir part, L_n, thus reads

$$L_n = \sum_{k\in\mathbb{Z}} c_k c_{n-k} - inQc_n = \sum_{k\neq 0,n} c_k c_{n-k} + i(2\hat{P} - nQ)c_n, \tag{7}$$

$$L_0 = \frac{Q^2}{4} - \hat{P}^2 + 2\sum_{k>0} c_{-k} c_k \,, \tag{8}$$

here, $c_0 = i\hat{P}$,

$$-ic_0|P\rangle = \hat{P}|P\rangle = P|P\rangle, \;\; \langle P|(-ic_0) = \langle P|\hat{P} = -P\langle P|.$$

By this construction, L_n defined in (7-8) is obviously unitary,

$$L_{-n} = L_n^\dagger.$$

In 2d CFT, one frequently meets another (more conventional) definition of the Virasoro generators,

$$L_n^0 = \sum_{k\in\mathbb{Z}} c_k c_{n-k} - iQ(n+1)c_n \,. \tag{9}$$

If (7) and (8) are combined in this way,

$$T(z) = \partial\varphi\partial\varphi + iQ\partial^2\varphi + iQz^{-1}\partial\varphi - \frac{Q^2}{4}\frac{1}{z^2} = \sum_n L_n z^{-n-2} \,, \tag{10}$$

then $T(z)$ differs from the more conventional one $T^0(z) = \partial\varphi\partial\varphi + iQ\partial^2\varphi$ by a similarity transformation

$$\begin{aligned}
T(z) &= e^{-iQq}(\partial\varphi\partial\varphi + iQ\partial^2\varphi)e^{iQq} = \sum_n \tilde{L}_n z^{-n-2} \\
\tilde{L}_n &= \sum_{k\in\mathbb{Z}} \tilde{c}_k \tilde{c}_{n-k} - iQ(n+1)\tilde{c}_n \,.
\end{aligned} \tag{11}$$

Comparing (10) and (11), we have

$$\tilde{c_n} = \begin{cases} c_n, & n \neq 0 \\ c_0 + \frac{i}{2}Q, & n = 0 \end{cases} \,. \tag{12}$$

Viewing the $Vir \oplus u(1)$ model as a 2d sigma model, since under conjugation c_n and a_n transform differently we recognize that $-i\varphi$ is spacelike and $\tilde{\varphi}$ timelike when they are considered as coordinates in target space. So the target space of the sigma model under consideration is curved in space and flat in time direction. The two scalars can also be linearly combined to form the "light-cone" scalars $\varphi^{\pm}(z)$,

$$\begin{aligned}
&\varphi^{\pm}(z) = \tilde{\varphi}(z) \pm \varphi(z)\,, \\
&\varphi^{\pm}(z)\varphi^{\pm}(z') = \log(z - z')\,, \\
&\varphi^{\pm}(z)\varphi^{\mp}(z') = 0\,, \\
&\varphi^{\pm\dagger}(z) = \varphi^{\mp}(z)\,.
\end{aligned} \tag{13}$$

The descendant states in the conformal family split into sub-spaces of different levels, which are measured by $I_2 = L_0 + \sum_{n=1}^{\infty} a_{-n}a_n$. Within the sub-space of given level $N \equiv \sum_{n>0}(a_{-n}a_n + c_{-n}c_n)$, states can be labeled either by linear conbinations of $a_{-X}L_{-Y}$'s or $J^{+}_{-X}J^{-}_{-Y}$'s, with X, Y the Young tableau, $|X| + |Y| = N$, $J^{\pm}_{\pm Y}$ the annihilator $(-ib)^{-1}a^{\pm}_{Y}$'s (or creator $(-ib)^{-1}a^{\pm}_{-Y}$'s) valued Jack symmetric functions. In either case, one can infer from AGT duality that there exist a Hermitian operator H, which commutes with I_2 and diagonizes this subspace. Hence the eigenstates of H form an orthogonal basis. We know I_2 acts on this subspace trivially like an identity operator. So in order to eliminate the degeneracy, the next candidate H we are looking for should be at least cubic in the oscillators a_n's and c_n's. Once H is introduced, the descendant states will organize themselves into an orthogonal basis labeled by two sets of Young tableaux $\{Y_1, Y_2\}$. In our opinion, it is better to start with the $J^{\pm}_{Y}$ system, since there is already a Hamiltonian structure $H^{\pm}$ acting separately on them. But $H_0 = H^{+} + H^{-}$ does not commute with the screening charges $S^{\pm}$ pertaining to L_n,

$$S^{\pm} = \oint e^{2ib^{\pm}\varphi(z)}\mathrm{d}z\,,$$

here $b^{+} \equiv b,\ b^{-} \equiv b^{-1}$. We then add a new term H_I to H_0, $H = H_0 + H_I$ and require that $[S^{\pm}, H] = 0$. If H_I are chosen correctly, the eigenstates of H will coincide with the unique orthogonal basis $|P\rangle_{\vec{Y}}$, which we call AFLT states and are defined to satisfy (1), in which the matrix elements $\frac{{}_{\vec{Y}'}\langle P'|V_\alpha|P\rangle_{\vec{Y}}}{\langle P'|V_\alpha|P\rangle}$ is factorized in a consistent way.

On the 4d theory side, one can decouple a single massless bifundamental matter[6] ($\vec{a} = (P, P), m = 0$). We shall show that under this condition the contributions can be written as the orthogonality condition for the $|P\rangle_{\vec{Y}}$'s, provided (1) is satisfied.

Proposition 1 [7] *If AGT conjecture is true, then the AFLT states, $|P\rangle_{\vec{Y}}$'s defined in (1), form an orthogonal basis.*

$${}_{Y_1',Y_2'}\langle P|P\rangle_{Y_1,Y_2} \;=\; Z^{U(2)inst}_{bifund}(\vec{a}, \vec{Y}, \vec{a}, \vec{Y}'; 0) \propto \delta_{\vec{Y},\vec{Y}'}\,. \tag{14}$$

[6]The massless condition implies $|Y_1'| + |Y_2'| = |Y_1| + |Y_2|$.

[7]This is actually Proposition 2.4 in [7], but here we give more details.

