Lattice quantum electrodynamics for graphene

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The effects of gauge interactions in graphene have been analyzed up to now in terms of effective models of Dirac fermions. However, in several cases lattice effects play an important role and need to be taken consistently into account. In this paper we introduce and analyze a lattice gauge theory model for graphene, which describes tight binding electrons hopping on the honeycomb lattice and interacting with a three-dimensional quantum $U(1)$ gauge field. We perform an exact Renormalization Group analysis, which leads to a renormalized expansion that is finite at all orders. The flow of the effective parameters is controlled thanks to Ward Identities and a careful analysis of the discrete lattice symmetry properties of the model. We show that the Fermi velocity increases up to the speed of light and Lorentz invariance spontaneously emerges in the infrared. The interaction produces critical exponents in the response functions; this removes the degeneracy present in the non interacting case and allow us to identify the dominant excitations. Finally we add mass terms to the Hamiltonian and derive by a variational argument the correspondent gap equations, which have an anomalous non-BCS form, due to the non trivial effects of the interaction.

keywords: Graphene, Lattice Gauge Theory, honeycomb lattice, Kekulé mass generation, Ward Identities, Renormalization Group, Critical exponents

CONTENTS

1. INTRODUCTION

Graphene, a two dimensional crystal of carbon atoms that has been recently experimentally realized [45, 46], has highly unusual electronic properties. The lattice in graphene has a honeycomb shape and the corresponding energy bands intersect at two points, close to which the effective dispersion relation is approximately conical and similar to a "relativistic" one. The low energy excitations of the half-filled system consist of hole-particle pairs created close to the tips of these two cones. These quasi-particles behave like two-dimensional (2D) massless Dirac fermions: for this reason the infrared (IR) properties of the system can be understood in terms of a model of 2D Dirac particles in the continuum [51, 56]. As a result, a number of concepts and features of high energy physics have a correspondence and can be observed at much lower energies in this crystal.

The description in terms of Dirac fermions is quite accurate in the free case, and it is also helpful in the presence of many-body interactions (see, e.g., [7] for a review), as it allows to translate and adopt a number of powerful methods from the realm of quantum field theory (QFT) to that of graphene. However, taking the effective description too seriously has some drawbacks, since a model of interacting 2D Dirac fermions in the continuum has spurious *ultraviolet divergences* due to the linear bands. In order to make the continuum theory well-defined, ad hoc *regularizations* must be introduced to cure the short distance singularities, which are obviously absent in the tight binding model, where the honeycomb lattice acts naturally as an ultraviolet (UV) cut-off. It is unfortunate that the computation within the Dirac model of certain physical observables, such as the conductivity, is sensitive to the specific choice of the regularization scheme, a fact that makes the comparison with experiments difficult or inaccurate, see, e.g., [35].

These ambiguities make an approximation-free analysis of the effects of the lattice and of the non-linear bands in graphene highly desirable. In this paper, we study a lattice gauge model for graphene that describes electrons hopping on the honeycomb lattice and weakly interacting with a three-dimensional (3D) quantum $U(1)$ gauge field. Our model has two independent parameters: the bare Fermi velocity v and the electric charge e (we use units such that the reduced Planck constant $\hbar$ and the speed of light c are equal to 1). The analysis is performed by using exact Renormalization Group (RG) methods, which allow us to express the physical observables in terms of renormalized expansions in the electric charge with *finite* coefficients at all orders, uniformly in the volume and in the temperature (more precisely, we show that the coefficient multiplying e^{2n} in the renormalized expansion grows at most as $n!$). Our main physical predictions are the following.

1. Thanks to the validity of exact lattice *Ward Identities*, the gauge field remains massless and the IR behavior of the system is characterized by a *line of fixed points* (i.e., the effective charge has vanishing beta function). Correspondingly, the physical parameters are strongly renormalized by the interaction. In particular, *the Fermi velocity increases up to the speed of light* and Lorentz invariance spontaneously emerges in the infrared. This is proved by fully taking into account the discrete lattice symmetries, which are used to exclude the presence of dangerous extra marginal or relevant terms in the RG flow.

2. *The wave function renormalization diverges at the Fermi points with an anomalous exponent.* This last properties strongly resembles one of the crucial features of one-dimensional *Luttinger liquids*; in this sense, the model considered in this paper is one of the very few established examples of Luttinger liquid behavior in *two* dimensions (it has been suggested that also the Hubbard model on the square lattice close to half filling is a Luttinger liquid, but this is still an unproven fact).

3. The response functions have an *anomalous behavior* expressed in terms of non trivial scaling exponents. The difference between the interacting and non-interacting exponents is small at small coupling. In particular, the response functions associated to fermionic bilinears, which decay as r^{-4} at large distances in the non-interacting case, remain integrable even in the presence of weak interactions; therefore, magnetic, phonon or superconducting susceptibilities are finite and *no evidence for quantum instabilities* is found, in agreement with the fact that the fixed point is close to the trivial one at weak coupling.

4. On the other hand, the interaction removes the degeneracy in the decay exponents of the response functions: some exponent increase and some other decrease and this depresses or enhances the effects of local perturbations associated to specific fermionic bilinears. This gives us a criterium to extrapolate the qualitative behavior of the system at larger values of the electric charge and decide what are the favorite *quantum instabilities at intermediate to strong coupling.* An explicit computation of the anomalous exponents shows that the *dominant excitations* correspond to: (i) *Kekulé distortions*, associated to a dimerized Peierls' pattern, (ii) *charge-density waves* associated to an excess/deficit of the electron density on the two sublattices of the honeycomb lattice, (iii) *Neél antiferromagnetism*, (iv) the *Haldane circulating currents* [27]. In all these cases, the logarithmic singularity at zero transferred momentum in the first derivative of the response function in momentum space is changed into a power law singularity with an anomalous exponent. The singularity in the other responses is either absent or weaker than in the four cases mentioned above.

5. If we add a symmetry breaking field coupled to a Kekulé distortion the induced energy gap in the spectrum is dramatically amplified by the interaction: the ratio between the energy gap and the amplitude Δ_0 of the external field diverges as $\Delta_0 \to 0$ with an anomalous power law.

6. The effect of the electronic repulsion on the Peierls-Kekulé instability, usually neglected, is evaluated by deriving an exact non-BCS gap equation, from which evidence is found that the gauge interaction facilitates the spontaneous distortion of the lattice and the gap generation. Similar conclusions can be drawn for the mass terms corresponding to the other dominant excitations.

The exact Wilsonian RG methods we use are based on ideas borrowed from constructive QFT and have been introduced in the context of interacting Fermi systems in [3] (and then extended in [47, 54]) at the beginning of the 90s, and since then successfully applied to various problems in solid state physics, see, e.g., [39] for an updated review. In particular, they appear to be very well suited to analyze the properties of graphene without any Dirac approximation, large-N approximations or unphysical regularization schemes. Such methods have been first applied in [15] to the Hubbard model on the honeycomb lattice, where we proved in a full *non-perturbative* fashion the analyticity of the ground state of the half-filled system; we used the same methods to rigorously establish the universality of the optical conductivity of graphene with weak short range interactions [20, 21], an issue that still needs to be fully understood in the case of electromagnetic interactions. Technically, the analysis in this paper extends the one in [18], the main differences being that here we fully exploit the lattice symmetries and, besides computing the reduced density matrices, we construct the response functions, we compute the critical exponents and discuss the effects of external symmetry breaking fields. Most of the results in this paper were announced in [19].

Let us add a comment on the range of applicability of our theory. Our analysis is based on resummations of perturbation theory in α, where $\alpha = e^2/(4\pi\varepsilon\hbar v)$ is the effective fine structure constant of graphene, which unfortunately is not small: e.g., in suspended graphene, $\alpha \simeq 2$, which makes graphene an intrinsically *strongly coupled* problem, apriori not accessible to approaches based on power series expansions. However, one needs to take into account that the effective Fermi velocity is considerably increased by the interactions (recent experiments [12] can observe a factor three amplification of the velocity close to the Fermi points and even a larger enhancement is expected at lower energies), an effect that goes in the direction of decreasing the effective fine structure constant. Therefore, our theory is valid close to the IR fixed point, with effective parameters e and v (v close to the speed of light) that should be thought as being obtained by the (non-perturbative) integration of the "first few IR scales", possibly by using numerical methods, like those of [9, 10]. Let us also remark that since we predict the emergence of anomalous critical exponents, our final results can be easily extrapolated to intermediate coupling, which would not be the case if the apparent logarithmic divergences emerging in perturbation theory were not correctly resummed at all orders.

The rest of the paper is organized as follows. In Section 2, we define the model, we present in detail our main results and compare them with existing literature. In Sections 3 to 9 we discuss the proof of our results: in Section 3 we discuss the functional integral representation of our model and derive the relevant Ward Identities (WIs); in Section 4 we describe our RG scheme to compute the functional integral; in Section 5 we show how to use WIs to prove the vanishing of the photon mass, the vanishing of the beta function for the effective charge, and how to control the flow of the renormalization parameters (Fermi velocity, wave function renormalization, vertex functions); in Sections 6 and 7 we compute the response functions associated to several fermionic bilinears; in Sections 8 and 9 we explain how to modify the general RG scheme to include the effects of an external field coupled to local order parameters and how to derive the non-BCS equation for the gap. A number of more technical aspects of the proof are deferred to the Appendices: in Appendix A we derive the functional integral representation, we derive the WIs and we explicitly show the equivalence between the Feynman and Coulomb gauges; in Appendices B and C we analyze the symmetry properties of the theory and use them to identify the symmetry structure of the relevant and marginal kernels; in Appendix D we perform the lowest order computations of the beta function and of the critical exponents; finally, for completeness, in Appendix E we check at lowest order the cancellation of the photon mass and of the charge beta function, which follows from the general gauge invariance of the model.

2. A LATTICE GAUGE THEORY FOR GRAPHENE

In this section we introduce the model, define the main quantities of interest and present our main results. A comparison with existing literature is also presented.

A. The model

We let $\Lambda = \{n_1\vec{l}_1 + n_2\vec{l}_2 : n_i = 0, \ldots, L-1\}$ be a periodic triangular lattice of period L, with basis vectors $\vec{l}_1 = \frac{1}{2}(3, \sqrt{3})$, $\vec{l}_2 = \frac{1}{2}(3, -\sqrt{3})$. We denote by $\Lambda_A = \Lambda$ and $\Lambda_B = \Lambda + \vec{\delta}_i$ the A- and B- sublattices of the honeycomb lattice, with $\vec{\delta}_i$ the nearest neighbors vectors defined as:

$$\vec{\delta}_1 = (1,0) \,, \qquad \vec{\delta}_2 = \frac{1}{2}(-1,\sqrt{3}) \,, \qquad \vec{\delta}_3 = \frac{1}{2}(-1,-\sqrt{3}) \,. \tag{2.1}$$

We introduce creation and annihilation fermionic operators for electron sitting at the sites of the A- and B- sublattices

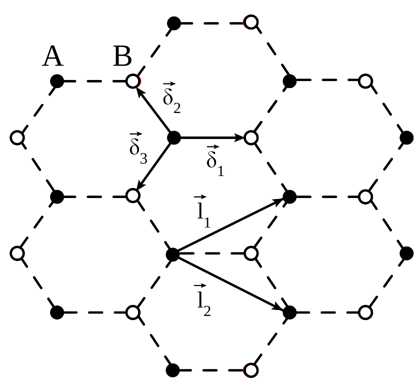

FIG. 1. The honeycomb lattice of graphene.

with spin index $\sigma =\uparrow\downarrow$ as

$$a^{\pm}_{\vec{x},\sigma} = L^{-2} \sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}\vec{x}} \hat{a}^{\pm}_{\vec{k},\sigma} \,, \qquad \vec{x} \in \Lambda_A \,,$$
$$b^{\pm}_{\vec{x},\sigma} = L^{-2} \sum_{\vec{k}\in\mathcal{B}_L} e^{\pm i\vec{k}(\vec{x}-\vec{\delta}_1)} \hat{b}^{\pm}_{\vec{k},\sigma} \,, \qquad \vec{x} \in \Lambda_B \,, \tag{2.2}$$

where $\mathcal{B}_L = \{\vec{k} = n_1\vec{G}_1/L + n_2\vec{G}_2/L : 0 \le n_i < L\}$, with $\vec{G}_{1,2} = \frac{2\pi}{3}(1, \pm\sqrt{3})$, is the first Brillouin zone; note that in the thermodynamic limit $L^{-2}\sum_{\vec{k}\in\mathcal{B}_L} \to |\mathcal{B}|^{-1}\int_{\mathcal{B}} d\vec{k}$, with $\mathcal{B} = \{\vec{k} = \xi_1\vec{G}_1 + \xi_2\vec{G}_2 : \xi_i \in [0,1)\}$ and $|\mathcal{B}| = 8\pi^2/(3\sqrt{3})$.

The operators $a^\pm_{\vec{x},\sigma}$, $b^\pm_{\vec{x},\sigma}$ satisfy the canonical anticommutation rules, and are periodic over Λ; their Fourier transforms are normalized in such a way that, if $\vec{k},\vec{k}'\in\mathcal{B}_L$,

$$\{\hat a^+_{\vec{k},\sigma},\hat a^+_{\vec{k}',\sigma'}\}=\{\hat a^-_{\vec{k},\sigma},\hat a^-_{\vec{k}',\sigma'}\}=\{\hat b^+_{\vec{k},\sigma},\hat b^+_{\vec{k}',\sigma'}\}=\{\hat b^-_{\vec{k},\sigma},\hat b^-_{\vec{k}',\sigma'}\}=0\;,\qquad \{\hat a^-_{\vec{k},\sigma},\hat a^+_{\vec{k}',\sigma'}\}=\{\hat b^-_{\vec{k},\sigma},\hat b^+_{\vec{k}',\sigma'}\}=L^2\delta_{\vec{k},\vec{k}'}\delta_{\sigma,\sigma'}\;. \tag{2.3}$$

Definition Eq.(2.2) implies that $\hat a^\pm_{\vec{k},\sigma}$, $\hat b^\pm_{\vec{k},\sigma}$ are periodic over the reciprocal lattice Λ^*.

