A NOTE ON ESSENTIAL SMOOTHNESS IN THE HESTON MODEL

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Abstract. This note studies an issue relating to essential smoothness that can arise when the theory of large deviations is applied to a certain option pricing formula in the Heston model. The note identifies a gap, based on this issue, in the proof of Corollary 2.4 in [2] and describes how to circumvent it. This completes the proof of Corollary 2.4 in [2] and hence of the main result in [2], which describes the limiting behaviour of the implied volatility smile in the Heston model far from maturity.

1. Introduction

In [2] the authors study the limiting behaviour of the implied volatility in the Heston model as maturity tends to infinity. The main aim of this note is to give a rigorous account of the relationship between the concept of essential smoothness and the large deviation principle for the family of random variables $(X_t/t \pm E_\lambda/t)_{t\geq 1}$, where the process X denotes the log-spot in Heston model (5) and E_λ is an exponential random variable with parameter $\lambda > 0$ independent of X. This note fills a gap in the proof of Corollary 2.4 in [2] and hence completes the proof of the main result in [2], which describes the limiting behaviour of the implied volatility smile in the Heston model far from maturity.

The note is organized as follows. Section 2 describes the relevant concepts of the large deviation theory and discusses how the effective domain changes when a family of random variables is perturbed by an independent exponential random variable. Section 3 discusses the failure of essential smoothness when the Heston model is perturbed by an independent exponential, which is what causes the gap in the proof of Corollary 2.4 in [2]. Section 3 also proves Theorem 3, which fills the gap.

2. The large deviation principle for random variables in $\mathbb{R}$

We briefly recall the basic facts of the large deviation theory in $\mathbb{R}$ (see monograph [1, Ch. 2] for more details). Let $(Z_t)_{t\geq 1}$ be a family of random variables with $Z_t \in \mathbb{R}$. J is a *rate function* if it is lower semicontinuous and $J(\mathbb{R}) \subset [0,\infty]$ holds. The family $(Z_t)_{t\geq 1}$ satisfies the *large deviation principle (LDP)* with the *rate function* J if for every Borel set $B \subset \mathbb{R}$ we have

$$(1) \qquad -\inf_{x\in B^\circ} J(x) \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathsf{P}\left[Z_t \in B\right] \leq \limsup_{t\to\infty} \frac{1}{t} \log \mathsf{P}\left[Z_t \in B\right] \leq -\inf_{x\in \overline{B}} J(x),$$

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with the convention $\inf \emptyset = \infty$ the relative notions of interior (interior B°, closure $\overline{B}$ and boundary $\overline{B} \setminus B^\circ$ are in the topology of $\mathbb{R}$).

The Gärtner-Ellis theorem (Theorem 1 below) gives sufficient conditions for a family $(Z_t)_{t\geq 1}$ to satisfy the LDP (see monograph [1, Section 2.3] for details). Let $\Lambda_t(u) := \log \mathsf{E}\left[e^{uZ_t}\right] \in (-\infty, \infty]$ be the cumulant generating function of Z_t. Assume that for every $u \in \mathbb{R}$

$$\Lambda(u) := \lim_{t\to\infty} \Lambda_t(tu)/t \quad \text{exists in } [-\infty,\infty] \qquad \text{and} \qquad 0 \in \mathcal{D}_\Lambda^\circ, \tag{2}$$

where $\mathcal{D}_\Lambda := \{u \in \mathbb{R} : \Lambda(u) < \infty\}$ is the *effective domain* of Λ and $\mathcal{D}_\Lambda^\circ$ is its interior. The *Fenchel-Legendre transform* Λ^* of the convex function Λ is defined by the formula

$$\Lambda^*(x) \quad := \quad \sup\{ux - \Lambda(u) \, : \, u \in \mathbb{R}\} \quad \text{for} \quad x \in \mathbb{R}. \tag{3}$$

Under the assumption in (2), Λ^* is lower semicontinuous with compact level sets $\{x : \Lambda^*(x) \leq \alpha\}$ (see [1, Lemma 2.3.9(a)]) and $\Lambda^*(\mathbb{R}) \subset [0, \infty]$ and hence satisfies the definition of a *good rate function*. We now state the Gärtner-Ellis theorem (see [1, Section 2.3] for its proof).