Proof: We proceed, from (1),

$$
\begin{aligned}
{}_{Y_1',Y_2'}\langle P|P\rangle_{Y_1,Y_2} &= {}_{Y_1',Y_2'}\langle P|V_{\alpha=0}|P\rangle_{Y_1,Y_2} \qquad (15)\\
&= \prod_{Y_1}\{Q-[(a_{Y_1}+1)b^{-1}-l_{Y_1'}b]\}\prod_{Y_1'}\{(a_{Y_1'}+1)b^{-1}-l_{Y_1}b\}\\
&\times \prod_{Y_1}\{Q-[2P+(a_{Y_1}+1)b^{-1}-l_{Y_2'}b]\}\prod_{Y_2'}\{-2P+(a_{Y_2'}+1)b^{-1}-l_{Y_1}b\}\\
&\times \prod_{Y_2}\{Q-[-2P+(a_{Y_2}+1)b^{-1}-l_{Y_1'}b]\}\prod_{Y_1'}\{2P+(a_{Y_1'}+1)b^{-1}-l_{Y_2}b\}\\
&\times \prod_{Y_2}\{Q-[(a_{Y_2}+1)b^{-1}-l_{Y_2'}b]\}\prod_{Y_2'}\{(a_{Y_2'}+1)b^{-1}-l_{Y_2}b\}\,.
\end{aligned}
$$

We shall prove now that under this situation, if the result is non-zero, one can conclude

$$\vec{Y}=\vec{Y}'.$$

If

$${}_{Y_1',Y_2'}\langle P|P\rangle_{Y_1,Y_2}\neq 0,$$

one gets

$$y_{1,1}\leq y_{1,1}'.$$

[8] Since otherwise there must exist a box in the tableau Y_1 satisfying

$$a_{Y_1}=0,\ \ l_{Y_1'}=-1\ .$$

This will lead to

$$Q-[(a_{Y_1}+1)b^{-1}-l_{Y_1'}b]=0, \qquad (16)$$

This argument cycles and one finally conclude:

$$y_{1,i}\leq y_{1,i}'\ ,i=1,2,\ldots\ .$$

For Y_2, similarly, the argument follows, and gives:

$$y_{2,i}\leq y_{2,i}'\ ,i=1,2,\ldots\ .$$

However, the original condition

$$|Y_1'|+|Y_2'|=|Y_1|+|Y_2|$$

then forces $Y_1=Y_1', Y_2=Y_2'$. Q.E.D.

The orthogonality condition, (14), strongly suggests the existence of mutually commuting Hermitian operators, whose common eigenstates form a complete orthogonal basis of the Hilbert space. One of the operators, called the energy operator, probably cubic in a_n's and c_n's (since this is most likely the case beyond I_2), is the first object we are going to construct. However, hermiticity alone is not enough to constrain the possible forms of the construction. For example, $H_0=H^++H^-$ is Hermitian, but does

[8]Here we use the notation $y_{1,r}$ to label the r-th part of the partition Y_1.

not belong to $Vir \oplus u(1)$. In fact, we shall show that altogether there should be at least 3 conditions H are to meet in order the orthogonality of the H_0 eigenstates play an important role here.

$$\begin{array}{ll} i) & Hermiticity \\ ii) & Triangularity \\ iii) & Reflection - invariance\,. \end{array}$$

Now we explain what the other two conditions means. Condition ii), triangularity, means that H, in its matrix form, $H_{\vec{Y}',\vec{Y}}(P) = \langle \vec{Y}'|H(P)|\vec{Y}\rangle$, where $|\vec{Y}\rangle$'s are the eigenstates of $H_0 = H^+ + H^-$(the "unperturbed" energy operator)[9], is lower(or upper)-triangular with $H_I \equiv H - H_0$ strictly triangular (with zero diagonal entries). Under such circumstances, the spectrum of H coincides with that of H_0, and the eigenstates of H can be expressed as $|P\rangle_{Y_1,Y_2} = \Omega_{Y_1,Y_2}(P)R(E)|\vec{Y},P\rangle$, here the normalization constant $\Omega_{Y_1,Y_2}(P)$ will be specified later on. $R(E) = 1 + \tilde{R}(E)$, a unitriangular matrix, is again triangular with identity diagonal entries following the triangularity of H. $H^\pm$ is the collective mode Hamiltonian for the Calogero-Sutherland model in terms of the oscillators $a_n^\pm$'s. Thus the eigenstate of H_0 is just $J^+_{-Y_1}J^-_{-Y_2}|P\rangle$. $H^\pm$ in our formalism (including the zero modes $a_0^\pm$) is defined as

$$H^\pm = -i\frac{1}{3}\oint(z\partial_z\varphi^\pm(z))^3 dz/z + \sum_{n=1}^{\infty} Qna^\pm_{-n}a^\pm_n$$

Since $\varphi^\pm\dagger = \varphi^\mp$, we have $J^+_Y\dagger = J^-_{-Y}$. There is a natural question on how to define the inner products between $J^\pm_Y$'s. The answer is that we need the condition iii) Reflection-invariance. Notice that $\langle P'|P\rangle$ not zero means $P + P' = 0$. In order to get a non-vanishing result, we need to shift $\langle P|$ to $\langle -P|$. We thus expect that there exists an operation which changes ${}_{Y_1,Y_2}\langle P|$ to ${}_{Y_2,Y_1}\langle -P|$. We call this operation reflection following the terminology in a similar situation in [29]. Actually, by looking closer to the NIC formula i.e. the r.h.s of (1), one can find that there exist an apparent symmetry

$$ {}_{Y_1,Y_2}\langle P| \leftrightarrow {}_{Y_2,Y_1}\langle -P|. \tag{17}$$

If we change either bra state $\langle P|_{Y_1,Y_2}$ to ${}_{Y_2,Y_1}\langle -P|$, or ket state $|P\rangle_{Y_1,Y_2}$ to $|-P\rangle_{Y_2,Y_1}$, on the l.h.s. of (1), the factors on the r.h.s. of eq(1) get reshuffling but the final result keep invariant. We may name this symmetry "reflection" or "flipping" symmetry. On the 2d CFT side, from general reasoning that such an operation should be conformally invariant, it is natural to identify the insertions of the screening charges as this "reflection" operation. For Liouville theory (or Coulomb gas model), we can attach to $V_{\alpha=0}$ some screening charges[10], such that

$$ {}_{Y_1',Y_2'}\langle P|V_0 S^n|P\rangle_{Y_1,Y_2} \neq 0 \tag{18}$$

$$ S = \oint e^{2ib\varphi(z)}\mathrm{d}z\,. \tag{19}$$

Now the neutrality condition forces $2P + nb = 0$. If this is satisfied, then Felder's contour for the integration of the screening charges actually closes and S^n becomes a floating charge[8, 9]. Now S^n

[9] Here we have fixed $\hat{P}$ eigenvalue equals P .

[10] We suppose originally there is no screening charge attached to $V_{\alpha=0}$ for simplicity.

can move away from V_α and communing through L_n's and finally acts on the vacuum state $\langle P|$. Since S^n acts by not changing the conformal weight, we deduce

$$\langle P|S^n = \langle -P|$$

for a suitable normalization of S^n. Similar arguments apply to the case of V_α, $\alpha \neq 0$, and one can always move a subset of screening charges, $S^{\frac{-2P}{b}}$ away from $V_\alpha(z)$. Since AGT duality formula is valid for any P, we may assume that n can take arbitrary real value, as analytical continuation away from integer n. This flipping is due to the fact that S^n can be detached from V_α, and act on the vacuum directly. Similar operation exists in Felder's BRST cohomology [8].