We also introduce a quantized photon field living in the 3D continuum. Let $\bar{\mathcal{S}}_{L,L'}=\mathcal{S}_L\times[-\frac{L'}{2},\frac{L'}{2})$ be a subset of $\mathbb{R}^3$ with periodic boundary conditions and $\mathcal{S}_L=\{\vec{x}=L\xi_1\vec{l}_1+L\xi_2\vec{l}_2:\xi_i\in[0,1)\}$; let also $\bar{\mathcal{D}}_{L,L'}=\mathcal{D}_L\times\frac{2\pi}{L'}\mathbb{Z}$ be the corresponding dual momentum space, with $\mathcal{D}_L=\{\vec{p}=n_1\vec{G}_1/L+n_2\vec{G}_2/L:n_i\in\mathbb{Z}\}$ (the honeycomb lattice can be thought as being contained in the section $\mathcal{S}_L\times 0$ at $z=0$ of $\bar{\mathcal{S}}_L$). For all $p=(\vec{p},p_3)\in\bar{\mathcal{D}}_{L,L'}$ we introduce bosonic creation and annihilation operators $\hat c^\pm_{p,r}$, with helicity index $r=1,2$; they satisfy the commutation relation

$$\left[\hat c^+_{p,r},\hat c^+_{p',r'}\right]=\left[\hat c^-_{p,r},\hat c^-_{p',r'}\right]=0\;,\qquad\left[\hat c^-_{p,r},\hat c^+_{p',r'}\right]=L'|\mathcal{S}_L|\delta_{p,p'}\delta_{r,r'}\;. \tag{2.4}$$

Let $A(x)=(\vec{A}(x),A_3(x))$ be the quantized vector potential on $\bar{\mathcal{S}}_L$ defined as:

$$A(x)=\frac{1}{L'|\mathcal{S}_L|}\sum_{p\in\bar{\mathcal{D}}_{L,L'}}\sum_{r=1,2}\sqrt{\frac{\chi(|p|)}{2|p|}}\varepsilon_{p,r}\left(\hat c^-_{p,r}e^{-ipx}+\hat c^+_{p,r}e^{ipx}\right)\;, \tag{2.5}$$

where $\varepsilon_{p,r}\in\mathbb{R}^3$ are polarization vectors satisfying the conditions

$$\varepsilon_{p,r}\cdot\varepsilon_{p,r'}=\delta_{r,r'}\;,\qquad\varepsilon_{p,r}\cdot p=0\;, \tag{2.6}$$

which reflect the choice of the *Coulomb gauge*. Note that, in the thermodynamic limit $|\bar{\mathcal{S}}_L|^{-1}\sum_{p\in\bar{\mathcal{D}}_L}\to(2\pi)^3\int_{\mathbb{R}^3}dp$. Moreover, the function $\chi(|p|)$ acts as an UV cutoff function: more exactly $\chi(t)$ is a smooth compact support function equal to 1 for $0\le t\le\frac{1}{2}$ and to 0 for $t\ge 1$. The photon UV cutoff is chosen to be on the same scale as the inverse lattice spacing. We expect that this cutoff should be removable, but this is not our main concern here.

The interacting electron-photon system we are interested in is described at half filling and in the Coulomb gauge by the following grandcanonical Hamiltonian:

$$H_\Lambda=H^h_\Lambda+H^f_\Lambda+V_\Lambda\;, \tag{2.7}$$

where the first term is the (gauge-invariant) hopping term, the second represents the field energy, and the third is the Coulomb interaction, namely:

$$H^h_\Lambda=-t\sum_{\substack{\vec{x}\in\Lambda_A\\ j=1,2,3}}\sum_{\sigma=\uparrow\downarrow}\left(e^{ie\int_0^1 ds\vec{A}(\vec{x}+s\vec{\delta}_j,0)\cdot\vec{\delta}_j}a^+_{\vec{x},\sigma}b^-_{\vec{x}+\vec{\delta}_j,\sigma}+e^{-ie\int_0^1 ds\vec{A}(\vec{x}+s\vec{\delta}_j,0)\cdot\vec{\delta}_j}b^+_{\vec{x}+\vec{\delta}_j,\sigma}a^-_{\vec{x},\sigma}\right)\;, \tag{2.8}$$

$$H^f_\Lambda=\frac{1}{L'|\mathcal{S}_L|}\sum_{\substack{p\in\bar{\mathcal{D}}_{L,L'}\\ r=1,2}}|p|\hat c^+_{p,r}\hat c^-_{p,r}\;,\qquad V_\Lambda=\frac{e^2}{2}\sum_{\vec{x},\vec{y}\in\Lambda_A\cup\Lambda_B}(n_{\vec{x}}-1)\varphi\big(\vec{x}-\vec{y}\big)(n_{\vec{y}}-1)\;,$$

where $t>0$ is the hopping strength, e is the electric charge, and $n_{\vec{x}}$ is equal to $\sum_{\sigma=\uparrow\downarrow}a^+_{\vec{x},\sigma}a^-_{\vec{x},\sigma}$ or to $\sum_{\sigma=\uparrow\downarrow}b^+_{\vec{x},\sigma}b^-_{\vec{x},\sigma}$ depending on whether $\vec{x}\in\Lambda_A,\Lambda_B$, respectively; moreover, $\varphi(\vec{x})$ is a regularized periodic version of the 3D Coulomb potential:

$$\varphi(\vec{x})=\frac{1}{L'|\mathcal{S}_L|}\sum_{p\in\bar{\mathcal{D}}_{L,L'}}\frac{\chi(|p|)}{p^2}e^{-i\vec{p}\cdot\vec{x}}\;, \tag{2.9}$$

where we remind the reader that $p=(\vec{p},p_3)$. Note that the electron-photon interaction is induced both by the complex hopping rate $t\exp\{ie\int_0^1 ds\vec{A}(\vec{x}+s\vec{\delta}_j)\cdot\vec{\delta}_j\}$ in H^h_Λ and by the static Coulomb interaction V_Λ; the combination of the two describes the retarded electromagnetic interaction mediated by 3D photons. If the electric charge e is 0, then the Hamiltonian decouples into a sum of two quadratic terms:

$$H_\Lambda\big|_{e=0}=-t\sum_{\substack{\vec{x}\in\Lambda\\ j=1,2,3}}\sum_{\sigma=\uparrow\downarrow}\left(a^+_{\vec{x},\sigma}b^-_{\vec{x}+\vec{\delta}_j,\sigma}+b^+_{\vec{x}+\vec{\delta}_j,\sigma}a^-_{\vec{x},\sigma}\right)+\frac{1}{L'|\mathcal{S}_L|}\sum_{\substack{p\in\bar{\mathcal{D}}_{L,L'}\\ r=1,2}}|p|\hat c^+_{p,r}\hat c^-_{p,r}=:H^0_\Lambda+H^f_\Lambda\;, \tag{2.10}$$

which can be explicitly diagonalized; the corresponding correlation functions can be computed exactly, via the Wick rule, in terms of the electron and photon propagators, which read as follows. Let $\Psi^+_{\mathbf{x},\sigma} = (a^+_{\mathbf{x},\sigma}, b^+_{\mathbf{x},+\boldsymbol{\delta}_1,\sigma})$ and $\Psi^-_{\mathbf{x},\sigma} = \begin{pmatrix} a^-_{\mathbf{x},\sigma} \\ b^-_{\mathbf{x}+\boldsymbol{\delta}_1,\sigma}\end{pmatrix}$ be row and column spinors, with $\boldsymbol{\delta}_1 = (0, \vec{\delta}_1)$, $\mathbf{x} = (x_0, \vec{x})$, $x_0 \in [0, \beta)$ an imaginary time, $\beta > 0$ the inverse temperature, and $a^\pm_{\mathbf{x},\sigma} = e^{H_\Lambda x_0} a^\pm_{\vec{x},\sigma} e^{-H_\Lambda x_0}$, $b^\pm_{\mathbf{x}+\boldsymbol{\delta}_1,\sigma} = e^{H_\Lambda x_0} b^\pm_{\vec{x}+\vec{\delta}_1,\sigma} e^{-H_\Lambda x_0}$ the imaginary time evolved of the creation/annihilation operators. Then the *free electron propagator* is [15]

$$S_0^{\beta,L}(\mathbf{x}) := \langle \mathbf{T}\{\Psi^-_{\mathbf{x},\sigma}\Psi^+_{\mathbf{0},\sigma}\}\rangle_{\beta,L}\big|_{e=0} = \frac{1}{\beta L^2}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}} e^{-i\mathbf{k}\mathbf{x}}\hat{S}_0(\mathbf{k})\ , \qquad \hat{S}_0(\mathbf{k}) := \frac{1}{k_0^2 + v^2|\Omega(\vec{k})|^2}\begin{pmatrix} ik_0 & -v\Omega^*(\vec{k}) \\ -v\Omega(\vec{k}) & ik_0 \end{pmatrix}\ , \tag{2.11}$$

where $\langle\cdot\rangle_{\beta,L}$ denotes the average with respect to $e^{-\beta H^0_\Lambda(t)}$, $\mathbf{T}$ is the fermionic time ordering, $\mathcal{B}_{\beta,L} := \frac{2\pi}{\beta}(\mathbb{Z}+\frac{1}{2})\times\mathcal{B}_L$, $v = \frac{3}{2}t$ is the *bare Fermi velocity* and $\Omega(\vec{k}) = \frac{2}{3}\sum_{j=1}^3 e^{i\vec{k}(\vec{\delta}_j - \vec{\delta}_1)}$ the *complex dispersion relation*. The function $\Omega(\vec{k})$ is vanishing if and only if $\vec{k} = \vec{p}_F^{\,\pm}$, where $\vec{p}_F^{\,\pm} = \left(\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}}\right)$ are the two *Fermi points*, close to which $\Omega(\vec{k}' + \vec{p}_F^{\,\pm}) = ik_1' \pm k_2' + O(|\vec{k}'|^2)$. Therefore, setting $\mathbf{p}_F^\omega = (0, \vec{p}_F^{\,\omega})$, $\mathbf{k}' = \mathbf{k} - \mathbf{p}_F^\omega$, $\omega = \pm$, the propagator in momentum space reads

$$\hat{S}_0(\mathbf{k}' + \mathbf{p}_F^\omega) = -\frac{1}{Z}\begin{pmatrix} ik_0 & v(-ik_1' + \omega k_2') \\ v(ik_1' + \omega k_2') & ik_0 \end{pmatrix}^{-1}\Big(1 + O(|\vec{k}'|^2)\Big)\ , \tag{2.12}$$

where $Z = 1$ is the *bare wave function renormalization.* Eq.(2.12) has the form of the propagator for *massless Dirac fermions* in $2+1$ dimensions.

Similarly, defining, for $\mathbf{x} = (x_0, \vec{x})$, $\vec{A}_\mathbf{x} = e^{H^f_\Lambda x_0}\vec{A}(\vec{x}, 0)\, e^{-H^f_\Lambda x_0}$, the in-plane *free photon propagator* is, for $i, j \in \{1, 2\}$,

$$w_{ij}^{\beta,L,L'}(\mathbf{x}) := \langle \mathbf{T}\big[(A_\mathbf{x})_i (A_\mathbf{0})_j\big]\rangle_{\beta,L}\big|_{e=0} = \frac{1}{L'|\mathcal{S}_L|}\sum_{\mathbf{p}\in\mathcal{D}_{\beta,L}}\ \sum_{p_3\in\frac{2\pi}{L'}\mathbb{Z}} e^{-i\mathbf{p}\mathbf{x}}\frac{\chi(|p|)}{\mathbf{p}^2 + p_3^2}\Big(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2 + p_3^3}\Big)\ , \tag{2.13}$$

where $\mathcal{D}_{\beta,L} := \frac{2\pi}{\beta}\mathbb{Z}\times\mathcal{D}_L$. In the limit $L' \to \infty$,

$$w_{ij}^{\beta,L}(\mathbf{x}) = \frac{1}{|\mathcal{S}_L|}\sum_{\mathbf{p}\in\mathcal{D}_{\beta,L}} e^{-i\mathbf{p}\mathbf{x}}\hat{w}_{ij}^{(C)}(\mathbf{p})\ , \qquad \hat{w}_{ij}^{(C)}(\mathbf{p}) := \int_\mathbb{R}\frac{dp_3}{2\pi}\frac{\chi(|p|)}{\mathbf{p}^2 + p_3^2}\Big(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2 + p_3^2}\Big)\ , \tag{2.14}$$

where the apex (C) reminds the choice of the Coulomb gauge. Note that the IR singularity of the in-plane photon propagator in momentum space is $\sim |\mathbf{p}|^{-1}$, rather than the usual $\sim |\mathbf{p}|^{-2}$ of standard quantum electrodynamics (QED). Therefore, interacting graphene at low energies is similar to a gas of massless 2D Dirac particles interacting via a modified $|\mathbf{p}|^{-1}$ photon propagator.