Theorem 1. *Let the random variables $(Z_t)_{t\geq 1}$ satisfy the assumption in (2). If Λ is essentially smooth and lower semicontinuous, then LDP holds for $(Z_t)_{t\geq 1}$ with the good rate function Λ^*.*

The function $\Lambda : \mathbb{R} \to (-\infty, \infty]$ defined in (2) is *essentially smooth* if it is (a) differentiable in $\mathcal{D}_\Lambda^\circ$ and (b) *steep*, i.e. $\lim_{n\to\infty} |\Lambda'(u_n)| = \infty$ for every sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_\Lambda^\circ$ that converges to a boundary point of $\mathcal{D}_\Lambda^\circ$. If $\mathcal{D}_\Lambda^\circ$ is a strict subset of $\mathbb{R}$, which is the case in the setting of [2] (see also Section 3 below), essential smoothness, which plays a key role in the proof of Theorem 1, is not automatic.

The following question is of central importance in [2]: does the LDP persist if a family of random variables $(Z_t)_{t\geq 1}$ is perturbed by an independent exponential random variable E_1? It is implicitly assumed in the proof of Corollary 2.4 in [2] (see the last line on page 17 and lines 4 and 14 on page 18) that if $(Z_t)_{t\geq 1}$ satisfies the assumptions of Theorem 1, then so do the families $(Y_t^{1+})_{t\geq 1}$ and $(Y_t^{1-})_{t\geq 1}$, where $Y_t^{1\pm} = Z_t \pm E_1/t$, and the LDP is applied. In particular the authors in [2] assume that the limiting cumulant generating functions of $(Y_t^{1\pm})_{t\geq 1}$ are essentially smooth. However the following simple lemma holds.

Lemma 2. *Let $(Z_t)_{t\geq 1}$ satisfy the assumption in (2) with a limiting cumulant generating function Λ. Let $\lambda > 0$ and E_λ an exponential random variable independent of $(Z_t)_{t\geq 1}$ with $\mathsf{E}[E_\lambda] = 1/\lambda$ and let $Y_t^{\lambda\pm} := Z_t \pm E_\lambda/t$. Then the families of random variables $(Y_t^{\lambda\pm})_{t\geq 1}$ satisfy the assumption in (2) and the corresponding limiting cumulant generating functions are given by*

$$\Lambda^{\lambda+}(u) = \begin{cases} \Lambda(u), & \text{if } u \in \mathcal{D}_\Lambda \cap (-\infty, \lambda), \\ \infty, & \text{otherwise}, \end{cases} \qquad \text{and} \quad \Lambda^{\lambda-}(u) = \begin{cases} \Lambda(u), & \text{if } u \in \mathcal{D}_\Lambda \cap (-\lambda, \infty), \\ \infty, & \text{otherwise}. \end{cases}$$

Remarks. **(a)** Let $(Z_t)_{t\geq 1}$ satisfy the assumption in (2) and assume further that Λ is differentiable in $\mathcal{D}_\Lambda^\circ$. If $1 \in \mathcal{D}_\Lambda^\circ$, then the right-hand boundary point of the interior of the effective domain $\mathcal{D}_{\Lambda^{1+}}^\circ$ is equal to 1 and Lemma 2 implies that the limiting cumulant generating function Λ^{1+} of $(Y_t^{1+})_{t\geq 1}$ is

- neither essentially smooth, since Λ^{1+} is not steep at 1,
- nor lower semicontinuous at 1, since it is differentiable in $\mathcal{D}_{\Lambda^{1+}}^\circ$ with $\Lambda^{1+}(1) = \infty$.

Loss of steepness and lower semicontinuity occurs also for $(Y_t^{1-})_{t\geq 1}$ in the case where $-1 \in \mathcal{D}_\Lambda^\circ$.