We are going to identify the reflection symmetry in NIC as the Hamiltonian symmetry in 2d CFT for the insertions of the screening charges S^n with $2P = -nb$. Since Hamiltonian $H \in Vir \oplus u(1)$, satisfies $[H, S^n] = 0$ it has the property of double degeneracy. So S^n with $2P = -nb$ should map one AFLT state to its partner state. If we require

$$\begin{aligned} {}_{Y_1,Y_2}\langle P|H &= {}_{Y_1,Y_2}\langle P|E_{Y_1,Y_2} && (20)\\ {}_{Y_1,Y_2}\langle P|S^n H &= {}_{Y_1,Y_2}\langle P|S^n E_{Y_1,Y_2}\,, && (21)\end{aligned}$$

then we can identify

$${}_{Y_1,Y_2}\langle P|S^n =_{Y_2,Y_1} \langle -P|\,,$$

since reflection symmetry means $E_{Y_1,Y_2}(P) = E_{Y_2,Y_1}(-P)$. Notice that nothing has changed for the $u(1)$ part. Define $P^\pm = -ia_0 \mp ic_0$, then we have[11]:

$$P^\pm|P\rangle = \pm P|P\rangle,\ \ \langle -P|P^\pm = \langle -P|(\pm P)\,,$$

which obviously shows that $\langle -P|P\rangle \neq 0$. So reflection invariance means that we can identify the inner product ${}_{Y_1',Y_2'}\langle P|P\rangle_{Y_1,Y_2}$ with either ${}_{Y_2',Y_1'}\langle -P|P\rangle_{Y_1,Y_2}$ or ${}_{Y_1',Y_2'}\langle P|-P\rangle_{Y_2,Y_1}$ by the incertions of screening charges satisfying $n = -2Pb^{-1}$.

Having determined that $|P\rangle_{\vec{Y}}$ form a normalizable orthogonal basis, the next step is the determination of their normalization. Before doing this, let's review the so-called Carlsson-Okounkov formula[10] which is useful for our formulation. First, define

$$E = 1 + e_1 + e_2 + \cdots = e^{-\sum_n \frac{(-)^n}{n} p_n} = e^{-\frac{1}{k}\varphi_{(-)}(-1)} \tag{22}$$

[12]which is a vertex operator, and also a generating function for J_{-1^n}

$$e^{-\frac{1}{k}\varphi_{(-)}(z)}|0\rangle = \sum_n (-)^n \frac{J_{-1^n}}{n!} z^n |0\rangle\,,$$

here e_i are elementary symmetric functions, p_n is the power sum symmetric function. Then

$$e^{-\frac{1}{k}\varphi_{(-)}(-1)}|0\rangle = \sum_n \frac{J_{-1^n}}{n!}|0\rangle \equiv \sum_n P_{-1^n}|0\rangle. \tag{23}$$

[11]We have set $a_0|P\rangle = 0$ throughout this paper.

[12]For infinitely many arguments z_i's, $i = 1, 2, \cdots, \infty$, one may identify $p_n \equiv \sum_i z_i^n$ with $\frac{a_{-n}}{k}$, $k^2 = \beta$ and $J_Y^{1/\beta}(\{p_n\})$ with $J_Y^{1/\beta}(\{\frac{a_{-n}}{k}\})$. Here our convention is that $\frac{a_{-n}}{k}|0\rangle$ creates a state $|p_n\rangle$. As a consequence, e_n is to be identified with $P_{-1^n} \equiv \dfrac{J_{1^n}^{1/\beta}(\{\frac{a_{-n}}{k}\})}{n!} \equiv J_{-1^n}/n!$. Such kind of identification is justified because they share the same values of their inner products. For more details see [36].

The conjugation of E reads

$$E^* = e^{\frac{1}{k}\varphi_{(+)}(-1)}\,, \tag{24}$$

and we have

$$\langle 0|E^* = \langle 0|\sum_n P_{1^n}\,.$$

Now the Carlsson-Okounkov formula reads

$$\begin{aligned}
&\langle E^m (E^*)^{\beta-m-1} J_{-Y_1}, J_{-Y_2}\rangle \qquad (25)\\
=\;& (-)^{|Y_1|}\beta^{-|Y_1|-|Y_2|}\prod_{Y_1}(m+(a_{Y_1}+1)+\beta l_{Y_2})\prod_{Y_2}(m-a_{Y_2}-\beta(l_{Y_1}+1))\\
=\;& \langle J_{Y_1} E^{\beta-m-1}(E^*)^m J_{-Y_2}\rangle\\
=\;& \langle J_{Y_1} e^{(-k+k^{-1}+\frac{m}{k})\varphi_{(-)}(-1)} e^{\frac{m}{k}\varphi_{(+)}(-1)} J_{-Y_2}\rangle\,.
\end{aligned}$$

For Liouville theory, $k=-ib$. If we set $\frac{m}{-ib}=-2i\alpha$, then the Carlsson-Okounkov formula reads

$$\begin{aligned}
&\langle J_{Y_1} e^{i(Q-2\alpha)\varphi_{(-)}(-1)} e^{-2i\alpha\varphi_{(+)}(-1)} J_{-Y_2}\rangle \qquad (26)\\
=\;& (-)^{|Y_2|} b^{-|Y_1|-|Y_2|}\prod_{Y_1}(-2\alpha+(a_{Y_1}+1)b^{-1}-l_{-Y_2}b)\prod_{Y_2}(-2\alpha-a_{Y_2}b^{-1}+(l_{Y_1}+1)b)\,.
\end{aligned}$$

The normalization of the AFLT states is inherited from AFLT's version of the AGT duality formula, (1) and the orthogonality condition, (15),