B. Response functions

Our goal is to understand the behavior of the system in the presence of a non-zero electron-photon coupling. We will be mainly concerned with the computation of the interacting correlations, in particular of the interacting electronic propagator and response functions. The latter are particularly relevant from a physical point of view, since we can read from their long distance behavior (or, equivalently, from their singularities in momentum space) the tendency of the system to develop quantum instabilities associated to several putative local order parameters, in the same spirit as [52]. For illustrative purposes, we restrict our attention to the response functions associated to the following bond fermionic bilinears:

$$\begin{aligned}
&\zeta^K_{\mathbf{x},j} = \textstyle\sum_\sigma \Big(e^{ie\int_0^1 ds\, \vec{\delta}_j \vec{A}_{\mathbf{x}+s\boldsymbol{\delta}_j}} a^+_{\mathbf{x},\sigma} b^-_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma} + c.c.\Big) && \text{(lattice distortion)}\\
&\zeta^{CDW}_{\mathbf{x},j} = \textstyle\sum_\sigma \Big(a^+_{\mathbf{x},\sigma} a^-_{\mathbf{x},\sigma} - b^+_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma} b^-_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma}\Big) && \text{(staggered density)}\\
&\zeta^{AF}_{\mathbf{x},j} = \textstyle\sum_\sigma \sigma\Big(a^+_{\mathbf{x},\sigma} a^-_{\mathbf{x},\sigma} - b^+_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma} b^-_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma}\Big) && \text{(staggered magnetization)}\\
&\zeta^{D}_{\mathbf{x},j} = \textstyle\sum_\sigma \Big(a^+_{\mathbf{x},\sigma} a^-_{\mathbf{x},\sigma} + b^+_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma} b^-_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma}\Big) && \text{(bond density)}\\
&\zeta^{J}_{\mathbf{x},j} = \textstyle\sum_\sigma \Big(ie^{ie\int_0^1 ds\, \vec{\delta}_j \vec{A}_{\mathbf{x}+s\boldsymbol{\delta}_j}} a^+_{\mathbf{x},\sigma} b^-_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma} + c.c.\Big) && \text{(bond current)}\\
&\zeta^{H}_{\mathbf{x},j} = \textstyle\sum_\sigma \Big(ie^{ie\int_0^1 ds\, \vec{m}_j \vec{A}_{\mathbf{x}+s\mathbf{m}_j}} a^+_{\mathbf{x},\sigma} a^-_{\mathbf{x}+\mathbf{m}_j,\sigma} - ie^{-ie\int_0^1 ds\, \vec{m}_j \vec{A}_{\mathbf{x}+s\mathbf{m}_j}}\ b^+_{\mathbf{x}+\boldsymbol{\delta}_j,\sigma} b^-_{\mathbf{x}+\boldsymbol{\delta}_j+\mathbf{m}_j,\sigma} + c.c.\Big)\\
& && \text{(Haldane circulating currents)}
\end{aligned} \tag{2.15}$$

where in the last line $\mathbf{m}_1 = \boldsymbol{\delta}_2 - \boldsymbol{\delta}_3$, $\mathbf{m}_2 = \boldsymbol{\delta}_3 - \boldsymbol{\delta}_1$ and $\mathbf{m}_3 = \boldsymbol{\delta}_1 - \boldsymbol{\delta}_2$ indicate next to nearest neighbor vectors. The corresponding response functions are defined as:

$$R^{(a)}_{ij}(\mathbf{x}-\mathbf{y}) = \langle \zeta^a_{\mathbf{x},i}; \zeta^a_{\mathbf{y},j} \rangle = \lim_{\beta\to\infty} \lim_{L\to\infty} \langle \zeta^a_{\mathbf{x},i}; \zeta^a_{\mathbf{y},j} \rangle_{\beta,L} \;, \tag{2.16}$$

where the semicolon in $\langle \cdot ; \cdot \rangle$ indicates truncated expectation: $\langle A; B\rangle := \langle AB\rangle - \langle A\rangle\langle B\rangle$. From the long distance behavior of $R^{(a)}_{ij}(\mathbf{x})$ we can read the possible emergence of long range order. For instance, if the lattice distortion response function behaved as $R^{(K)}_{ij}(\mathbf{x}) \simeq C\cos(\vec{p}^{+}_F(\vec{x} - \vec{\delta}_i + \vec{\delta}_j))$ for some $C \neq 0$ asymptotically as $|\mathbf{x}| \to \infty$, this would signal the spontaneous emergence of a Peierls' instability in the form of the dimerized Kekulé pattern of Fig.2a, which is one of the possible distortion patterns of graphene [13, 33]. In fact, $\zeta^K_{\mathbf{x},j}$ is the local order parameter coupled in the Hamiltonian to the hopping strength, whose mean value heuristically represents the intensity of the hopping from a given site to its nearest neighbor. Having $C \neq 0$ means that far away bonds are strongly correlated and that the joint distribution of their hopping rates is non uniform, but rather oscillating with a cosine dependence. With reference to Fig.2a, this oscillation can be heuristically understood by associating a factor proportional to 1 (to $-\frac{1}{2}$) to the double (single) bonds and by averaging over the three equiprobable configurations obtained by rotating Fig.2a by $0, \frac{2\pi}{3}, \frac{4\pi}{3}$. In momentum space, this would correspond to the appearance of a delta singularity (i.e., of a "condensate") in $\hat{R}^{(K)}_{ij}(\mathbf{p})$ at $\mathbf{p} = \mathbf{p}^{\pm}_F$. Similarly, a condensate in the $\mathbf{p} = \mathbf{0}$ mode of $\hat{R}^{(CDW)}_{ij}(\mathbf{p})$, $\hat{R}^{(AF)}_{ij}(\mathbf{p})$ or $\hat{R}^{(H)}_{ij}(\mathbf{p})$ would signal the spontaneous emergence of a staggered density pattern (*charge density wave*), of a staggered magnetization pattern (*Neél order*) as in Fig.2b and Fig.2c, respectively, or of the specific pattern of circulating currents discussed in [27]. Even in the absence of condensation, that is of delta-like singularities in $\hat{R}^{(a)}_{ij}(\mathbf{p})$, the possible loss of regularity in $\hat{R}^{(a)}_{ij}(\mathbf{p})$ due to the interaction can be interpreted as a tendency of the system to develop quasi-long range order in the corresponding channel.

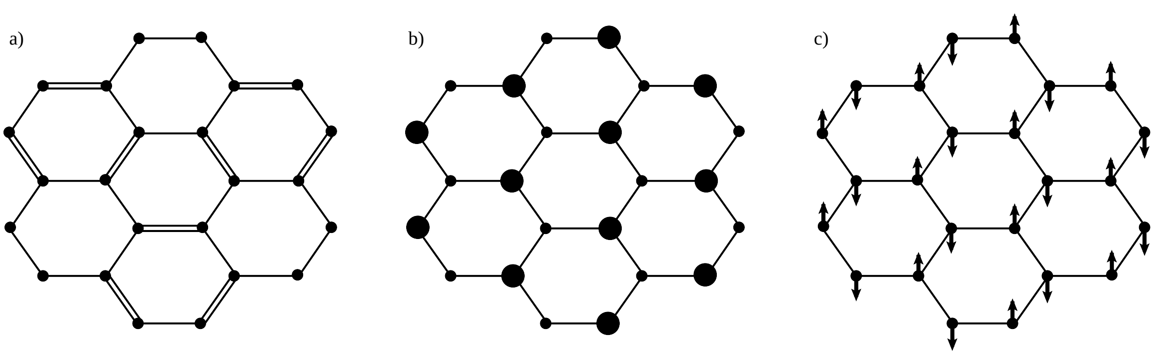

FIG. 2. *a*) The *Kekulé distortion*; the double and single bonds correspond to higher and smaller hopping rates, respectively. *b*) The *charge density wave instability*; big dots correspond to a charge excess, while small dots correspond to a charge deficit. *c*) The *antiferromagnetic instability*; the arrows represent the spins of the electrons, and they have to be understood as lying on the axis orthogonal to the honeycomb lattice.

C. Critical behavior and anomalous exponents

We are now ready to state our main results. As shown in the following, after systematic resummations of perturbation theory, we are able to express the observables of our theory as renormalized series in the electric charge, with finite (and explicitly bounded) coefficients at all orders; the coefficient of e^{2n} grows at most as $n!$, a behavior compatible with Borel summability of the theory at weak enough coupling strength.

In particular, in the limit $\beta, L \to \infty$, the interacting two-point function in the Feynman gauge (see Section 3) is given by:

$$S(\mathbf{x}) = \langle \mathbf{T}(\psi^-_{\mathbf{x},\sigma}\psi^+_{\mathbf{0},\sigma})\rangle = \int_{-\infty}^{\infty} \frac{dk_0}{(2\pi)} \int_{\mathcal{B}} \frac{d\vec{k}}{|\mathcal{B}|} e^{-i\mathbf{p}\mathbf{x}} \hat{S}(\mathbf{k}) \;,$$
$$\hat{S}(\mathbf{k}' + \mathbf{p}^{\omega}_F) = -\frac{1}{Z(\mathbf{k}')} \begin{pmatrix} ik_0 & v(\mathbf{k}')(-ik'_1 + \omega k'_2) \\ v(\mathbf{k}')(ik'_1 + \omega k'_2) & ik_0 \end{pmatrix}^{-1} \big(1 + B(\mathbf{k}')\big) \tag{2.17}$$

where $Z(\mathbf{k}')$ and $v(\mathbf{k}')$ are the interacting wave function renormalization and Fermi velocity, while $B(\mathbf{k})$ is a subdominant term, vanishing at the Fermi points $\mathbf{k}' = \mathbf{0}$. The interacting propagator close to the Fermi points has a structure very reminiscent of the free propagator, Eq.(2.12). However, $Z(\mathbf{k}')$ and $v(\mathbf{k}')$ are *strongly renormalized* by the interaction:

$$Z(\mathbf{k}') \simeq |\mathbf{k}'|^{-\eta} , \qquad 1 - v(\mathbf{k}') \simeq (1-v)|\mathbf{k}'|^{\tilde{\eta}} , \tag{2.18}$$

where

$$\eta = \frac{e^2}{12\pi^2} + O(e^4) , \qquad \tilde{\eta} = \frac{2e^2}{5\pi^2} + O(e^4) , \tag{2.19}$$

are anomalous critical exponents, well defined at all orders in renormalized perturbation theory (the $O(e^4)$ remainders in Eq.(2.19) are written below in terms of a series in e^{2n}, with coefficients growing at most as $n!$). Eqs.(2.17)-(2.19) are very reminiscent of the IR behavior of the 2-point function of a Luttinger liquid [28, 41, 52], and are consistent with results obtained in a model of interacting Dirac fermions in the continuum [22].

On the contrary, the interacting 2-point function for the photon has the same IR singularity $\sim |\mathbf{p}|^{-1}$ as the free case; this means that the gauge field remains *massless* (no screening).

Regarding the response functions associated to fermionic bilinears, we find that in general the interaction changes their decay exponents at large distances as compared to the non-interacting case, where all the responses decay as $\sim r^{-4}$ at large distances. The presence of the interaction makes these exponents non trivial functions of e (anomalous dimensions). In particular we prove that

$$R^{(K)}_{ij}(\mathbf{x}) = \frac{27}{8\pi^2} A_K \frac{\cos\left(\vec{p}^{\,+}_F(\vec{x} - \vec{\delta}_i + \vec{\delta}_j)\right)}{|\mathbf{x}|^{4-\xi^{(K)}}} + r^{(K)}_{ij}(\mathbf{x}) , \tag{2.20}$$

$$R^{(CDW)}_{ij}(\mathbf{x}) = \frac{27}{8\pi^2} A_{CDW} \frac{1}{|\mathbf{x}|^{4-\xi^{(CDW)}}} + r^{(CDW)}_{ij}(\mathbf{x}) , \tag{2.21}$$

$$R^{(AF)}_{jj'}(\mathbf{x}) = \frac{27}{8\pi^2} A_{AF} \frac{1}{|\mathbf{x}|^{4-\xi^{(AF)}}} + r^{(AF)}_{ij}(\mathbf{x}) , \tag{2.22}$$

$$R^{(H)}_{jj'}(\mathbf{x}) = \frac{81}{8\pi^2} A_{H} \frac{1}{|\mathbf{x}|^{4-\xi^{(H)}}} + r^{(H)}_{ij}(\mathbf{x}) , \tag{2.23}$$

where

$$\xi^{(K)} = \frac{4e^2}{3\pi^2} + O(e^4); \quad \xi^{(CDW)} = \frac{4e^2}{3\pi^2} + O(e^4); \quad \xi^{(AF)} = \frac{4e^2}{3\pi^2} + O(e^4); \quad \xi^{(H)} = \frac{4e^2}{3\pi^2} + O(e^4) \tag{2.24}$$

and $A_\# = A_\#(v,e)$ are constants that are equal to 1 at the free Dirac point, i.e., $A_\#\big|_{v=1,e=0} = 1$ (in particular, for v close to 1, $A_\# = 1 + O(1-v) + O(e^2)$). Moreover, the correction terms $r^{(a)}_{ij}(\mathbf{x})$ are subdominant contributions, decaying at infinity faster than $|\mathbf{x}|^{-4+\xi^{(a)}}$; they include both the effects coming from the irrelevant terms in a RG sense and the effects proportional to $1 - v(\mathbf{k}')$ (see Eq.(2.18)) coming from the Lorentz symmetry breaking terms. From Eqs.(2.20)–(2.22), we see that the decay of the interacting responses in the K, CDW, AF, H channels is slower than the corresponding non-interacting functions; i.e., the responses to K, CDW, AF, H are strongly enhanced by the interaction. On the contrary, all other responses decay at infinity faster than $|\mathbf{x}|^{-4+(\mathrm{const.})e^4}$, i.e., if $a = D, J$, (and similar bounds are valid for other observables like the Cooper pairs, see Section 7 below)

$$|R^{(a)}_{ij}(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|^{4-Ce^4}} , \tag{2.25}$$

for some constant $C > 0$. These results can be naturally extrapolated to larger values of the electric charge, in which case they suggest that the lattice distortion, staggered density, staggered magnetic order and the "Haldane circulating currents" are the dominant quantum instabilities at intermediate to strong coupling strength.

As we noticed, quantum instabilities are also signaled by divergences in the Fourier transform of the response functions. Even in the presence of non trivial exponent, the power law decay remains integrable at weak coupling: therefore, no divergence is found in the Fourier transform of the response function. On the other hand,

$$\partial_{\mathbf{p}} \hat{R}^{(K)}_{ij}(\mathbf{p}' + \mathbf{p}^{\pm}_F) \sim |\mathbf{p}'|^{-\xi^{(K)}} \quad \text{and} \quad \partial_{\mathbf{p}} \hat{R}^{(a)}_{ij}(\mathbf{p}) \sim |\mathbf{p}|^{-\xi^{(a)}} , \quad \text{with} \quad a = CDW, AF, H , \tag{2.26}$$

so that the singularity in the first derivative of the response functions in momentum space, which appeared as a discontinuity or at most as a logarithmic divergence in the non interacting case, is enhanced and turned into a power law singularity by the interaction for these four responses. The singularity at different momenta or for other response functions is either weaker or absent. Also in this respect, *the conclusion is that the system shows a tendency towards Kekulé, charge density wave, Neél ordering* or to the formation of the *Haldane gap*.