(b) Lemma 2 implies that if $(Z_t)_{t\geq 1}$ satisfies the assumptions of Theorem 1 *and* $\mathcal{D}_\Lambda$ is contained in $(-\infty, \lambda)$, for some $\lambda > 0$, then $(Y_t^{\lambda+})_{t\geq 1}$ also satisfies the assumptions of Theorem 1 and hence the LDP with a good rate function Λ^*. An analogous statement holds for $(Y_t^{\lambda-})_{t\geq 1}$.

Proof. Note that $\log \mathsf{E}\left[e^{uE_\lambda}\right]$ is finite and equal to $\log\left(\lambda/(\lambda-u)\right)$ if and only if $u \in (-\infty, \lambda)$. For all large t and $u \in \mathcal{D}_\Lambda \cap (-\infty, \lambda)$, the assumption in (2) implies that $\Lambda_t^{\lambda+}(tu) = \log \mathsf{E}\left[\exp\left(tuY_t^{\lambda+}\right)\right]$ is finite and that the formula holds

$$\Lambda_t^{\lambda+}(tu) = \Lambda_t(tu) + \log\frac{\lambda}{\lambda - u}, \qquad \text{where} \qquad \Lambda_t(tu) = \log \mathsf{E}\left[\exp\left(tuZ_t\right)\right]. \tag{4}$$

The inequality $u \geq \lambda$ implies that, since $\Lambda_t(tu) > -\infty$, we have $\Lambda_t^{\lambda+}(tu) = \infty$ for all t and hence $\Lambda^{\lambda+}(u) = \infty$. If $u \in (\mathbb{R} \setminus \mathcal{D}_\Lambda) \cap (-\infty, \lambda)$, then (4) yields $\Lambda^{\lambda+}(u) = \lim_{t\nearrow\infty} \Lambda_t^{\lambda+}(tu)/t = \infty$. This proves the lemma for $(Y_t^{\lambda+})_{t\geq 1}$. The case of $(Y_t^{\lambda-})_{t\geq 1}$ is analogous. □

3. Essential smoothness can fail

The Heston model $S = e^X$ is a stochastic volatility model with the log-stock process X given by

$$dX_t = -\frac{Y_t}{2}dt + \sqrt{Y_t}dW_t^1 \quad \text{and} \quad dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t^2, \tag{5}$$

where $\kappa, \theta, \sigma > 0$, $Y_0 = y_0 > 0$, $X_0 = x_0 \in \mathbb{R}$ and W^1, W^2 are standard Brownian motions with correlation $\rho \in (-1, 1)$. The standing assumption

$$\rho\sigma - \kappa < 0, \tag{6}$$

is made in [2] (see equation (2.2) in Theorem 2.1 on page 5 of [2]). In particular the inequality in (6) implies that S is a strictly positive true martingale and allows the definition of the share measure $\widetilde{\mathsf{P}}$ via the Radon-Nikodym derivative $d\widetilde{\mathsf{P}}/d\mathsf{P} = e^{X_t - x_0}$.

The authors' aim in [2] is to obtain the limiting implied volatility smile as maturity tends to infinity at the strike $K = S_0e^{xt}$ for any $x \in \mathbb{R}$ in the Heston model. Their main formula is given in Corollary 3.1 of [2]. A key step in the proof of [2, Corollary 3.1] is given by [2, Corollary 2.4]. In the proof of [2, Corollary 2.4] (see last line on page 17 and lines 4 and 14 on page 18) it is implicitly assumed that the LDP for $(X_t/t_{t\geq 1}$ implies the LDP for the family $(X_t/t \pm E_1/t)_{t\geq 1}$. However, as we have seen in Section 2 (see remarks following Lemma 2), Theorem 1 cannot be applied directly to the family $(X_t/t \pm E_1/t)_{t\geq 1}$, even if $(X_t/t)_{t\geq 1}$ satisfies its assumptions. We start with a precise description of the problem and present the solution in Theorem 3.