$$\begin{aligned}
&{}_{Y_1,Y_2}\langle P|P\rangle_{Y_1,Y_2} ={}_{Y_2,Y_1}\langle -P|P\rangle_{Y_1,Y_2} \qquad (27)\\
=\;& \prod_{Y_1}\{-a_{Y_1}b^{-1}+(l_{Y_1}+1)b\}\{(a_{Y_1}+1)b^{-1}-l_{Y_1}b\}\\
\times\;& \prod_{Y_2}\{-a_{Y_2}b^{-1}+(l_{Y_2}+1)b\}\{(a_{Y_2}+1)b^{-1}-l_{Y_2}b\}\\
\times\;& \prod_{Y_1}\{-2P-a_{Y_1}b^{-1}+(l_{Y_2}+1)b)\}\prod_{Y_2}\{-2P+(a_{Y_2}+1)b^{-1}-l_{Y_1}b\}\\
\times\;& \prod_{Y_2}\{2P-a_{Y_2}b^{-1}+(l_{Y_1}+1)b)\}\prod_{Y_1}\{2P+(a_{Y_1}+1)b^{-1}-l_{Y_2}b\}\\
=\;& (-)^{|Y_1|+|Y_2|} j_{Y_1} j_{Y_2}\\
\times\;& \prod_{Y_1}\{-2Pb-a_{Y_1}+(l_{Y_2}+1)b^2\}\prod_{Y_2}\{-2Pb+(a_{Y_2}+1)-l_{Y_1}b^2\}\\
\times\;& \prod_{Y_2}\{2Pb-a_{Y_2}+(l_{Y_1}+1)b^2\}\prod_{Y_1}\{2Pb+(a_{Y_1}+1)-l_{Y_2}b^2\}\\
\equiv\;& j_{Y_1} j_{Y_2}\Omega_{Y_2,Y_1}(-P)\Omega_{Y_1,Y_2}(P)\\
=\;& j_{Y_1} j_{Y_2}\langle J_{Y_2} e^{i(Q-2P)\varphi_{(-)}(-1)} e^{-i2P\varphi_{(+)}(-1)} J_{-Y_1}\rangle\\
\times\;& (b^4)^{|Y_1|+|Y_2|}\langle J_{Y_1} e^{i(Q+2P)\varphi_{(-)}(-1)} e^{i2P\varphi_{(+)}(-1)} J_{-Y_2}\rangle\,.
\end{aligned}$$

In reaching the last line in the above eqation, Carlsson-Okounkov formula has been applied, and we have defined

$$\begin{aligned}
\Omega_{Y_1,Y_2}(P) &= (-)^{|Y_1|} b^{|Y_1|+|Y_2|} \prod_{Y_1} \left(2P + (a_{Y_1}+1)b^{-1} - l_{Y_2} b\right) \prod_{Y_2} \left(2P - a_{Y_2} b^{-1} + (l_{Y_1}+1)b\right) \\
&= (-b^2)^{(|Y_1|+|Y_2|)} \langle J_{Y_1} e^{i(Q+2P)\varphi_{(-)}(-1)} e^{i2P\varphi_{(+)}(-1)} J_{-Y_2} \rangle \\
&= b^{2(|Y_1|+|Y_2|)} \langle J_{Y_1} e^{i(Q+2P)\varphi_{(-)}(1)} e^{i2P\varphi_{(+)}(1)} J_{-Y_2} \rangle \\
&\equiv \Omega_{\vec{Y}}(P)\,.
\end{aligned} \tag{28}$$

Notice that $\Omega_{\vec{Y}}(P)$ is just a generalization of $\Omega_Y(P)$ defined in [7].

3 The Construction of the AFLT States

Now we come to our main problem of the construction of the Hamiltonian H with the requirement that its eigenstates be identified with ALFT states satisfying (1). We prefer to work first on the basis of Jack symmetric functions $J_{\vec{Y}}$, which already form an orthogonal basis. We found that if H_I matrix elements are strictly triangular on this basis, then the orthogonality of the $H = H_0 + H_I$ eigenstates follows immediately from the orthogonality of the H_0 eigenstates. This is just the simplest way to go from one orthogonal basis to another one. To see this, let's introduce an operator $R(E)$ which map H_0 eigenstates to H eigenstates, with $\Omega_{Y_1,Y_2}(P)$ the normalization constant

$$\begin{aligned}
|P\rangle_{Y_1,Y_2} &= R(E) J^+_{-Y_1} J^-_{-Y_2} |P\rangle \Omega_{Y_1,Y_2}(P) \\
{}_{Y_2',Y_1'}\langle -P| &= \langle -P| J^-_{Y_2'} J^+_{Y_1'} R(E')^\dagger \Omega_{Y_2' Y_1'}(-P) \\
R(E) &= 1 + \cdots = 1 + \tilde{R}(E)\,,
\end{aligned} \tag{29}$$

where the reflection symmetry has been applied to the AFLT states

$${}_{Y_1',Y_2'}\langle P|S^n = \langle P|S^n J^-_{Y_1'} J^+_{Y_2'} R(E')^\dagger \Omega_{Y_1' Y_2'}(P) = {}_{Y_2',Y_1'}\langle -P| = \langle -P| J^-_{Y_2'} J^+_{Y_1'} R(E')^\dagger \Omega_{Y_2' Y_1'}(-P),$$

and $\tilde{R}(E)$ is strictly lower(or upper)-triangular $\Rightarrow \tilde{R}_{\vec{Y},\vec{Y}}(E) = 0$. The way $R(E)$ is expanded in (29) follows from the normalization condition, (27-28) as we shall see in (33).

The Hermitian operator H should satisfy:

$$\begin{aligned}
H|P\rangle_{Y_1,Y_2} &= E_{Y_1,Y_2}(P) |P\rangle_{Y_1,Y_2} \\
{}_{Y_2',Y_1'}\langle -P|H &= {}_{Y_2',Y_1'}\langle -P| E_{Y_2',Y_1'}(-P),
\end{aligned} \tag{30}$$

where the energy eigenvalue has the double degeneracy:

$$E_{Y_2,Y_1}(-P) = E_{Y_1,Y_2}(P)\,, \tag{31}$$

due to the orthogonality condition,

$${}_{Y_1',Y_2'}\langle P|P\rangle_{Y_1,Y_2} = {}_{Y_2',Y_1'}\langle -P|P\rangle_{Y_1,Y_2} \propto \delta_{\vec{Y},\vec{Y}'}\,. \tag{32}$$

Then the next step is to determine if we get the right normalization for $|P\rangle_{Y_1,Y_2}$

$$\begin{aligned} {}_{Y_2,Y_1}\langle -P|P\rangle_{Y_1,Y_2} &= \langle -P|J^-_{Y_2}J^+_{Y_2}R(E)^\dagger R(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle\Omega_{Y_1,Y_2}(P)\Omega_{Y_2,Y_1}(-P) \\ &= \langle -P|J^-_{Y_2}J^+_{Y_1}(1+\tilde{R}(E)^\dagger)(1+\tilde{R}(E))J^+_{-Y_1}J^-_{-Y_2}|P\rangle\Omega_{Y_1,Y_2}(P)\Omega_{Y_2,Y_1}(-P) \\ &= \Omega_{Y_1,Y_2}(P)\Omega_{Y_2,Y_1}(-P) \\ &\times \left[j_{Y_1}j_{Y_2} + \langle -P|J^-_{Y_2}J^+_{Y_1}(\tilde{R}(E)^\dagger + \tilde{R}(E) + \tilde{R}(E)^\dagger\tilde{R}(E))J^+_{-Y_1}J^-_{-Y_2}|P\rangle\right] \\ &= \Omega_{Y_1,Y_2}(P)\Omega_{Y_2,Y_1}(-P)j_{Y_1}j_{Y_2}\,. \end{aligned} \tag{33}$$