D. Mass renormalization and gap equation

The enhancement of the response functions suggests that the effects of small external staggered fields coupled to the K, CDW, AF, H local order parameters are dramatically enhanced by the interactions. This is in fact the case. Let us, for instance, add an external staggered field coupled to the lattice distortion local order parameter, with the same cosine dependence as the long distance decay of $R^{(K)}_{ij}(\mathbf{x})$, see Eq.(2.20). Physically, this can be interpreted as a fixed distortion of the lattice into a Kekulé pattern as in Fig.2a. In fact, if we allow distortions of the honeycomb lattice, the hopping becomes a function of the bond length $\ell_{\vec{x},j}$ that, for small deformations, can be approximated by the linear function $t_{\vec{x},j} = t + g(\ell_{\vec{x},j} - \bar{\ell}) =: t + \phi_{\vec{x},j}$, where $\bar{\ell}$ is the equilibrium length of the bonds and $\phi_{\vec{x},j}$ plays the role of a classical phonon field. If a Kekulé distortion of amplitude proportional to Δ_0/g is present, then $\phi_{\vec{x},j} = \frac{\Delta_0}{3} 2\cos(\vec{p}_F^{\,+}(\vec{x} - \vec{\delta}_j + \vec{\delta}_{j_0}))$ for some $j_0 \in \{1, 2, 3\}$ and the Hamiltonian becomes

$$H^{\Delta_0}_\Lambda = H_\Lambda - \frac{\Delta_0}{3} \sum_{j=1,2,3} \sum_{\substack{\vec{x}\in\Lambda \\ \omega=\pm}} e^{i\vec{p}_F^{\,\omega}(\vec{x}-\vec{\delta}_j+\vec{\delta}_{j_0})} \zeta^K_{\vec{x},j} \,. \tag{2.27}$$

If the electron-photon coupling e is equal to 0, then the fermionic 2-point function has a mass proportional to Δ_0. If we switch on the interaction, the mass (i.e., the decay constant describing the exponential decay of the 2-point function at large distances) becomes

$$\Delta = \Delta_0^{1/(1+\eta_\Delta)} \,, \qquad \eta_\Delta = \frac{2e^2}{3\pi^2} + \cdots \,, \tag{2.28}$$

that is, the Kekulé mass is strongly amplified by the interaction (note that the ratio between the interacting and bare masses diverges with an anomalous exponent as $\Delta_0 \to 0$). This phenomenon is very reminiscent of the spontaneous mass generation phenomenon in QFT, the main difference being that while in a truly relativistic theory in the continuum the flow of the mass can be studied also in the UV region and the bare mass can be let to zero with the UV cutoff (still keeping the same dressed mass at fixed IR scale), see [25], here the lattice acts as a fixed UV cut-off, so that the mass is amplified but not spontaneously generated. Similar arguments can be repeated for the other dominant excitations.

Let us finally discuss a possible mechanism for the spontaneous generation of a Kekulé instability in our model. Rather then fixing the distortion pattern $\phi_{\vec{x},j}$ once and for all, we can let $\underline{\phi} = \{\phi_{\vec{x},j}\}^{j=1,2,3}_{\vec{x}\in\Lambda}$ be a classical field to be fixed self-consistently, in such a way that the total energy in the Born-Oppenheimer approximation is minimal, i.e.,

$$\underline{\phi} = \text{argmin}\Big\{ E_0(\underline{\phi}) + \frac{\kappa}{2g^2} \sum_{\substack{\vec{x}\in\Lambda \\ j=1,2,3}} \phi^2_{\vec{x},j} \Big\} \,, \tag{2.29}$$

where $E_0(\underline{\phi})$ is the ground state energy of $H^{\phi}_\Lambda = H_\Lambda - \sum_{\substack{\vec{x}\in\Lambda \\ j=1,2,3}} \phi_{\vec{x},j}\zeta^{(K)}_{\vec{x},j}$. We find that the Kekulé distortion pattern

$$\phi^{(j_0)}_{\vec{x},j} = \phi_0 + \frac{2}{3}\Delta_0 \cos\big(\vec{p}_F^{\,+}(\vec{x} - \vec{\delta}_j + \vec{\delta}_{j_0})\big) \tag{2.30}$$

is a stationary point of the total energy, provided that $\phi_0 = c_0 g^2/\kappa + \cdots$ for a suitable constant c_0 and that Δ_0 satisfies the following non-BCS gap equation:

$$\Delta_0 \simeq 6\frac{g^2}{\kappa} \int\limits_{\Delta\lesssim|\mathbf{k}'|\lesssim 1} d\mathbf{k}' \frac{Z^{-1}(\mathbf{k}')\Delta(\mathbf{k}')}{k_0^2 + v^2(\mathbf{k}')|\Omega(\vec{k}' + \vec{p}_F^{\,\omega})|^2 + |\Delta(\mathbf{k}')|^2} \,, \tag{2.31}$$

where $\Delta = \Delta_0^{1/(1+\eta_\Delta)}$ and, for $\Delta \lesssim |\mathbf{k}'| \ll 1$, $Z(\mathbf{k}') \sim |\mathbf{k}'|^{-\eta}$, $v(\mathbf{k}') \sim 1 - (1-v)|\mathbf{k}'|^{\tilde{\eta}}$ and $\Delta(\mathbf{k}') \sim \Delta_0\,|\mathbf{k}'|^{-\eta_\Delta}$. Our gap equation has the same qualitative properties of the simpler equation:

$$1 = g^2 \int_\Delta^1 d\rho \frac{\rho^{\eta-\eta_\Delta}}{1-(1-v)\rho^{\tilde{\eta}}} \tag{2.32}$$

from which it is apparent that at small e, the equation admits a non trivial solution only for g larger than a critical coupling g_c; remarkably, $g_c \sim \sqrt{v}$, with v the free Fermi velocity, even though the effective Fermi velocity tends to the speed of light. Therefore, at weak coupling, the prediction for g_c is qualitatively the same as in the free case [33]; this can be easily checked by noting that the Fermi velocity $v(\mathbf{k}')$ is sensibly different from v only for momentum scales exponentially small in v/e^2. See Section 9 for more comments about this point.

Even more interestingly, the value of g_c decreases as e increases; i.e., interactions facilitate the formation of a Kekulé pattern. Eqs.(2.31)-(2.32) can be naturally extrapolated to intermediate coupling: if in such a regime $\eta_\Delta - \eta = \frac{7e^2}{12\pi^2} + \cdots$ exceeds 1, then the integrand in the r.h.s. of the gap equation diverges as $\Delta \to 0$, a fact that guarantees the existence of a non-trivial solution *for arbitrarily small g*. In other words, the larger the electron-photon interaction, the easier is to form a Kekulé patterned state; it is even possible that at intermediate coupling $g_c = 0$, which would imply a spontaneous generation of the Peierls'-Kekulé instability. Note the non BCS-like form of the gap, similar to the one appearing in certain Luttinger superconductors [40]. A similar analysis can be repeated for the gap generated by the staggered density, the magnetization or the Haldane mass.

E. A comparison with existing literature

Before we enter the technical part of our work, let us conclude this expository section by a comparison of our model and our results with existing literature. We do not pretend to give a full account of the rapidly expanding literature on the effects of interactions in graphene; several excellent reviews already exists, like [7, 38], which we refer to for extensive bibliography. Here we focus on the difference between the approaches and results based on the effective models of Dirac gas in the continuum, which is the most popular and widely studied model of graphene, and ours, which is based on a tight binding lattice model.

Short range interactions. Graphene with electron-electron screened interactions has been studied in terms of an effective model of 2D massless Dirac fermions interacting with a local quartic potential: in the weak coupling regime this interaction is irrelevant in the Renormalization Group sense [24], while at strong coupling analyses based on large N expansions found some evidence for quantum critical points [29, 30, 49, 50, 53]. This model requires an UV regularization and some observables, like the conductivity, appear to be sensitive to the specific UV regularization scheme used: different results are found [31, 32] depending on whether momentum or dimensional regularizations is chosen.

A more realistic model for graphene with short range interactions is a tight binding model that keeps into full account lattice effects, such as the half-filled *Hubbard model on the honeycomb lattice*. Formally, the Hubbard model reduces to the continuum Dirac gas in the limit as the lattice spacing goes to zero; in this sense the latter can be thought as a scaling limit approximation of the former. One important advantage of the lattice model as compared to the continuum one is that within the former no ambiguities arise in the computation of the conductivity. In particular, all the interaction correction to the conductivity *exactly cancel out* in the optical limit [20], in agreement with experimental results, [43], and in disagreement with the Dirac model with momentum regularization.

Gauge-invariant electromagnetic interactions. In the early paper [22] (written much before the actual realization of graphene) an effective model for interacting of graphene was proposed, in which massless Dirac fermions in the 2D continuum are coupled to a quantum 3D photon field, with the fermionic propagation speed much smaller than the speed of light. The main result of [22], based on second order perturbation theory (and, therefore, valid in the weak coupling regime), is that at low energies the Fermi velocity tends to the speed of light and the wave function renormalization diverges with an anomalous exponent. No computation of the response function exponents was performed.

In [18], we revisited the model proposed in [22] and, rather than using dimensional regularization as in [22], we used an UV momentum cut-off; this is a much more natural choice, since the Dirac continuum model should be though of as an effective model emerging in a Wilsonian RG after the integration of the high energy degrees of freedom. Using this model, we extended the results in [22] at all orders, but still we computed neither the response functions nor the gap equation. On the other hand, the momentum cutoffs, necessary to avoid spurious UV divergences, break gauge invariance and this imposed the introduction of counterterms in order to keep the photon mass vanishing and in order to have one (rather than three) effective charge [18].

In the more realistic model considered in this paper the electrons live on a honeycomb lattice and interact with a quantum photon field living in the 3D continuum. The fact that gauge invariance is not broken has the effect that no unphysical counterterms need to be introduced. The critical exponents and the gap equation have been computed here for the first time. Moreover, lattice gauge invariance prevents the generation of several potentially dangerous marginal and relevant terms. As in the case of Hubbard interactions, the conductivity computed in the continuum model show an unphysical dependence from the conductivity [35], while considering the model and the formalism introduced in this model will resolve such ambiguities.

Static Coulomb interactions. The most popular model used to describe graphene with unscreened electromagnetic interactions is a Dirac gas with static density-density Coulomb interactions; this is an apparently sensible approximation, since the (bare) propagation speed of quasi-particles in graphene is much smaller than the speed of light, so that retardation effects should be negligible at least in a wide range of energy scales. However, in the weak coupling regime, a second order RG analysis predicts an unbounded growth of the Fermi velocity in the IR and the vanishing of the effective charge at the Fermi points [23]. Therefore, at low energy scales the Fermi velocity becomes comparable with the speed of light and the model with static interactions loses its significance, a fact that can be seen as a dramatic manifestation of its "uncompletness", see [55]. This is also consistent with the fact that the theory with static Coulomb interactions does not appear to be renormalizable at all orders, see [19, p.1425]. In conclusion, retardation effects are important to understand the nature of the IR fixed point of the theory.

The effective Dirac model with Coulomb interactions has been also extensively analyzed in the strong coupling regime. It has been argued that, at large enough coupling, an excitonic gap spontaneously opens [25, 36, 37], by a mechanism similar to mass generation in QED_{2+1} [42]: in these works, the gap equation is derived by a self-consistence argument; the corresponding solution is shown to be momentum-dependent and vanishing in the limit of high momenta. However, these findings rely on several approximations, in particular: in [25, 36] the vertex, wave function and velocity renormalizations are neglected; in [37] the renormalization of the velocity is taken into account, but the corresponding flow is IR-unbounded and UV-cutoff dependent. In our work, the gap equation is obtained by using an *exact* energy optimization problem and fully takes into account the renormalization of all relevant and marginal operators; moreover, since we do not neglect effects of the honeycomb lattice, our results are free from the ambiguities related to the presence of the spurious UV divergences typical of the Dirac approximation.

In [26, 30] a systematic classifications of the possible interaction and mass terms allowed by symmetry is performed in the Dirac model with Coulomb interactions; in the present paper a similar analysis is carried out, with the difference that the lattice discrete symmetries rather than the continuum symmetries are taken into account.

RG analyses based on large N expansions [11, 29, 30, 53] and Quantum Montecarlo analyses [9, 10] in the presence of Coulomb interactions have identified critical exponents for the response functions and found evidence for the presence of excitonic phase transitions (like CDW and Kekulé instabilities). Again, these results are in qualitative agreement with our finding that the effective Kekulé mass (or the CDW, AF, H mass) grows at low momenta with an anomalous power law, although the model and the method are quite different (expansion in the charge and retarded interactions versus $1/N$ expansions and instantaneous interactions). A strong coupling expansion for a lattice gauge theory for graphene has been also performed in [1, 2] and evidence for spontaneous Kekulé mass generation was found. It is worth stressing that the lattice used for the Quantum Montecarlo analysis in [9, 10] is a *square* lattice, rather than the original *honeycomb* lattice; it would be interesting to repeat a similar analysis for the more realistic honeycomb lattice gauge theory introduced in the present paper.

Let us finally note that the Peierls-Kekulé instability was first discussed in the non-interacting case in [33], as a key ingredient for the emergence of electron fractionalization without the breaking of time-reversal symmetry (on this issue, see also [8, 34]). Our gap equation generalizes the one of [33] to the interacting case.