Remarks. **(i)** Under (6), a simple calculation implies that Λ and $\mathcal{D}_\Lambda$ of the family $(X_t/t)_{t\geq 1}$ are:

$$\Lambda(u) = -\frac{\theta\kappa}{\sigma^2}\left(u\rho\sigma - \kappa + \sqrt{\Delta(u)}\right) \quad \text{for} \quad u \in \mathcal{D}_\Lambda \qquad \text{and} \qquad \mathcal{D}_\Lambda = [u_-, u_+] \quad \text{where} \tag{7}$$

$$u_\pm = \left(1/2 - \rho\kappa/\sigma \pm \sqrt{(\kappa/\sigma - \rho)\,\kappa/\sigma + 1/4}\right) / \left(1-\rho^2\right) \quad \text{with} \quad u_- < 0 < 1 < u_+. \tag{8}$$

In (7) the function Δ is a quadratic $\Delta(u) = (u\rho\sigma - \kappa)^2 - \sigma^2(u^2 - u)$ and the boundary points u_+ and u_- of the effective domain $\mathcal{D}_\Lambda$ are its zeros. Elementary calculations show that Λ is essentially smooth and that the unique minimum of Λ^* is attained at $\Lambda'(0) = -\theta/2$. Therefore $(X_t/t)_{t\geq 1}$ satisfies the LDP with the good rate function Λ^*, defined in (3), by Theorem 1.

(ii) Under the share measure $\widetilde{\mathsf{P}}$, given by $\mathrm{d}\widetilde{\mathsf{P}}/\mathrm{d}\mathsf{P} = \mathrm{e}^{X_t - x_0}$, we have $\widetilde{\mathsf{E}}\left[\mathrm{e}^{uX_t}\right] = \mathrm{e}^{-x_0}\mathsf{E}\left[\mathrm{e}^{(u+1)X_t}\right]$ for all $u \in \mathbb{R}$ and $t > 0$ and hence the family $(X_t/t)_{t\geq 1}$ under $\widetilde{\mathsf{P}}$ satisfies the assumption in (2) with the limiting cumulant generating function $\widetilde{\Lambda}(u) = \Lambda(u+1)$, $\mathcal{D}_{\widetilde{\Lambda}} = [u_- - 1, u_+ - 1]$. As before, $(X_t/t)_{t\geq 1}$ satisfies the LDP under $\widetilde{\mathsf{P}}$ with the strictly convex good rate function $\widetilde{\Lambda}$, which satisfies $\widetilde{\Lambda}^*(x) = \Lambda^*(x) - x$ for all $x \in \mathbb{R}$ and attains its unique minimum at $\widetilde{\Lambda}'(0) = \Lambda'(1) = \theta\kappa/(\kappa - \rho\sigma)$.

Theorem 3. *Let the process X be given by* (5) *and assume that* (6) *holds. Let E_1 be the exponential random variable with $\mathsf{E}[E_1] = 1$, which is independent of X. Then the following limits hold:*

$$\lim_{t\nearrow\infty}\frac{1}{t}\log\mathsf{P}\left[X_t - x_0 + E_1 < xt\right] = -\Lambda^*(x) \qquad \textit{for} \quad x \leq \Lambda'(0) = -\theta/2; \tag{9}$$

$$\lim_{t\nearrow\infty}\frac{1}{t}\log\widetilde{\mathsf{P}}\left[X_t - x_0 - E_1 > xt\right] = x - \Lambda^*(x) \qquad \textit{for} \quad x \geq \Lambda'(1) = \theta\kappa/(\kappa - \rho\sigma); \tag{10}$$

$$\lim_{t\nearrow\infty}\frac{1}{t}\log\widetilde{\mathsf{P}}\left[X_t - x_0 - E_1 \leq xt\right] = x - \Lambda^*(x) \qquad \textit{for} \quad x \in \left[\Lambda'(0), \Lambda'(1)\right]; \tag{11}$$

where Λ is given in (7)*, its Fenchel-Legendre transform Λ^* is defined in* (3) *and* $\mathrm{d}\widetilde{\mathsf{P}}/\mathrm{d}\mathsf{P} = \mathrm{e}^{X_t - x_0}$.