It is in agreement with (27). In deriving this we have used the fact that if $R(E)$ is a unitriangular matrix[13],

$$R(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle = J^+_{-Y_1}J^-_{-Y_2}|P\rangle + \sum_{\substack{|Y'_1|>|Y_1| \\ |Y'_2|<|Y_2| \\ |Y'_1|+|Y'_2|=|Y_1|+|Y_2|}} R^{Y'_1Y'_2}_{Y_1,Y_2}(E)J^+_{-Y'_1}J^-_{-Y'_2}|P\rangle\,, \tag{34}$$

then it is easy to check that

$$\begin{aligned} &\langle -P|J^-_{Y_2}J^+_{Y_1}\tilde{R}(E)^\dagger J^+_{-Y_1}J^-_{-Y_2}|P\rangle \\ &= \langle -P|J^-_{Y_2}J^+_{Y_1}\tilde{R}(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle \\ &= \langle -P|J^-_{Y_2}J^+_{Y_1}\tilde{R}(E)^\dagger\tilde{R}(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle \\ &= 0\,. \end{aligned}$$

Now we summarize the requirements for $R(E)$

i) $R(E)$ is unitriangular

ii) $R(E)$ creates the eigenstate for H

$$HR(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle = E_{Y_1,Y_2}(P)R(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle$$

iii) Reflection invariant

$$S^nR(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle\Omega_{Y_1,Y_2}(P) = R(E)J^+_{-Y_2}J^-_{-Y_1}|-P\rangle\Omega_{Y_2,Y_1}(-P)$$

$$[S^n,H]=0,\quad E_{Y_1,Y_2}(P) = E_{Y_2,Y_1}(-P)$$

This means that the Hamiltonian H should also be triangular, and H_I strictly triangular,

$$H = H^+ + H^- + H_I \tag{35}$$

$$H^\pm = \frac{-i}{3}\oint (z\partial_z\varphi^\pm)^3 + \sum_{n>0} Qna^\pm_{-n}a^\pm_n\,, \tag{36}$$

[13]A unitriangular matrix is a triangular matrix with the diagonal entries equal to 1 .

here H_I is to be determined later on.

$$
\begin{aligned}
H^+ + H^- &= \frac{-i}{3}\oint\left[(z\partial_z(\varphi+\tilde{\varphi}))^3 + (z\partial_z(\varphi-\tilde{\varphi}))^3\right]\frac{\mathrm{dz}}{z} + \\
&\quad \sum_{n>0} Qna^+_{-n}a^+_n + \sum_{n>0} Qna^-_{-n}a^-_n \\
&= \frac{-i}{3}\oint\left[2\,(z\partial_z\tilde{\varphi})^3 + 6(z\partial_z\tilde{\varphi})(z\partial_z\varphi)^2\right]\frac{\mathrm{dz}}{z} \\
&+ \sum_{n>0} 2Qn(a_{-n}a_n + c_{-n}c_n) \\
&= \frac{-2i}{3}\oint (z\partial_z\tilde{\varphi})^3\,\frac{\mathrm{dz}}{z} + \sum_{n>0} 2Qna_{-n}a_n - 2i\oint (z\partial_z\tilde{\varphi})\,(z\partial_z\varphi)^2\,\frac{\mathrm{dz}}{z} + 2\sum_{n>0} Qnc_{-n}c_n\,.
\end{aligned}
$$

Now the requirement that H commute with S^n is equivalent to say that H can be written in terms of L_n's and a_n's. To make H triangular, we may try

$$
\begin{aligned}
H_I &\propto \sum_n Qna^+_{-n}a^-_n \\
&= \sum_n Qn(a_{-n}a_n - c_{-n}c_n - a_{-n}c_n + c_{-n}a_n) \\
&= \sum_n Qn(a_{-n}a_n - c_{-n}c_n) + Q\oint z\partial_z\tilde{\varphi}(z\partial_z)^2\varphi\frac{\mathrm{dz}}{z}\,.
\end{aligned}
$$

If we now make use of (10) and choose

$$
H_I = \sum_n 2Qna^+_{-n}a^-_n,
$$

then we get

$$
\begin{aligned}
H &= -\frac{2i}{3}\oint(z\partial_z\tilde{\varphi})^3\frac{\mathrm{dz}}{z} + 4Q\sum_{n\in\mathbb{N}^+} na_{-n}a_n - 2i\oint(z\partial_z\tilde{\varphi})z^2T(\mathrm{z})\frac{\mathrm{dz}}{z} + 2ia_0\frac{Q^2}{4} &(37)\\
&= -\frac{2i}{3}\oint(z\partial_z\tilde{\varphi})^3\frac{\mathrm{dz}}{z} + 4Q\sum_{n\in\mathbb{N}^+} na_{-n}a_n - 2i\sum_{n\in\mathbb{Z}} a_{-n}L_n + 2ia_0\frac{Q^2}{4} \\
&= -i\left\{\sum_{n,m\in\mathbb{N}^+}(a^+_{-n-m}a^+_na^+_m + a^-_{-n-m}a^-_na^-_m)\right. \\
&\quad + \left.\sum_{n,m\in\mathbb{N}^+}(a^+_{-n}a^+_{-m}a^+_{n+m} + a^-_{-n}a^-_{-n}a^-_{n+m})\right\} \\
&\quad + \sum_{n\in\mathbb{N}^+} Qn(a^+_{-n}a^+_n + a^-_{-n}a^-_n + 2a^+_{-n}a^-_n) &(38)\\
&\quad + \sum_{n\in\mathbb{N}^+} -2ia^+_0a^+_{-n}a^+_n - 2ia^-_0a^-_{-n}a^-_n - \frac{i}{3}((a^+_0)^3 + (a^-_0)^3)\,. &(39)
\end{aligned}
$$

Clearly, H indeed satisfies the three requirements proposed in the previous section.

$$\begin{array}{ll} i) & Hermitian \\ ii) & Triangular \\ iii) & Reflection-invarint\,. \end{array}$$

Besides, we found that $H \propto I_3$, where I_3 is defined in Appendix C of [7] as one of the infinitely many commuting operators which may makes the system integrable. The authors of [7] have checked for the first a few levels that the AFLT states, $|P\rangle_{Y_1,Y_2}$, which satisfies AGT duality formula, (1), are also the eigenstates of I_3. But a general formula for the I_3 eigenstates is missing in [7].