3. FUNCTIONAL INTEGRAL REPRESENTATION AND WARD IDENTITIES

The correlation functions introduced in the previous section can be conveniently expressed in terms of a functional integral. We introduce the generating functional of correlations in the ξ-gauge with infrared cutoff on the photon propagator as

$$e^{\mathcal{W}^{\xi,h^*}(\Phi,J,\lambda)} = \int P(d\Psi)P^{\xi,h^*}(dA)e^{\mathcal{V}(\Psi,A+J)+\mathcal{B}(\Psi,A+J,\Phi)+(\lambda,\Psi)}\,, \tag{3.1}$$

where:

1. $\Psi = \{\Psi^{\pm}_{\mathbf{x},\sigma}\}$ are two-components Grassmann fields, the components being denoted by $\Psi^{\pm}_{\mathbf{x},\sigma,\rho}$, $\rho = 1, 2$, the first corresponding to the $a^{\pm}$-fields, the second to the $b^{\pm}$-fields. Moreover, $A = \{A_{\mu,\mathbf{x}}\}$, with $\mu = 0, 1, 2$, are real fields. The convention on the Fourier transform of the fields that we use is the following:

$$\Psi^{\pm}_{\mathbf{x},\sigma,\rho} = \frac{1}{\beta L^2}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}} e^{\pm i\mathbf{k}\mathbf{x}}\hat{\Psi}_{\mathbf{k},\sigma,\rho}\,, \qquad A_{\mu,\mathbf{x}} = \frac{1}{\beta|\mathcal{S}_L|}\sum_{\mathbf{p}\in\mathcal{D}_{\beta,L}} e^{-i\mathbf{p}\mathbf{x}}\hat{A}_{\mu,\mathbf{p}}\,. \tag{3.2}$$

2. If $n_\mu = \delta_{\mu,0}$, $\mu = 0, 1, 2$, and

$$\hat{w}^{\xi,h^*}_{\mu\nu}(\mathbf{p}) = \int_{\mathbb{R}}\frac{dp_3}{2\pi}\frac{\chi(|p|)-\chi(2^{-h^*}|\mathbf{p}|)}{\mathbf{p}^2+p_3^2}\Big(\delta_{\mu\nu}-\xi\frac{p_\mu p_\nu - p_0(p_\mu n_\nu + p_\nu n_\mu)}{|\vec{p}|^2+p_3^2}\Big) \tag{3.3}$$

is the photon propagator in the ξ-gauge (the Coulomb gauge corresponding to $\xi=1$ and the *Feynman gauge* corresponding to $\xi=0$) with infrared cutoff at momenta of the order 2^{h^*}, $h^*<0$, then $P(d\Psi)$ and $P^\xi(d\vec{A})$ are the gaussian "measures" associated to the propagators $\hat{S}_0(\mathbf{k})$ and $\hat{w}^{\xi,h^*}(\mathbf{p})$, respectively:

$$P(d\Psi)=\frac{1}{\mathcal{N}_\Psi}\prod_{\mathbf{k}\in\mathcal{B}_{\beta,L}}\prod_{\substack{\rho=1,2\\ \sigma=\uparrow,\downarrow}}d\hat{\Psi}^+_{\mathbf{k},\sigma,\rho}d\hat{\Psi}^-_{\mathbf{k},\sigma,\rho}\exp\Big\{-\frac{1}{\beta L^2}\sum_{\substack{\mathbf{k}\in\mathcal{B}_{\beta,L}\\ \sigma=\uparrow\downarrow}}\hat{\Psi}^+_{\mathbf{k},\sigma}\big[\hat{S}_0(\mathbf{k})\big]^{-1}\hat{\Psi}^-_{\mathbf{k},\sigma}\Big\}\,, \tag{3.4}$$

$$P^{\xi,h^*}(dA)=\frac{1}{\mathcal{N}_\xi}\prod_{\mathbf{p}\in\mathcal{D}^+_{\beta,L}}\prod_{\mu=0,1,2}d\mathrm{Re}A_{\mu,\mathbf{p}}d\mathrm{Im}A_{\mu,\mathbf{p}}\exp\Big\{-\frac{1}{2\beta|\mathcal{S}_L|}\sum^{*}_{\substack{\mathbf{p}\in\mathcal{D}_{\beta,L}\\ \mu,\nu=0,1,2}}\hat{A}_{\mu,\mathbf{p}}\big[\hat{w}^{\xi,h^*}(\mathbf{p})\big]^{-1}_{\mu\nu}\hat{A}_{\nu,-\mathbf{p}}\Big\}\,, \tag{3.5}$$

where $\mathcal{N}_\Psi$, $\mathcal{N}_\xi$ two normalization factors; in the second line, the product over $\mathbf{p}$ runs over the subset $\mathcal{D}^+_{\beta,L}$ of $\mathcal{D}_{\beta,L}$ such that $\mathbf{p}>0$ (here $\mathbf{p}>0$ means that either $p_0>0$, or $p_0=0$ and $p_1>0$, or $p_0=p_1=0$ and $p_2>0$) and $\mathbf{p}$ is in the support of $\chi(|\mathbf{p}|)$. Similarly, the $*$ in the sum at exponent indicates that the summation runs over the momenta in the support of $\chi(|\mathbf{p}|)$.

Remark. Note that, by the very definition of the cutoff function χ (see the lines following Eq.(2.6)), the modes $\hat{A}_{\mu,\mathbf{p}}$ associated to momenta close to $\pm(\mathbf{p}^+_F-\mathbf{p}^-_F)$ and to their images over the dual lattice Λ^* (i.e., to the points $(0,\pm(\vec{p}^{\,+}_F-\vec{p}^{\,-}_F)+n_1\vec{G}_1+n_2\vec{G}_2)$, with $n_1,n_2\in\mathbb{Z}$) are vanishing. In other words, the UV cutoff on the photon field is chosen so small that *umklapp processes are suppressed.*

3. The interaction is

$$\begin{aligned}\mathcal{V}(\Psi,A)=&t\sum_{\substack{\sigma=\uparrow,\downarrow\\ j=1,2,3}}\int d\mathbf{x}\Big[\big(e^{ie\int_0^1 ds\,\vec{\delta}_j\vec{A}_{\mathbf{x}+s\boldsymbol{\delta}_j}}-1\big)\Psi^+_{\mathbf{x},\sigma,1}\Psi^-_{\mathbf{x}+\boldsymbol{\delta}_j-\boldsymbol{\delta}_1,\sigma,2}+\big(e^{-ie\int_0^1 ds\,\vec{\delta}_j\vec{A}_{\mathbf{x}+s\boldsymbol{\delta}_j}}-1\big)\Psi^+_{\mathbf{x}+\boldsymbol{\delta}_j-\boldsymbol{\delta}_1,\sigma,2}\Psi^-_{\mathbf{x},\sigma,1}\Big]\\ &-ie\sum_{\sigma=\uparrow\downarrow}\int d\mathbf{x}\big(A_{0,\mathbf{x}}\Psi^+_{\mathbf{x},\sigma,1}\Psi^-_{\mathbf{x},\sigma,1}+A_{0,\mathbf{x}+\boldsymbol{\delta}_1}\Psi^+_{\mathbf{x},\sigma,2}\Psi^-_{\mathbf{x},\sigma,2}\big)\,,\end{aligned} \tag{3.6}$$

where $\int d\mathbf{x}$ is a shorthand for $\int_{-\beta/2}^{\beta/2}dx_0\sum_{\vec{x}\in\Lambda}$. The interaction can be equivalently rewritten in momentum space, see Eq.(B.1).

4. The external sources are

$$\mathcal{B}(\Psi,A,\Phi)=\int d\mathbf{x}\sum_a\sum_{j=1,2,3}\Phi^a_{j,\mathbf{x}}\zeta^a_{j,\mathbf{x}}\,,\qquad(\lambda,\Psi)=\int d\mathbf{x}\sum_{\sigma=\uparrow\downarrow}(\lambda^+_{\mathbf{x},\sigma}\Psi^-_{\mathbf{x},\sigma}+\Psi^+_{\mathbf{x},\sigma}\lambda^-_{\mathbf{x},\sigma})\,, \tag{3.7}$$

where the sum over the index a runs over the choices $a=K,CDW,AF,D,J,C$, and $\zeta^a_{i,\mathbf{x}}$ are given by the same expressions Eqs.(2.15) after the replacement of the femionic and bosonic operators by the Grassmann and real fields Ψ and A. The external fields $\lambda^\pm_{\mathbf{x},\sigma}$ are Grassmann (two-components) fields, while $\Phi^a_{j,\mathbf{x}}$ and $J_{\mu,\mathbf{x}}$ are real fields, the first defined on the lattice, the second in the continuum with the same UV cutoff as the photon field (therefore, as discussed in the previous remark, the UV cutoff on the J field is chosen so small that the modes $\hat{J}_{\mu,\mathbf{p}}$ with $\mathbf{p}$ sufficiently close to $\pm(\mathbf{p}^+_F-\mathbf{p}^-_F)$ and to their images over Λ^* are vanishing). The source term can be equivalently rewritten in momentum space, see Eq.(B.1).

The response functions introduced in Section 2 can be written as functional derivatives of the generating function,

$$R^{(a)}_{ij}(\mathbf{x}-\mathbf{y})=\lim_{\beta\to\infty}\lim_{L\to\infty}\lim_{h^*\to-\infty}\frac{\partial^2}{\partial\Phi^a_{i,\mathbf{x}}\partial\Phi^a_{j,\mathbf{y}}}\mathcal{W}^{\xi,h^*}(\Phi,0,0)\big|_{\Phi=0} \tag{3.8}$$

that, remarkably, are *independent* of the choice of the gauge, $\xi\in[0,1]$ A sketch of the proof of the functional integral representation of the observables of our theory, as well as a proof of the independence of Eq.(3.8) on the specific choice of the gauge, is given in Appendix A. Since the response functions are independent of ξ, from now on we choose to evaluate them in the Feynman gauge, i.e., $\xi=0$.

Similarly, the *Schwinger functions* in the ξ-gauge at finite volume and finite temperature are defined as the $h^*\to-\infty$ limit of

$$S^\xi_{n,m;\underline{\varepsilon},\underline{\sigma},\underline{\rho},\underline{\mu}}(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{y}_1,\ldots,\mathbf{y}_m)=\frac{\partial^{n+m}\mathcal{W}^{\xi,h^*}(0,J,\lambda)}{\partial\lambda^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1}\cdots\partial\lambda^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n}\partial J_{\mu_1,\mathbf{x}_{n+1}}\cdots\partial J_{\mu_m,\mathbf{x}_{n+m}}}\Big|_{\lambda=J=0}\,. \tag{3.9}$$

Contrary to the response functions, the Schwinger functions (at least the way they are defined in Eq.(3.9)) are not gauge invariant and, therefore, they depend on the specific choice of the gauge. Nevertheless, we decide to compute them in the Feynman gauge, which is technically the simplest where to perform computations. Although the Schwinger functions in the Feynman gauge do not have an obvious Hamiltonian counterpart, we believe that they are a source of valuable information on the behavior of the system. In particular, as we will see, the 2-point functions and the vertex function, defined as

$$\begin{aligned} \big[\hat S^{\xi,h^*}_{2,0}(\mathbf{k})\big]_{\rho\rho'} &= \beta L^2 \frac{\partial^2 \mathcal{W}^{\xi,h^*}(0,0,\lambda)}{\partial\hat\lambda^-_{\mathbf{k},\sigma,\rho'}\partial\hat\lambda^+_{\mathbf{k},\sigma,\rho}}\,, \qquad \hat S^{\xi,h^*}_{0,2;(\mu,\nu)}(\mathbf{p}) = \beta|\mathcal{S}_L|\frac{\partial^2\mathcal{W}^{\xi,h^*}(0,J,0)}{\partial\hat J^-_{\mu,\mathbf{p}}\partial\hat J_{\nu,-\mathbf{p}}}\,,\\ \big[\hat S^{\xi,h^*}_{2,1;\mu}(\mathbf{k},\mathbf{p})\big]_{\rho\rho'} &= \beta^2L^2|\mathcal{S}_L|\frac{\partial^3\mathcal{W}^{\xi,h^*}(0,J,\lambda)}{\partial\hat J_{\mu,\mathbf{p}}\partial\hat\lambda^-_{\mathbf{k},\sigma,\rho'}\partial\hat\lambda^+_{\mathbf{k}+\mathbf{p},\sigma,\rho}}\,, \end{aligned} \tag{3.10}$$

will play a crucial role in the study of the flow of the effective couplings (and, therefore, of the response function themselves).

The independence of the gauge invariant observables on the specific choice of ξ is strictly related to the gauge invariance of the generating functional $\mathcal{W}^{\xi,h^*}$ with respect to $U(1)$ gauge transformations. Namely, for all $\xi\in[0,1]$ and $h^*<0$, we have:

$$0 = \frac{\partial}{\partial\hat\alpha_{\mathbf{p}}}\mathcal{W}^{\xi,h^*}(\Phi, J+\partial\alpha, \lambda e^{ie\alpha})\Big|_{\alpha=0}\,, \tag{3.11}$$

see Appendix A for a proof. By taking derivatives with respect to the external fields in Eq.(3.11), we can generates infinitely many identities between correlations, also known as *Ward identities*. In fact, Eq.(3.11) is equivalent to

$$\sum_{\mu=0}^{2} p_\mu \frac{\partial\mathcal{W}^{\xi,h^*}(\Phi,J,\lambda)}{\partial\hat J_{\mu,\mathbf{p}}} = \frac{e}{\beta|\mathcal{S}_L|}\sum_{\substack{\mathbf{k}\in\mathcal{B}_{\beta,L}\\ \sigma=\uparrow\downarrow}}\Big[\hat\lambda^+_{\mathbf{k}+\mathbf{p},\sigma}\Gamma_0(\mathbf{p})\frac{\partial\mathcal{W}^{\xi,h^*}(\Phi,J,\lambda)}{\partial\hat\lambda^+_{\mathbf{k},\sigma}} + \frac{\partial\mathcal{W}^{\xi,h^*}(\Phi,J,\lambda)}{\partial\hat\lambda^-_{\mathbf{k}+\mathbf{p},\sigma}}\Gamma_0(\mathbf{p})\hat\lambda^-_{\mathbf{k},\sigma}\Big]\,, \tag{3.12}$$

with $\Gamma_0(\mathbf{p}) = \begin{pmatrix}1 & 0\\ 0 & e^{-i\mathbf{p}\boldsymbol{\delta}_1}\end{pmatrix}$. Taking, e.g., one derivative with respect to J or two derivatives with respect to λ, we find:

$$\sum_{\mu=0}^{2} p_\mu \hat S^{\xi,h^*}_{0,2;(\mu,\nu)}(\mathbf{p}) = 0\,, \qquad \sum_{\mu=0}^{2} p_\mu \hat S^{\xi,h^*}_{2,1;\mu}(\mathbf{k},\mathbf{p}) = e\Big(\Gamma_0(\mathbf{p})\hat S^{\xi,h^*}_{2,0}(\mathbf{k}) - \hat S^{\xi,h^*}_{2,0}(\mathbf{k}+\mathbf{p})\Gamma_0(\mathbf{p})\Big)\,, \tag{3.13}$$

which will be used below to deduce that the dressed photon mass is zero and the dressed electric charge is close to the bare one within $O(e^3)$. Note the crucial fact that Eqs.(3.11)–(3.13) are valid at finite volume, finite temperature and *for any value of* h^*.

4. RENORMALIZATION GROUP ANALYSIS

In this Section we start the evaluation of the generating functional Eq.(3.1). In the following, for simplicity, we set the external fermionic and bosonic fields to 0, $J=\lambda=0$, and $h^*=-\infty$ (dropping the h^* index in the formulas). The effect of the external fields J and λ has been discussed several times in the literature, see, e.g., [18, Appendix B]. The presence of a finite bosonic infrared cutoff on scale 2^{h^*} will be discussed at the beginning of the next section. Moreover, we set all the external fields $\Phi^{(a)}$ *but the one coupled to the Kekulé distortion* to zero. The presence of the external field $\Phi^{(K)}$ will be discussed in detail, for illustrative purposes. The effect of its addition is non trivial, particularly because of the new marginal terms it can generate. The effect of other external fields $\Phi^{(a)}$, $a\neq K$, can be studied along the same lines and will be discussed in Section 7.