Remark. The limits in Theorem 3 are precisely the limits that arise in the proof of [2, Corollary 2.4] (see the last line on page 17 and lines 4 and 14 on page 18) and are claimed to hold since the family $(X_t/t)_{t\geq 1}$ satisfies the LDP under P and $\widetilde{\mathsf{P}}$ by Remarks (i) and (ii) above and Theorem 1. However Lemma 2 implies that the limiting cumulant generating function Λ^{1+} of the family of random variables $(Z_t + E_1/t)_{t\geq 1}$, where $Z_t = (X_t - x_0)/t$, is neither lower semicontinuous nor essentially smooth. Hence Theorem 1 cannot be applied to $(Z_t + E_1/t)_{t\geq 1}$. An anologous issue arises under the measure $\widetilde{\mathsf{P}}$.

Proof. The basic idea of the proof is simple: for (9) we sandwich the probability $\mathsf{P}\left[X_t - x_0 + E_1 < xt\right]$ between two tail probabilities of two families of random variables, which satisfy the LDP with the same rate function Λ^* by Lemma 2 and Theorem 1. The limits in (10) and (11) follow similarly.

For given parameter values in the Heston model pick $\lambda > u_+$, where u_+ is defined in (8). Let E_λ be an exponential random variable with $\mathsf{E}[E_\lambda] = 1/\lambda$, defined on the same probability space as X and E_1 and independent of both. Since $u_+ > 1$, we have the elementary inequality

$$\mathsf{P}\left[E_\lambda < \alpha\right] = I_{\{\alpha>0\}}\left(1-\mathrm{e}^{-\lambda\alpha}\right) \leq I_{\{\alpha>0\}}\left(1-\mathrm{e}^{-\alpha}\right) = \mathsf{P}\left[E_1 < \alpha\right] \qquad \text{for any} \quad \alpha \in \mathbb{R}. \tag{12}$$

The inequality

$$\mathsf{P}\left[X_t - x_0 + E_\lambda < xt\right] \quad \leq \quad \mathsf{P}\left[X_t - x_0 + E_1 < xt\right] \tag{13}$$

follows by conditioning on X_t and applying (12). On the other hand, since $E_1 > 0$ a.s., we have

$$\mathsf{P}\left[X_t - x_0 + E_1 < xt\right] \quad \leq \quad \mathsf{P}\left[X_t - x_0 < xt\right]. \tag{14}$$

Lemma 2 implies that the families of random variables $(Z_t + E_\lambda/t)_{t\geq 1}$ and $(Z_t)_{t\geq 1}$, where $Z_t = (X_t - x_0)/t$, both have the limiting cumulant generating function equal to Λ given in (7) with the effective domain $\mathcal{D}_\Lambda = [u_-, u_+]$. Since Λ is essentially smooth and lower semicontinuous on $\mathcal{D}_\Lambda$ and the assumption in (2) is satisfied, Theorem 1 implies that $(Z_t + E_\lambda/t)_{t\geq 1}$ and $(Z_t)_{t\geq 1}$, satisfy the LDP with the good rate function Λ^*. Since x in (9) is assumed to be less or equal to the unique minimum $\Lambda'(0) = -\theta/2$ of Λ^* (see Remark (i) above) and Λ^* is non-negative and strictly convex, the LDP (see the inequalities in (1)) and the inequalities in (13) and (14) imply the limit in (9).

To prove (10) pick $\lambda > 1 - u_-$ and note that the inequality in (12) and conditioning on X_t yield

$$\widetilde{\mathsf{P}}\left[X_t - x_0 > xt\right] \geq \widetilde{\mathsf{P}}\left[X_t - x_0 - E_1 > xt\right] \geq \widetilde{\mathsf{P}}\left[X_t - x_0 - E_\lambda > xt\right]. \tag{15}$$

As before, Lemma 2 and Theorem 1 imply that $(Z_t - E_\lambda/t)_{t\geq 1}$ and $(Z_t)_{t\geq 1}$ satisfy the LDP with the convex rate function $\widetilde{\Lambda}^*$, which by Remark (ii) above attains its unique minimum at $\Lambda'(1) = \theta\kappa/(\kappa - \rho\sigma)$. Since $x \geq \Lambda'(1)$ in (10), the limit follows. A similar argument implies the limit in (11) for all $x \in [\Lambda'(0), \Lambda'(1)]$, which concludes the proof. □

References

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