Now the next question is : how to find all the eigenstates of H? First, let's consider H^+

$$\begin{aligned} H^+ &= -i \sum_{n,m\in\mathbb{N}^+} \{a^+_{-n-m}a^+_n a^+_m + a^+_{-n}a^+_{-m}a^+_{n+m}\} \\ &+ \sum_{n\in\mathbb{N}^+} \left\{nQa^+_{-n}a^+_n + 2a^+_0(-i)a^+_{-n}a^+_n\right\} - \frac{i(a^+_0)^3}{3}\,, \end{aligned} \tag{40}$$

Its eigenvalue

$$H^+J^+_{-Y}|P^+\rangle = E^+_Y(P^+)J^+_{-Y}|P^+\rangle$$

$$E^+_Y(P^+) = \sum_i \left\{y_i^2 b^{-1} + (2i-1)y_i b\right\} + 2P^+|Y| - \frac{(P^+)^3}{3}.$$

Here we have assumed the zero modes take the following eigenvalues,

$$a_0 = iP^a\,,\ c_0 = iP^c\,,\ a^\pm_0 = iP^\pm = i(P^a \pm P^c) \tag{41}$$

For the bi-Jack system, we have the following eigenequation,

$$HR(E)J^+_{-Y_1}J^-_{-Y_2}|P^+,P^-\rangle = E_{Y_1,Y_2}(P^+,P^-)R(E)J^+_{-Y_1}J^-_{-Y_2}|P^+,P^-\rangle\,.$$

Triangularity means

$$\begin{aligned} E_{Y_1,Y_2}(P^+,P^-) = &\quad E^+_{Y_1}(P^+) + E^-_{Y_2}(P^-) \\ = &\quad \sum_i \left\{y^2_{1,i}b^{-1} + (2i-1)y^2_{1,i}b\right\} + \sum_i \left\{y^2_{2,i}b^{-1} + (2i-1)y^2_{2,i}b\right\} \\ &\quad +2P^+|Y_1| + 2P^-|Y_2| - \frac{(P^+)^3 + (P^-)^3}{3} \end{aligned}$$

Since H can be constructed in terms of L_n's and a_n's, so $S^n|P^+,P^-\rangle_{Y_1,Y_2}$ dose not change the eigenvalue. But S^n changes $P^c \to -P^c$ and $P^+ \leftrightarrow P^-$ and from $E_{Y_1,Y_2}(P^+,P^-) = E_{Y_2,Y_1}(P^-,P^+)$. We conclude

$$S^n|P^+,P^-\rangle_{Y_1,Y_2} \propto |P^-,P^+\rangle_{Y_2,Y_1}. \tag{42}$$

Next, since P^a does not play any important role, we may consider it as a gauge symmetry and can be fixed to any desired value. For convenience, we fix $P^a = 0$, hence, $P^+ = P^c \equiv P$, $P^- = -P^c \equiv -P$ and

$$\begin{aligned} E_{Y_1,Y_2}(P,-P) &\equiv E_{Y_1,Y_2}(P) \\ &= E_{Y_1} + E_{Y_2} + 2P(|Y_1| - |Y_2|) \end{aligned}$$

Here

$$E_Y = \sum_i \left\{ y_i^2 b^{-1} + (2i-1) y_i b \right\}$$

If we define

$$\begin{aligned} |P\rangle &\equiv |P, -P\rangle \\ |P\rangle_{Y_1,Y_2} &\equiv R(E) J^+_{-Y_1} J^-_{-Y_2} |P\rangle \Omega_{Y_1,Y_2}(P) \end{aligned}$$

Then we infer from from (42)

$$S^n |P\rangle_{Y_1,Y_2} = |-P\rangle_{Y_2,Y_1}$$

with the proper normalization for S^n. Now we are going to determine $R(E)$ which satisfies

$$HR(E) J^+_{-Y_1} J^-_{-Y_2} |P\rangle = E_{Y_1,Y_2}(P) R(E) J^+_{-Y_1} J^-_{-Y_2} |P\rangle.$$

Proposition 2

$$R(E) J^+_{-Y_1} J^-_{-Y_2} |P\rangle = \frac{1}{1 - \frac{1}{E-H_0} H_I} J^+_{-Y_1} J^-_{-Y_2} |P\rangle,$$

here $H_0 = H^+ + H^-, E = E_{Y_1,Y_2}(P)$. $R(E)$ *defined in such a way should be understood as*

$$\begin{aligned} R(E) &= \frac{1}{1 - \frac{1}{E-H_0} H_I} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{E_{Y_1,Y_2}(P) - H_0} H_I\right)^n \end{aligned}$$

Proof: First, we rewrite H as

$$\begin{aligned} H &= H_0 + H_I \\ &= E + H_0 + H_I - E \\ &= E + (H_0 - E)(1 + \frac{1}{H_0 - E} H_I)\,. \end{aligned}$$

Then from

$$\begin{aligned} HR(E) &= (E + (H_0 - E)(1 + \frac{1}{H_0 - E} H_I)) \frac{1}{1 - \frac{1}{E-H_0} H_I} \\ &= E \frac{1}{1 - \frac{1}{E-H_0} H_I} + H_0 - E\,, \\ &= ER(E) + H_0 - E\,, \end{aligned}$$

one gets

$$HR(E) J^+_{-Y_1} J^-_{-Y_2} |P\rangle = ER(E) J^+_{-Y_1} J^-_{-Y_2} |P\rangle + (H_0 - E) J^+_{-Y_1} J^-_{-Y_2} |P\rangle\,.$$

Since $J^+_{-Y_1}J^-_{-Y_2}|P\rangle$ is an eigenstate of H_0 with eigenvalue $E = E_{Y_1,Y_2}(P)$, we have

$$(H_0 - E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle = 0.$$

Hence, we conclude that $R(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle$ is an eigenstate of H with eigenvalue E,

$$E \equiv E_{Y_1,Y_2}(P) = \left(\sum_{i=1,l=1}^{i=2,l=y^t_{i,1}} (y_{i,l})^2 + (2i-1)y_{i,l}\right) + 2P(|Y_1| - |Y_2|)\,,$$

Q.E.D.

Now we shall address the question raised in [7] on the possible degeneracy of H. The authors of [7], argued that I_3 has some degeneracy at level 4 and higher. We have analyzed what causes such kind of degeneracy. After analyzing the spectrum of H, we believe that such degeneracy happens when $|Y_1| = |Y_2|$, and we have $2P(|Y_1| - |Y_2|) = 0$,

$$E_{Y_1,Y_2}(P) = E_{Y_1} + E_{Y_2} = E_{Y_2} + E_{Y_1} = E_{Y_2,Y_1}(P)$$

This can happen, for $Y_1 \neq Y_2$, first at level 4, $|Y_1| + |Y_2| \equiv |\vec{Y}| = 4$, and $Y_1 = 2$, $Y_2 = 1^2$. Such degeneracy can happen at any even level higher or equal to 4. For example at level = 6:

$$Y_1 = 3, Y_2 = 1^3, \text{or } Y_1 = 3, Y_2 = \{2,1\}, \text{or } Y_1 = 1^3, Y_2 = \{2,1\},$$

or simply, we have (Y_1, Y_2) pair

$$(3, 1^3), (3, \{2,1\}), (1^3, \{2,1\})$$

Such degeneracy does not cause any problem in constructing the eigenstate of H for the following reasons.