As mentioned above, from now on we will work in the Feynman gauge, $\xi=0$, in which case, the bosonic propagator simply reads

$$\hat w(\mathbf{p})\delta_{\mu\nu} := \hat w^{\xi}_{\mu\nu}(\mathbf{p})\Big|_{\xi=0} = \int_{\mathbb{R}}\frac{dp_3}{2\pi}\frac{\chi(|p|)}{\mathbf{p}^2+p_3^2}\delta_{\mu\nu} \tag{4.1}$$

The bosonic propagator $\hat w(\mathbf{p})$ is singular at $\mathbf{p}=\mathbf{0}$, while the fermionic propagator is singular at the two Fermi points $\mathbf{k}=\mathbf{p}^{\pm}_F$. The first step of the RG analysis consists in rewriting both the fermionic and the bosonic propagators as sums of two propagators, one supported close to the singularity (infrared propagator) and one in the complementary region (ultraviolet propagator), that is,

$$\hat{S}_0(\mathbf{k}) = \hat{g}^{(\leq 0)}(\mathbf{k}) + \hat{g}^{(1)}(\mathbf{k}) \; , \qquad \hat{w}(\mathbf{p}) = \hat{w}^{(\leq 0)}(\mathbf{p}) + \hat{w}^{(1)}(\mathbf{p}) \; . \tag{4.2}$$

where

$$g^{(\leq 0)}(\mathbf{k}) = \sum_{\omega=\pm} \chi(|\mathbf{k}-\mathbf{p}_F^\omega|)\hat{S}_0(\mathbf{k}) =: \sum_{\omega=\pm} \hat{g}_\omega^{(\leq 0)}(\mathbf{k}-\mathbf{p}_F^\omega) \; , \qquad \hat{w}^{(\leq 0)}(\mathbf{p}) = \chi(|\mathbf{p}|) \int_{\mathbb{R}} \frac{dp_3}{2\pi} \frac{1}{\mathbf{p}^2+p_3^2} = \frac{\chi(|\mathbf{p}|)}{2|\mathbf{p}|} \; . \tag{4.3}$$

Note that in the first formula the support functions $\chi(|\mathbf{k}-\mathbf{p}_F^+|)$ and $\chi(|\mathbf{k}-\mathbf{p}_F^-|)$ have disjoint supports. Correspondingly, we rewrite the Gaussian measures as

$$P(d\Psi) = \Big[\prod_{\omega=\pm} P(d\Psi_\omega^{(\leq 0)}) \Big] P(d\Psi^{(1)}) \, , \qquad P^\xi(dA)\Big|_{\xi=0} = P(dA^{(\leq 0)}) P(dA^{(1)}) \; , \tag{4.4}$$

where $\hat{\Psi}_\omega^{(\leq 0)}$, $\hat{\Psi}^{(1)}$, $\hat{A}^{(\leq 0)}$ and $\hat{A}^{(1)}$ have propagators given by $\hat{g}_\omega^{(\leq 0)}$, $\hat{g}^{(1)}$, $\hat{w}^{(\leq 0)}$ and $\hat{w}^{(1)}$, respectively. The fields $\hat{\Psi}_\omega^{(\leq 0)}$ are called *quasi-particle* fields, and the index $\omega = \pm$ is called quasi-particle or valley index.

Using this decomposition of the fermionic and bosonic fields, the generating functional $\mathcal{W}(\Phi) = \mathcal{W}^{\xi,-\infty}(\Phi, 0, 0)\big|_{\xi=0}$ at $J = \lambda = 0$ can be rewritten as

$$\begin{aligned} e^{\mathcal{W}(\Phi)} &= \int P(d\Psi^{(\leq 0)})P(dA^{(\leq 0)}) \int P(d\Psi^{(1)})P(dA^{(1)}) e^{\mathcal{V}(\Psi^{(\leq 0)}+\Psi^{(1)},A^{(\leq 0)}+A^{(1)})+\mathcal{B}(\Psi^{(\leq 0)}+\Psi^{(1)},A^{(\leq 0)}+A^{(1)},\Phi)} = \\ &= e^{-\beta L^2 F_0 + S^{(\geq 0)}(\Phi)} \int P(d\Psi^{(\leq 0)})P(dA^{(\leq 0)}) e^{\mathcal{V}^{(0)}(\Psi^{(\leq 0)},A^{(\leq 0)})+\mathcal{B}^{(0)}(\Psi^{(\leq 0)},A^{(\leq 0)},\Phi)} \; , \end{aligned} \tag{4.5}$$

where the expression in the second line is obtained by an explicit integration of the UV degrees of freedom, which is very simple: in fact, the UV theory for the imaginary time variable in the presence of a fixed UV cutoff on the spatial variables is trivially convergent, see [15, Appendix C] or [48] for more technical details on this issue. The quantities F_0 and $S^{(\geq 0)}(\Phi)$ (normalized so that $S^{(\geq 0)}(0) = 0$) are the contributions to the specific free energy and to the generating function of response functions, respectively, coming from the ultraviolet integration. $\mathcal{V}^{(0)}$ and $\mathcal{B}^{(0)}$ (that are both normalized in such a way that $\mathcal{V}^{(0)}(0,0) = \mathcal{B}^{(0)}(0,0,\Phi) = 0$) are the effective interaction and the effective source term, whose structure will be explicitly spelled out below.

The integration of the IR effective theory is performed by using an iterative procedure, based on the following decomposition:

$$\begin{aligned} \hat{g}_\omega^{(\leq 0)}(\mathbf{k}') &= \sum_{h\leq 0} \hat{g}_\omega^{(h)}(\mathbf{k}') \; , \qquad \hat{g}_\omega^{(h)}(\mathbf{k}') = f_h(\mathbf{k}')\hat{S}_0(\mathbf{k}'+\mathbf{p}_F^\omega) \; , \\ \hat{w}^{(\leq 0)}(\mathbf{p}) &= \sum_{h\leq 0} \hat{w}^{(h)}(\mathbf{p}) \; , \qquad \hat{w}^{(h)}(\mathbf{p}) = \frac{f_h(\mathbf{p})}{2|\mathbf{p}|} \; , \end{aligned} \tag{4.6}$$

where $f_h(\mathbf{p}) = \chi(2^{-h}|\mathbf{p}|) - \chi(2^{-h+1}|\mathbf{p}|)$. At each step we integrate the propagators $\hat{g}^{(h)}$ and $\hat{w}^{(h)}$, $h = 0, -1, -2, \ldots$, corresponding to degrees of freedom on momentum scale of order $2^0, 2^{-1}, 2^{-2}, \ldots$, the result of the integration defining the new effective interaction and source terms at scale h. At each step, we identify the marginal and relevant terms in these effective potentials and correspondingly define the effective coupling constants at scale h. Finally, after having inserted the quadratic fermionic relevant and marginal terms in the gaussian Grassmann integration (so defining a flowing dressed fermionic propagator), we proceed to the next integration step. After the integration of the scales $0, -1, \ldots, -h+1$, we get (see below for an inductive proof)

$$e^{\mathcal{W}(\Phi)} = e^{-\beta L^2 F_h + S^{(\geq h)}(\Phi)} \int P(d\Psi^{(\leq h)})P(dA^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\Psi^{(\leq h)},A^{(\leq h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\Psi^{(\leq h)},A^{(\leq h)},\Phi)} \; , \tag{4.7}$$

where $P(d\Psi_\omega^{(\leq h)})$ and $P(dA^{(\leq h)})$ have propagators

$$\hat{g}_\omega^{(\leq h)}(\mathbf{k}') = -\frac{\chi_h(\mathbf{k}')}{Z_h(\mathbf{k}')} \begin{pmatrix} ik_0 & v_h(\mathbf{k}')\Omega^*(\vec{k}'+\vec{p}_F^{\,\omega}) \\ v_h(\mathbf{k}')\Omega(\vec{k}'+\vec{p}_F^{\,\omega}) & ik_0 \end{pmatrix}^{-1} \; , \qquad \hat{w}^{(\leq h)}(\mathbf{p}) = \frac{\chi_h(\mathbf{p})}{2|\mathbf{p}|} \; , \tag{4.8}$$

with $\chi_h(\mathbf{p}) = \chi(2^{-h}|\mathbf{p}|)$and $Z_h(\mathbf{k}'), v_h(\mathbf{k}')$ the effective wave function renormalization and Fermi velocity at scale h, to be inductively defined below. Moreover, if $Z_h = Z_h(\mathbf{0})$, the effective interaction and the effective source can be written as sums over monomials of the fields (we recall that all the external fields $\Phi^{(a)}$ with $a \neq K$ are set to zero, for notational simplicity):

$$\mathcal{V}^{(h)}(\sqrt{Z_h}\Psi, A) = \sum_{\substack{n,m\geq 0:\\ n+m\geq 1}} (Z_h)^n \int \Big[\prod_{i=1}^{2n} \hat{\Psi}^{\varepsilon_i}_{\mathbf{k}'_i,\sigma_i,\rho_i,\omega_i}\Big]\Big[\prod_{i=1}^{m} \hat{A}_{\mu_i,\mathbf{p}_i}\Big] \hat{W}^{(h)}_{2n,m,0}(\underline{\mathbf{k}}',\underline{\mathbf{p}})\delta_{\underline{\omega}}(\underline{\mathbf{k}}',\underline{\mathbf{p}}) \,, \tag{4.9}$$

$$\mathcal{B}^{(h)}(\sqrt{Z_h}\Psi, A, \Phi) = \sum_{\substack{n,m\geq 0:\\ n+m\geq 1}} \sum_{p\geq 1} (Z_h)^n \int \Big[\prod_{i=1}^{2n} \hat{\Psi}^{\varepsilon_i}_{\mathbf{k}'_i,\sigma_i,\rho_i,\omega_i}\Big]\Big[\prod_{i=1}^{m} \hat{A}_{\mu_i,\mathbf{p}_i}\Big]\Big[\prod_{i=1}^{p} \hat{\Phi}^{K}_{j_i,\mathbf{q}_i}\Big] \hat{W}^{(h)}_{2n,m,p}(\underline{\mathbf{k}}',\underline{\mathbf{p}},\underline{\mathbf{q}})\delta_{\underline{\omega}}(\underline{\mathbf{k}}',\underline{\mathbf{p}},\underline{\mathbf{q}}) \,,$$

where the integral sign is a shorthand for the sum over momenta and over the field labels, the underlined variables indicate a collection of variables (e.g., $\underline{\mathbf{k}}' = (\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n})$) and $\delta_{\underline{\omega}}$ enforces momentum conservation; note that, precisely because of momentum conservation, $\hat{W}^{(h)}_{2n,m,p}(\underline{\mathbf{k}}',\underline{\mathbf{p}},\underline{\mathbf{q}})$ explicitly depends on $2n+m+p-1$ variables rather than on $2n+m+p$ (the "missing" momentum, which can be eliminated using the delta, can be chosen arbitrarily among the variables $(\underline{\mathbf{k}}',\underline{\mathbf{p}},\underline{\mathbf{q}})$). In Eq.(4.9) the kernels $\hat{W}^{(h)}_{2n,m,p}$ also depend on the choice of the field labels $\underline{\varepsilon},\underline{\sigma},\underline{\rho},\underline{\omega},\underline{\mu}$, but we dropped these indices to avoid an overwhelming notation.

A. Emergent relativistic theory

In order to make the emergent relativistic structure of the theory apparent, it is convenient to rewrite the fermionic propagator in Eq.(4.8) as

$$\hat{g}^{(\leq h)}_{\omega}(\mathbf{k}') = \frac{1}{\beta L^2}\langle \Psi^-_{\mathbf{k}',\sigma,\omega}\Psi^+_{\mathbf{k}',\sigma,\omega}\rangle_h = \frac{\chi_h(\mathbf{k}')}{Z_h(\mathbf{k}')}\frac{1}{ik_0\Gamma^0_\omega + iv_h(\mathbf{k}')\vec{k}'\cdot\vec{\Gamma}_\omega}\big(1+R_{h,\omega}(\mathbf{k}')\big) \tag{4.10}$$

where $\mathbf{k}' = (k_0,\vec{k}')$, $|\bar{R}_{h,\omega}(\mathbf{k}')| \leq (\text{const.})|\mathbf{k}'|$ and

$$\Gamma^0_\omega = -\mathbb{1}\,, \qquad \Gamma^1_\omega = i\sigma_2\,, \qquad \Gamma^2_\omega = i\omega\sigma_1\,, \tag{4.11}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\,, \qquad \sigma_2 = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\,. \tag{4.12}$$

the standard Pauli matrices. We can introduce a "Dirac" 4-spinors, which makes the relation between the quasi-particle fields $\Psi^\pm_\omega$ and a theory of massless Dirac fermions more transparent:

$$\overline{\psi}^{(\leq h)}_{\mathbf{k}',\sigma} = \big(-\Psi^{(\leq h)+}_{\mathbf{k}',\sigma,2,-}, -\Psi^{(\leq h)+}_{\mathbf{k}',\sigma,1,-}, \Psi^{(\leq h)+}_{\mathbf{k}',\sigma,1,+}, \Psi^{(\leq h)+}_{\mathbf{k}',\sigma,2,+}\big)\,, \qquad \psi^{(\leq h)}_{\mathbf{k}',\sigma} = \begin{pmatrix}\Psi^{(\leq h)-}_{\mathbf{k}',\sigma,1,+}\\ \Psi^{(\leq h)-}_{\mathbf{k}',\sigma,2,+}\\ \Psi^{(\leq h)-}_{\mathbf{k}',\sigma,2,-}\\ \Psi^{(\leq h)-}_{\mathbf{k}',\sigma,1,-}\end{pmatrix}\,. \tag{4.13}$$

The propagator of the $\psi,\overline{\psi}$ fields reads:

$$\frac{1}{\beta L^2}\langle\psi^{(\leq h)}_{\mathbf{k}',\sigma}\,\overline{\psi}^{(\leq h)}_{\mathbf{k}',\sigma}\rangle_h = \frac{\chi_h(\mathbf{k}')}{Z_h(\mathbf{k}')}\frac{1}{ik_0\gamma_0 + iv_h(\mathbf{k}')\,\vec{k}'\cdot\vec{\gamma}}(1+R_h(\mathbf{k}'))\,, \tag{4.14}$$

where $|R_h(\mathbf{k}')| \leq (\text{const.})|\mathbf{k}'|$ and γ_μ, $\mu = 0,1,2$, are euclidean gamma matrices:

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}\,, \qquad \gamma_1 = \begin{pmatrix} 0 & i\sigma_2\\ i\sigma_2 & 0\end{pmatrix}\,, \qquad \gamma_2 = \begin{pmatrix} 0 & i\sigma_1\\ i\sigma_1 & 0\end{pmatrix}\,, \tag{4.15}$$

satisfying the anticommutation relations: $\{\gamma_\mu,\gamma_\nu\} = -2\delta_{\mu\nu}$. For what follows, it is also useful to define $\gamma_3 = \begin{pmatrix} 0 & -i\sigma_3\\ -i\sigma_3 & 0\end{pmatrix}$ and the corresponding fifth gamma matrix:

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\,, \tag{4.16}$$

which anticommutes with all the other gamma matrices: $\{\gamma_\mu, \gamma_5\} = 0$, $\forall \mu = 0, 1, 2, 3$.

Modulo the correction term R_h, the propagator in Eq.(4.14) is the same as the one for euclidean massless Dirac fermions in $2+1$ dimensions. The analysis is therefore very similar to the one performed in [18] for a system of interacting Dirac fermions coupled to with a massless gauge field. In particular, the scaling dimension of the kernels $\hat W^{(h)}_{2n,m,p}$ of the effective potential is the same, see [18]:

$$D = 3 - 2n - m - p \; , \tag{4.17}$$

where we use the convention that positive scaling dimensions correspond to relevant operators and viceversa.

B. Lattice symmetries

An important difference between the present case and the one studied in [18] is that here the propagator is not exactly equal to the Dirac one: on the contrary, it differs from it by the correction term proportional to $R_h(\mathbf{k}')$. Of course, this correction term is dimensionally negligible: therefore, it does not change the power counting. However, it violates some continuous relativistic symmetries used in [18] to exclude the presence of several relevant and marginal terms in the RG flow. One may fear that the lack of such symmetries might be responsible for the generation of new marginal or relevant terms, which are absent in the relativistic Dirac model. These potentially dangerous terms can be controlled through a careful analysis of the honeycomb lattice symmetries. In particular, it is proved in Appendix B that the effective interaction, the effective source and the gaussian integrations at all scales $h \le 0$ are separately invariant under the following symmetry transformations. Again, we spell out the symmetries only in the presence of the external field Φ^K, the effect of the other external fields being discussed in Appendix B. In the following formulas, we drop the scale label h for notational simplicity; moreover, we think of $\Psi^-_{\mathbf{k}',\sigma,\omega}$ ($\Psi^+_{\mathbf{k}',\sigma,\omega}$) as being the the column (row) vector of components $\Psi^-_{\mathbf{k}',\sigma,\rho,\omega}$ ($\Psi^+_{\mathbf{k}',\sigma,\rho,\omega}$), $\rho = 1, 2$, and we think of $\hat A_{\mathbf{p}}$ as being the column vector of components $\hat A_{\mu,\mathbf{p}}$, $\mu = 0, 1, 2$.

(1) Spin flip: $\hat\Psi^\varepsilon_{\mathbf{k}',\sigma,\omega} \to \hat\Psi^\varepsilon_{\mathbf{k}',-\sigma,\omega}$, and $A_{\mu,\mathbf{p}}, \hat\Phi^K_{j,\mathbf{p}}$ are left invariant;

(2) Global $U(1)$: $\hat\Psi^\varepsilon_{\mathbf{k}',\sigma,\omega} \to e^{i\varepsilon\alpha_\sigma}\hat\Psi^\varepsilon_{\mathbf{k}',\sigma,\omega}$, with $\alpha_\sigma \in \mathbb{R}$ independent of $\mathbf{k}'$, and $A_{\mu,\mathbf{p}}, \hat\Phi^K_{j,\mathbf{p}}$ are left invariant;

(3) Spin SO(2): $\begin{pmatrix}\hat\Psi^\varepsilon_{\mathbf{k}',\uparrow,\rho,\omega}\\ \hat\Psi^\varepsilon_{\mathbf{k}',\downarrow,\rho,\omega}\end{pmatrix} \to e^{i\theta\sigma_2}\begin{pmatrix}\hat\Psi^\varepsilon_{\mathbf{k}',\uparrow,\rho,\omega}\\ \hat\Psi^\varepsilon_{\mathbf{k}',\downarrow,\rho,\omega}\end{pmatrix}$, with θ independent of $\mathbf{k}'$, and $A_{\mu,\mathbf{p}}, \hat\Phi^K_{j,\mathbf{p}}$ are left invariant;

(4) Discrete spatial rotations: if $T\mathbf{k} = (k_0, e^{-i\frac{2\pi}{3}\sigma_2}\vec k)$ and $n_- = (1-\sigma_3)/2$,

$$\hat\Psi^-_{\mathbf{k}',\sigma,\omega} \to e^{i(\mathbf{p}_F^\omega+\mathbf{k}')(\boldsymbol{\delta}_3-\boldsymbol{\delta}_1)n_-}\hat\Psi^-_{T\mathbf{k}',\sigma,\omega}\;, \qquad \hat\Psi^+_{\mathbf{k}',\sigma,\omega} \to \hat\Psi^+_{T\mathbf{k}',\sigma,\omega}e^{-i(\mathbf{p}_F^\omega+\mathbf{k}')(\boldsymbol{\delta}_3-\boldsymbol{\delta}_1)n_-}\;,$$
$$\hat A_{\mathbf{p}} \to T^{-1}\hat A_{T\mathbf{p}}\;, \qquad \hat\Phi^K_{j,\mathbf{p}} \to \hat\Phi^K_{j+1,T\mathbf{p}}$$

(5) Complex conjugation: if c is a generic constant appearing in the effective potentials or in the gaussian integrations:

$$c \to c^*\;, \qquad \hat\Psi^\varepsilon_{\mathbf{k}',\sigma,\omega} \to \hat\Psi^\varepsilon_{-\mathbf{k}',\sigma,-\omega}\;, \qquad \hat A_{\mathbf{p}} \to -\hat A_{-\mathbf{p}}\;, \qquad \hat\Phi^K_{j,\mathbf{p}} \to \hat\Phi^K_{j,-\mathbf{p}}\;;$$

(6.a) Horizontal reflections: if $R_h\mathbf{k} = (k_0, -k_1, k_2)$ and $r_h 1 = 1$, $r_h 2 = 3$, $r_h 3 = 2$,

$$\hat\Psi^-_{\mathbf{k}',\sigma,\omega} \to \sigma_1\hat\Psi^-_{R_h\mathbf{k}',\sigma,\omega}\;, \quad \hat\Psi^+_{\mathbf{k}',\sigma,\omega} \to \hat\Psi^+_{R_h\mathbf{k}',\sigma,\omega}\sigma_1\;, \quad \hat A_{\mathbf{p}} \to R_h\hat A_{R_h\mathbf{p}}e^{i\mathbf{p}\boldsymbol{\delta}_1}\;, \quad \hat\Phi^K_{j,\mathbf{p}} \to \hat\Phi^K_{r_h j,R_h\mathbf{p}}e^{-i\mathbf{p}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\;;$$

(6.b) Vertical reflections: if $R_v\mathbf{k} = (k_0, k_1, -k_2)$ and $r_v 1 = 1$, $r_v 2 = 3$, $r_v 3 = 2$,

$$\hat\Psi^\varepsilon_{\mathbf{k}',\sigma,\omega} \to \hat\Psi^\varepsilon_{R_v\mathbf{k}',\sigma,-\omega}\;, \qquad \hat A_{\mathbf{p}} \to R_v\hat A_{R_v\mathbf{p}}\;, \qquad \hat\Phi^K_{j,\mathbf{p}} \to \hat\Phi^K_{r_v j,R_v\mathbf{p}}\;;$$

(7) Particle-hole: if $P\mathbf{k} = (k_0, -\vec k)$,

$$\hat\Psi^\varepsilon_{\mathbf{k}',\sigma,\omega} \to i\hat\Psi^{-\varepsilon}_{P\mathbf{k}',\sigma,-\omega}\;, \qquad \hat A_{\mathbf{p}} \to P\hat A_{-P\mathbf{p}}\;, \qquad \hat\Phi^K_{j,\mathbf{p}} \to \hat\Phi^K_{j,-P\mathbf{p}}\;;$$

(8) Time-reversal: if $I\mathbf{k} = (-k_0, \vec k)$,

$$\hat\Psi^-_{\mathbf{k}',\sigma,\omega} \to -i\sigma_3\hat\Psi^-_{I\mathbf{k}',\sigma,\omega}\;, \qquad \hat\Psi^+_{\mathbf{k}',\sigma,\omega} \to -i\hat\Psi^+_{I\mathbf{k}',\sigma,\omega}\sigma_3\;, \qquad \hat A_{\mathbf{p}} \to I\hat A_{I\mathbf{p}}\;, \qquad \hat\Phi^K_{j,\mathbf{p}} \to \hat\Phi^K_{j,I\mathbf{p}}\;.$$

In the following subsection, the implications of these symmetries on the structure of the marginal and relevant terms are discussed.

C. Localization and the symmetry properties of the local terms

In order to inductively prove Eq.(4.7), we write $\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$ and $\mathcal{B}^{(h)} = \mathcal{L}\mathcal{B}^{(h)} + \mathcal{R}\mathcal{B}^{(h)}$, where the $\mathcal{L}$ operator isolates the *local terms*, while $\mathcal{R}$ isolates the *irrelevant terms*; according to Eq.(4.17), we define

$$\mathcal{L}\hat{W}^{(h)}_{2n,m,p;\underline{\omega},\underline{\mu},\underline{j}}(\underline{\mathbf{k}}',\underline{\mathbf{p}},\underline{\mathbf{q}}) = \begin{cases} \hat{W}^{(h)}_{2n,m,p;\underline{\omega},\underline{\mu},\underline{j}}(\underline{\mathbf{0}},\underline{\mathbf{0}},\underline{\mathbf{0}}) \;, & \text{if } 2n+m+p=3 \;, \\ \big[1+(\underline{\mathbf{k}}',\underline{\omega})\cdot\partial_{(\underline{\mathbf{k}}',\underline{\omega})}+\underline{\mathbf{p}}\cdot\partial_{\underline{\mathbf{p}}}+\underline{\mathbf{q}}\cdot\partial_{\underline{\mathbf{q}}}\big)\hat{W}^{(h)}_{2n,m,p;\underline{\omega},\underline{\mu},\underline{j}}(\underline{\mathbf{0}},\underline{\mathbf{0}},\underline{\mathbf{0}}) \;, & \text{if } 2n+m+p=2 \;, \\ 0 \;, & \text{otherwise.} \end{cases} \tag{4.18}$$

In the second line, $(\underline{\mathbf{k}}',\underline{\omega})\cdot\partial_{(\underline{\mathbf{k}}',\underline{\omega})} = \sum_{i=1}^{2n}(\mathbf{k}'_i,\omega_i)\cdot\partial_{(\mathbf{k}'_i,\omega_i)}$, where $(\mathbf{k}',\omega)\cdot\partial_{(\mathbf{k}',\omega)}$ is a shorthand for

$$\begin{aligned} &(\mathbf{k}',\omega)\cdot\partial_{(\mathbf{k}',\omega)} = k_0\partial_{k_0} + \Omega(\vec{p}_F^{\,\omega}+\vec{k}')\,\partial_{\vec{k}',\omega} + \Omega^*(\vec{p}_F^{\,\omega}+\vec{k}')\,\partial^*_{\vec{k}',\omega} \;, \\ &\text{with} \qquad \partial_{\vec{k}',\omega} = \frac{1}{2}(-i\partial_{k'_1}+\omega\partial_{k'_2}) \;, \quad \partial^*_{\vec{k}',\omega} = \frac{1}{2}(i\partial_{k'_1}+\omega\partial_{k'_2}) \;. \end{aligned} \tag{4.19}$$

In Appendix C it is proved that, thanks to the symmetry properties (1)–(8) listed in the previous subsection, the only non-vanishing local terms with $2n+m+p=3$ are either those with $(2n,m,p)=(0,2,1)$ (i.e., terms of the form $\Phi^K AA$) or the *vertices*, with $2n=2$ and $m+p=1$ (i.e., terms of the form $A\Psi^+\Psi^-$ or $\Phi^K\Psi^+\Psi^-$). The latter have the following explicit structure:

$$\begin{aligned} &\mathcal{L}\hat{W}^{(h)}_{2,1,0;(\omega,\omega),\mu}(\mathbf{k}',\mathbf{p}) = i\lambda_{\mu,h}\Gamma^\mu_\omega \;, \qquad \mathcal{L}\hat{W}^{(h),K}_{2,0,1;(\omega,\pm\omega),j}(\mathbf{k}',\mathbf{p}) = \frac{Z^\pm_{K,h}}{Z_h}\Gamma^\pm_{\omega,j} \;, \\ &\text{where} \qquad \Gamma^\pm_{\omega,j} := \begin{pmatrix} 0 & e^{\pm i\omega\frac{2\pi}{3}(j-1)} \\ e^{-i\omega\frac{2\pi}{3}(j-1)} & 0 \end{pmatrix} \end{aligned} \tag{4.20}$$

and the apex K added to the kernels with $p \neq 0$ is meant to remind the reader that the external field Φ is of type K. The constants $\lambda_{\mu,h}$, $Z^\pm_{K,h}$, Z_h are real and $\lambda_{1,h}=\lambda_{2,h}$. Note that the kernel with $2n=2$ and $m=1$ with different omegas is zero simply because the photon field A, as well as the external field J, has an UV cutoff that makes the modes corresponding to momenta $\mathbf{p}$ close to $\pm(\mathbf{p}_F^+ - \mathbf{p}_F^-)$ and to their images over Λ^* vanishing, see Remark after item 2 in Section 3.

Moreover, the only non-vanishing local terms with $2n+m+p=2$ are either those with $(2n,m,p)=(2,0,0)$ (i.e., terms of the form $\Psi^+\Psi^-$) or those with $(2n,m,p)=(0,2,0)$ (i.e., terms of the form AA). They have the following explicit structure (see Appendix C for a proof):

$$\mathcal{L}\hat{W}^{(h)}_{2,0,0;(\omega,\omega)}(\mathbf{k}') = \begin{pmatrix} iz_{0,h} & z_{1,h}\Omega^*(\vec{p}_F^{\,\omega}+\vec{k}') \\ z_{1,h}\Omega(\vec{p}_F^{\,\omega}+\vec{k}') & iz_{0,h} \end{pmatrix} \;, \qquad \mathcal{L}\hat{W}^{(h)}_{0,2,0;(\mu,\nu)}(\mathbf{p}) = \nu_{\mu,h}\delta_{\mu\nu} \;, \tag{4.21}$$

where $z_{\mu,h}$ and $\nu_{\mu,h}$ are real and $\nu_{1,h}=\nu_{2,h}$.