i) The mother state $J^+_{-Y_1}J^-_{-Y_2}|P\rangle$ is uniquely determined by the Young diagram, even for the degenerate E.

ii) Consider power expansion

$$R(E) = \sum_{n=0}^{\infty}\Big(\frac{1}{E_{Y_1,Y_2}(P) - H_0}H_I\Big)^n.$$

For an intermediate state.

$$\begin{aligned} E_{Y_1,Y_2}(P) - H_0 \quad &\sim E_{Y_1,Y_2}(P) - E_{Y'_1,Y'_2}(P) \\ &= E_{Y_1} + E_{Y_2} - E_{Y'_1} - E_{Y'_2} + 2P(|Y_1| - |Y_2| - |Y'_1| + |Y'_2|) \end{aligned}$$

Since $|Y'_1| > |Y_1|$, $|Y'_2| < |Y_2|$ and $|Y_1| - |Y'_1| + |Y'_2| - |Y_2| < 0$ because of strictly triangularity of H_I, so for a general value of P, $\frac{1}{E_{Y_1,Y_2}(P)-H_0}$ is not singular, and $R(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle$ is well defined.

iii) The construction given above leads to the orthogonality of the state $|P\rangle_{Y_1,Y_2}$ for distinct Y_1, Y_2 even for the degenerate values of E, cf. eqs.(27,28,33).

iv) It can be proven that the eigenstate of H, constructed as in proposition 2, is actually the common eigenstate for all the conseved charges which commute with H, with the mild assumption that all the conserved charges are triangular in a similar way as H is. Due to lack of space for the present paper, we shall give a proof on this statement elsewhere.

Finally, we shall make a comment on the possible poles of $R(E)$ in the complex $p-$plane. The $R(E)$ matrix elements is calculated based on the following formula,

$$\begin{aligned} |P\rangle_{Y_1,Y_2} &= R(E)J^+_{-Y_1}J^-_{-Y_2}|P\rangle\Omega_{Y_1,Y_2}(P) \\ &= \sum_{n=0}^{\infty}(\frac{1}{E_{Y_1,Y_2}(P)-H_0}H_I)^n J^+_{-Y_1}J^-_{-Y_2}|P\rangle\Omega_{Y_1,Y_2}(P)\,. \end{aligned}$$

which always ends up with finite order perturbation because H_I is strictly triangular. We found, by the explicit calculations carried out so far, that there is no pole in the finite P complex plane. The poles in $R(E)$ either cancels the zeros in $\Omega_{Y_1,Y_2}(P)$ or simply cancels by summing over all the relevant terms. Of course, this property is also the necessary condition if $|P\rangle_{Y_1,Y_2}$'s satisfy (1). Now the general AFLT state can be written as

$$\begin{aligned} |P\rangle_{Y_1,Y_2} &= \Bigg\{\Omega_{Y_1,Y_2}(P)J^+_{-Y_1}J^-_{-Y_2} + \sum_{\substack{|Y_1'|=|Y_1|+1\\|Y_2'|=|Y_2|-1}} C^{Y_1',Y_2'}_{Y_1,Y_2}J^+_{-Y_1'}J^-_{-Y_2'} \\ &+ \sum_{\substack{|Y_1''|=|Y_1|+2\\|Y_2''|=|Y_2|-2}} C^{Y_1'',Y_2''}_{Y_1,Y_2}J^+_{-Y_1''}J^-_{-Y_2''} \\ &+ \cdots + \sum_{|Y|=|Y_1|+|Y_2|} C^{Y,\emptyset}_{Y_1,Y_2}J^+_{-Y}\Bigg\}|P\rangle, \end{aligned} \tag{43}$$

here $C^{Y_3,Y_4}_{Y_1,Y_2}$ is the transition coefficient which measures the changing from the Young tableau vector (Y_1,Y_2) to (Y_3,Y_4).

We have calculated those coefficients up to level 4, the explicit results(up to level 3) are included in Appdix A. With the coefficients we calculated, one can check that:

$$\begin{aligned} &Z_{bif}(\alpha|P',\vec{X};P,\vec{Y}) = \\ &\sum_{(X_1',X_2'),(Y_1',Y_2')}\langle P'|J^-_{X_1'}J^+_{X_2'}C^{X_1',X_2'}_{X_1,X_2}V_\alpha C^{Y_1',Y_2'}_{Y_1,Y_2}J_{-Y_1'}J_{-Y_2'}|P\rangle\,, \end{aligned} \tag{44}$$

holds true, thus (1) is verified. Here for simplicity, we have only verified the cases without the incertions of the screening charges, i.e. $P+P'+\alpha=0$.

4 Conclusion and Perspective

The present work can be generalized in different ways. First, since the one parameter Jack symmetric function is a special limit of the two parameter Macdonald symmetric function, we expect that much of our work can be generalized to the cases where Macdonald symmetric function plays a role. In that case, we expect a similar relation to the NIC for 5d theory. Second, the Calogero-Sutherland model is an integrable system. And consequently, Jack symmetric function is the common eigenstate of the infinitely many commuting charges which are deformed W^{∞} charges. And for the construction of the AFLT states, the conserved charges are further deformed from those for the Jack symmetric functions. The final construction should give the same results as I_n proposed in [7], which are constructed from integrable KdV equations. We find in this case, the AFLT states remain to be the eigenstates for all the conserved charges. However, it is desirable to have infinitely many conserved charges constructed explicitly. Third, the reflection symmetry studied in this paper is actually powerful enough to give a closed form for the construction of the AFLT states. We shall present this result in our future work. Another interesting idea related to our work is to consider the Jack function as a perturbation away from the Schur function, we have found that similar formalism applies [37]. Finally, it is very interesting to see how we present the full pants diagram for the conformal blocks, comparing to the one we have considered with one external leg.

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A Coefficients for AFLT States(up to level 3)

Now we give the explicit construction of the AFLT states up to level 3. The transition coefficients $C^{Y_1',Y_2'}_{Y_1,Y_2}$ are defined as,

$$C^{Y_1',Y_2'}_{Y_1,Y_2} \equiv R^{Y_1',Y_2'}_{Y_1,Y_2}(E)\Omega_{Y_1,Y_2}(P), \quad C^{Y_1,Y_2}_{Y_1,Y_2} \equiv \Omega_{Y_1,Y_2}(P)\,.$$

Level 2 coefficients:

$$C^{1,1}_{0,1^2} = C^{1,1}_{0,2} = -4b\left(1+b^2\right)P,$$
$$C^{1^2,0}_{1,1} = \frac{\left(1+b^2\right)(1+2bP)}{-1+b^2},$$
$$C^{2,0}_{1,1} = -\frac{b^2\left(1+b^2\right)(1+2bP)}{-1+b^2},$$
$$C^{12,0}_{0,1^2} = 1+b^2\left(3+2b\left(b-\frac{2\left(1+b^2\right)P}{-1+b^2}\right)\right),$$
$$C^{2,0}_{0,1^2} = \frac{4b^3\left(1+b^2\right)P}{-1+b^2},$$
$$C^{2,0}_{0,2} = \frac{\left(1+b^2\right)P^2\left(-2+b^2+b^4+4bP\right)}{-1+b^2},$$
$$C^{1^2,0}_{0,2} = \frac{4b\left(1+b^2\right)P}{-1+b^2}$$

Level 3 coefficients:

$$C_{2,1}^{3,0} = \left\{ -\frac{b^3(1+b^2)(b+2P)(1+b^2+2bP)}{-2+b^2} \right\},$$
$$C_{2,1}^{\{2,1\},0} = \left\{ \frac{4(1+b^2)(1+bP)(1+b^2+2bP)}{-2+b^2} \right\},$$
$$C_{1^2,1}^{1^3,0} = \left\{ \frac{(1+b^2)(1+2bP)(1+b^2+2bP)}{-1+2b^2} \right\},$$
$$C_{1^2,1}^{\{2,1\},0} = \left\{ -\frac{4b^3(1+b^2)(b+P)(1+b^2+2bP)}{-1+2b^2} \right\},$$
$$C_{1,2}^{2,1} = \left\{ -\frac{2b^2(1+b^2)(1+2bP)(-1+b^2+2bP)}{-1+b^2} \right\},$$
$$C_{1,2}^{1^2,1} = \left\{ \frac{4b(1+b^2)P(1+2bP)}{-1+b^2} \right\},$$
$$C_{1,2}^{3,0} = \left\{ \frac{b^3(1+b^2)(b+b^3+4P)(-1+b^2+2bP)}{2-3b^2+b^4} \right\},$$
$$C_{1,2}^{\{2,1\},0} = \left\{ -\frac{4(1+b^2)^2(1+2bP)(-1+2b(b+P))}{2-5b^2+2b^4} \right\},$$
$$C_{1,2}^{1^3,0} = \left\{ \frac{4b(1+b^2)P(1+2bP)}{1-3b^2+2b^4} \right\},$$
$$C_{1,1^2}^{2,1} = \left\{ -\frac{4b^4(1+b^2)P(b+2P)}{-1+b^2} \right\},$$
$$C_{1,1^2}^{1^2,1} = \left\{ -\frac{2b(1+b^2)(b+2P)(-1+b^2-2bP)}{-1+b^2} \right\},$$
$$C_{1,1^2}^{3,0} = \left\{ \frac{4b^6(1+b^2)P(b+2P)}{2-3b^2+b^4} \right\},$$
$$C_{1,1^2}^{\{2,1\},0} = \left\{ \frac{4b^3(1+b^2)^2(b+2P)(-2+b^2-2bP)}{2-5b^2+2b^4} \right\},$$
$$C_{1,1^2}^{1^3,0} = \left\{ -\frac{(1+b^2)(-1+b^2-2bP)(1+b^2+4b^3P)}{1-3b^2+2b^4} \right\},$$
$$C_{0,3}^{1,2} = \left\{ -6b\left(1+b^2\right)P(-1+2bP) \right\},$$
$$C_{0,3}^{2,1} = \left\{ \frac{6b(1+b^2)P(-4+b^2+b^4+4bP)}{-1+b^2} \right\},$$
$$C_{0,3}^{1^2,1} = \left\{ -\frac{12b(1+b^2)P(-1+2bP)}{-1+b^2} \right\},$$
$$C_{0,3}^{3,0} = \left\{ -\frac{(1+b^2)(12+b(b(1+b^2)(-8+b^2+b^4)+12(-3+b^2+b^4)P+24bP^2))}{2-3b^2+b^4} \right\},$$
$$C_{0,3}^{1^3,0} = \left\{ -\frac{12b(1+b^2)P(-1+2bP)}{1-3b^2+2b^4} \right\},$$
$$C_{0,3}^{\{2,1\},0} = \left\{ \frac{12b(1+b^2)P(-5+3b^2+2b^4+6bP)}{2-5b^2+2b^4} \right\},$$
$$C_{0,1^3}^{1,1^2} = \left\{ 6b^2\left(1+b^2\right)(b-2P)P \right\},$$
$$C_{0,1^3}^{2,1} = \left\{ -\frac{12b^4(1+b^2)(b-2P)P}{-1+b^2} \right\},$$
$$C_{0,1^3}^{1^2,1} = \left\{ \frac{6b(1+b^2)(-1+b^2(-1+4b(b-P)))P}{-1+b^2} \right\},$$
$$C_{0,1^3}^{3,0} = \left\{ \frac{12b^6(1+b^2)(b-2P)P}{2-3b^2+b^4} \right\},$$
$$C_{0,1^3}^{1^3,0} = \left\{ -\frac{1}{1-3b^2+2b^4}\left(1+b^2\right)\left(1+b^2(2+b(12P+b(-7+4b(b(-2+3(b-2P)(b-P))+3P))))\right) \right\},$$
$$C_{0,1^3}^{\{2,1\},0} = \left\{ -\frac{12b^3(1+b^2)P(-2+b^2(-3+5b^2-6bP))}{2-5b^2+2b^4} \right\},$$

$$C^{1,1^2}_{0,\{2,1\}} = \left\{\frac{2b^2\left(-2+b^2\right)\left(1+b^2\right)(b-2P)P}{-1+b^2}\right\},$$
$$C^{1,2}_{0,\{2,1\}} = \left\{-\frac{2b\left(-1+b^2+2b^4\right)P(-1+2bP)}{-1+b^2}\right\},$$
$$C^{2,1}_{0,\{2,1\}} = \left\{\frac{4b\left(1+b^2\right)^2P\left(-1+b^2+2bP\right)}{-1+b^2}\right\},$$
$$C^{1^2,1}_{0,\{2,1\}} = \left\{\frac{4b\left(1+b^2\right)^2P\left(-1+b^2-2bP\right)}{-1+b^2}\right\},$$
$$C^{3,0}_{0,\{2,1\}} = \left\{-\frac{2b^3\left(1+b^2\right)P\left(-3+2b\left(b+b^3+3P\right)\right)}{2-3b^2+b^4}\right\},$$
$$C^{\{2,1\},0}_{0,\{2,1\}} = \left\{-\frac{\left(1+b^2\right)\left(4-17b^4+4b^8-2b\left(4+3b^2+3b^4+4b^6\right)P-36b^4P^2\right)}{2-5b^2+2b^4}\right\},$$
$$C^{1^3,0}_{0,\{2,1\}} = \left\{\frac{2b\left(1+b^2\right)\left(-2+b^2(-2+3b(b-2P))\right)P}{1-3b^2+2b^4}\right\}$$

In the above expressions, 0 labels $\{\varnothing\}$.

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