D. The single-scale RG step: the inductive integration procedure

The splitting into local and irrelevant terms is used in the inductive integration of the generating functional in the following way: we rewrite the integral in the r.h.s. of Eq.(4.7) as

$$\begin{aligned} &\int P(d\Psi^{(\le h)})P(dA^{(\le h)})e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\Psi^{(\le h)},A^{(\le h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\Psi^{(\le h)},A^{(\le h)},\Phi)} = \\ &= e^{\beta L^2 t_h}\int \widetilde{P}(d\Psi^{(\le h)})P(dA^{(\le h)})e^{\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\Psi^{(\le h)},A^{(\le h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\Psi^{(\le h)},A^{(\le h)},\Phi)} \;, \end{aligned} \tag{4.22}$$

where $\widetilde{\mathcal{V}}^{(h)} = \mathcal{V}^{(h)} - \mathcal{L}_\psi\mathcal{V}^{(h)}$, with $\mathcal{L}_\psi\mathcal{V}^{(h)}$ the contribution to $\mathcal{L}\mathcal{V}^{(h)}$ that is quadratic in the fermionic fields (i.e., the one corresponding to the first term in Eq.(4.21)) and t_h is a normalization constant. Moreover, $\widetilde{P}(d\Psi^{(\le h)})$ has a propagator given by the same expression as Eq.(4.8) but for the fact that $Z_h(\mathbf{k}')$ and $v_h(\mathbf{k}')$ are replaced by $Z_{h-1}(\mathbf{k}')$ and $v_{h-1}(\mathbf{k}')$, respectively, where

$$Z_{h-1}(\mathbf{k}') = Z_h(\mathbf{k}') + Z_h z_{0,h}\chi_h(\mathbf{k}') \;, \quad Z_{h-1}(\mathbf{k}')v_{h-1}(\mathbf{k}') = Z_h(\mathbf{k}')v_h(\mathbf{k}') + Z_h z_{1,h}\chi_h(\mathbf{k}') \;, \tag{4.23}$$

and $Z_h = Z_h(\mathbf{0})$, $v_h = v_h(\mathbf{0})$. After this, defining $Z_{h-1} = Z_{h-1}(\mathbf{0})$, we *rescale* the fermionic field by setting

$$\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\Psi^{(\leq h)}, A^{(\leq h)}) =: \hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\Psi^{(\leq h)}, A^{(\leq h)})$$
$$\mathcal{B}^{(h)}(\sqrt{Z_h}\Psi^{(\leq h)}, A^{(\leq h)}, \Phi) =: \hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\Psi^{(\leq h)}, A^{(\leq h)}, \Phi)\ . \tag{4.24}$$

By using Eqs.(4.20)-(4.21), the local part of the rescaled effective potential can be written as

$$\mathcal{L}\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\Psi^{(\leq h)}, A^{(\leq h)}) = \frac{1}{\beta|\mathcal{S}_L|}\sum_{\mu,\mathbf{p}}\left[Z_{h-1}e_{\mu,h}\hat{\jmath}^{(\leq h)}_{\mu,\mathbf{p}}\hat{A}^{(\leq h)}_{\mu,\mathbf{p}} - 2^h\nu_{\mu,h}\hat{A}^{(\leq h)}_{\mu,-\mathbf{p}}\hat{A}^{(\leq h)}_{\mu,\mathbf{p}}\right] \tag{4.25}$$

where $e_{\mu,h} := \frac{Z_h}{Z_{h-1}}\lambda_{\mu,h}$ and, if $v_{h-1} := v_{h-1}(\mathbf{0})$,

$$\hat{\jmath}^{(\leq h)}_{0,\mathbf{p}} := \frac{i}{\beta L^2}\sum_{\omega,\sigma,\mathbf{k}'}\hat{\Psi}^{(\leq h)+}_{\mathbf{k}'+\mathbf{p},\sigma,\omega}\Gamma^0_\omega\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\sigma,\omega}\ ,\quad \vec{\jmath}^{(\leq h)}_{\mathbf{p}} := \frac{iv_{h-1}}{\beta L^2}\sum_{\omega,\sigma,\mathbf{k}'}\hat{\Psi}^{(\leq h)+}_{\mathbf{k}'+\mathbf{p},\sigma,\omega}\vec{\Gamma}_\omega\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\sigma,\omega}\ . \tag{4.26}$$

Note that, by using the notation defined in Eqs.(4.13),(4.15), the density and the current in Eq.(4.26) can be rewritten in the familiar relativistic form as

$$\hat{\jmath}^{(\leq h)}_{0,\mathbf{p}} := \frac{i}{\beta L^2}\sum_{\sigma,\mathbf{k}'}\overline{\psi}^{(\leq h)}_{\mathbf{k}'+\mathbf{p},\sigma}\gamma_0\psi^{(\leq h)}_{\mathbf{k}',\sigma}\ ,\quad \vec{\jmath}^{(\leq h)}_{\mathbf{p}} := \frac{iv_{h-1}}{\beta L^2}\sum_{\sigma,\mathbf{k}'}\overline{\psi}^{(\leq h)}_{\mathbf{k}'+\mathbf{p},\sigma}\vec{\gamma}\psi^{(\leq h)}_{\mathbf{k}',\sigma}\ . \tag{4.27}$$

Finally, by using Eq.(4.20) and the properties stated right before this equation, we find that the local part of the effective source term is given by

$$\mathcal{L}\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\Psi^{(\leq h)}, A^{(\leq h)}, \Phi) = \frac{1}{\beta^2|\mathcal{S}_L|^2}\sum_{\substack{\mathbf{p},\mathbf{q}\\ j,\mu_1,\mu_2}}\lambda^K_{j,(\mu_1,\mu_2),h}\hat{\Phi}^K_{j,\mathbf{q}}\hat{A}^{(\leq h)}_{\mu_1,\mathbf{p}}\hat{A}^{(\leq h)}_{\mu_2,-\mathbf{p}-\mathbf{q}} +$$
$$+\frac{1}{\beta^2L^2|\mathcal{S}_L|}\sum_{\substack{\mathbf{k}',\mathbf{p}\\ \omega,\sigma,j}}\left[Z^+_{K,h}\hat{\Phi}^K_{j,\mathbf{p}}\hat{\Psi}^{(\leq h)+}_{\mathbf{k}'+\mathbf{p},\sigma,\omega}\Gamma^+_{\omega,j}\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\sigma,\omega} + Z^-_{K,h}\hat{\Phi}^K_{j,\mathbf{p}^\omega_F-\mathbf{p}^{-\omega}_F+\mathbf{p}}\hat{\Psi}^{(\leq h)+}_{\mathbf{k}'+\mathbf{p},\sigma,\omega}\Gamma^-_{\omega,j}\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\sigma,-\omega}\right]\ . \tag{4.28}$$

Note that, in contrast to what happens in a relativistic QFT, the effective source term Eq.(4.28) contains marginal terms that were not present in the original functional integral, i.e., the terms $\Phi^K AA$ in the first line of Eq.(4.28). These potentially dangerous terms can be shown to be harmless by using the lattice symmetries (1) − (8), see Section 5 B for a discussion of this point (nevertheless, let us anticipate that the reason why these terms do not create troubles is that they are "almost zero", precisely because they vanish in the relativistic approximation; therefore, their naive dimensional bound can be improved and they can be shown to be effectively irrelevant).

After the rescaling Eq.(4.24), we rewrite the r.h.s. of Eq.(4.22) as

$$e^{-\beta L^2 t_h}\int P(d\Psi^{(\leq h-1)})P(dA^{(\leq h-1)})\int P(d\Psi^{(h)})P(dA^{(h)})e^{\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\Psi^{(\leq h)},A^{(\leq h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\Psi^{(\leq h)},A^{(\leq h)},\Phi)}\ , \tag{4.29}$$

where $P(d\Psi^{(\leq h-1)})$, $P(dA^{(\leq h-1)})$ have propagators given by (4.8) with h replaced by $h-1$, while $P(dA^{(h)})$, $P(d\Psi^{(h)})$ have propagators

$$\hat{w}^{(h)}(\mathbf{p}) := \frac{f_h(\mathbf{p})}{2|\mathbf{p}|}\ ,\qquad \frac{\hat{g}^{(h)}_\omega(\mathbf{k}')}{Z_{h-1}} := \frac{\tilde{f}_h(\mathbf{k}')}{Z_{h-1}}\frac{1}{ik_0\Gamma^0_\omega + iv_{h-1}(\mathbf{k}')\vec{k}'\cdot\vec{\Gamma}_\omega}(1+R'_{h,\omega}(\mathbf{k}'))\ , \tag{4.30}$$

where $\tilde{f}_h(\mathbf{k}') := Z_{h-1}f_h(\mathbf{k}')/Z_{h-1}(\mathbf{k}')$ and $|R'_{h,\omega}(\mathbf{k}')| \leq (\text{const.})|\mathbf{k}'|$.

Remark. The single scale propagator can be decomposed as a sum of a Dirac-like propagator $\hat{g}^{(h)}_{D,h}(\mathbf{k}')$, which is the propagator obtained by setting $R'_{h,\omega} = 0$ in the second definition in Eq.(4.30), plus a rest, which has a better infrared behavior. We shall correspondingly write $\hat{g}^{(h)}_\omega(\mathbf{k}') = \hat{g}^{(h)}_{D,\omega}(\mathbf{k}') + r^{(h)}_\omega(\mathbf{k}')$. This decomposition will be useful in the following, as already anticipated by the comment after Eq.(4.28).

At this point, we can finally integrate the fields on scale h and, defining

$$e^{-\beta L^2 F_{h-1}+S^{(\geq h-1)}(\Phi)}e^{\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\Psi^{(\leq h-1)},A^{(\leq h-1)})+\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\Psi^{(\leq h-1)},A^{(\leq h-1)},\Phi)} := \tag{4.31}$$
$$= e^{-\beta L^2(F_h+t_h)+S^{(\geq h)}(\Phi)}\int P(d\Psi^{(h)})P(dA^{(h)})e^{\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\Psi^{(\leq h)},A^{(\leq h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\Psi^{(\leq h)},A^{(\leq h)},\Phi)}\ ,$$

our inductive assumption Eq.(4.7) is reproduced at scale $h-1$. Note that Eq.(4.31) can be thought as a recursive definition for the effective potential. The integration in Eq.(4.31) is performed by expanding in series the exponential in the r.h.s. and by integrating term by term with respect to the gaussian integration $P(d\psi^{(h)})P(dA^{(h)})$. This procedure gives rise to an expansion for the effective interaction and source terms in terms of the renormalized parameters $\{e_{\mu,k}, \nu_{\mu,k}, Z_{k-1}, v_{k-1}, Z^{\pm}_{K,k}, \lambda^K_{j,\underline{\mu},k}\}_{h<k\le 0}$, which can be conveniently represented as a sum over Gallavotti-Nicolò (GN) trees [14]; the value of each GN tree can be thought of as a sum over connected labelled Feynman diagrams, i.e., every GN tree represents a set of Feynman diagrams characterized by the same hierarchical structure of the scale labels associated to the propagators, see [18, Section 2.2] for a thorough discussion of this expansion.

We will call $\{e_{\mu,k}, \nu_{\mu,k}\}_{k\le 0}$ the *effective couplings* or *running coupling constants*. The constants $e_{\mu,h}$ play the role of *effective charges*, while $\nu_{\mu,h}$ play the role of *effective photon masses*. It will be shown below that the effective charges stay constant under the RG flow (more precisely, $e_{\mu,h}$ are essentially independent of μ and h) and that the effective photon mass is vanishing (more precisely, $\nu_{\mu,h}$ is small uniformly in h, on the "right scale").

Remark. Note the unusual dependence of the effective charges on the index μ: the global symmetries (1)–(8) discussed above ensure that $e_{1,h} = e_{2,h}$ but they do not a priori guarantee that $e_{0,h} = e_{1,h}$. This situation is in striking contrast with what happens in QFT, where Lorentz invariance guarantees such a property to be valid at all scales. However, in the next section we will show that, thanks to lattice WIs, $e_{0,h}$ and $e_{1,h}$, even if not exactly equal to each other at all scales, admit the same limit as $h\to-\infty$; namely, $e_{0,-\infty} = e_{1,-\infty} = e + O(e^2)$.

The expansion in GN trees and labelled Feynman diagrams allows us to obtain the following inductive estimate on the kernels $\hat W_{2n,m,p}$ in Eq.(4.9). Let $\bar\varepsilon_h = \max_{h<k\le 0}\{|e_{\mu,k}|, |\nu_{\mu,k}|\}$ be small enough. If $Z_k/Z_{k-1} \le e^{C\bar\varepsilon_h^2}$ and $C^{-1} \le v_{k-1} \le 1$, for all $h<k\le 0$ and a suitable constant $C>0$, then the N-th order contribution to $\hat W_{2n,m,p}$ in the effective couplings (to be denoted by $\hat W^{N;(h)}_{2n,m,p}$) admits the following bound (the "N! bound"):

$$||W^{N;(h)}_{2n,m,p}|| \le (\text{const.})^N \bar\varepsilon_h^N \Big(\frac{N}{2}\Big)! \,, \tag{4.32}$$

where $||W^{N;(h)}_{2n,m,p}|| := \; 2^{-h(3-2n-m-p)} \sup |W^{N;(h)}_{2n,m,p}(\underline{\mathbf{k}}', \underline{\mathbf{p}}, \underline{\mathbf{q}})|$ and the sup is performed with respect to the momenta and the field labels. The proof of Eq.(4.32) can be found in [18, Section 2.4]. The basic ingredient in the proof is a dimensional estimate of the kernels, which follows from the bounds:

$$\begin{aligned} &|\hat w^{(h)}(\mathbf{p})| \le (\text{const.})2^{-h}\,, \qquad \frac{1}{\beta|\mathcal{S}_L|}\sum_{\mathbf{p}} |\hat w^{(h)}(\mathbf{p})| \le (\text{const.})2^{2h}\,,\\ &\|\hat g^{(h)}_\omega(\mathbf{k}')\| \le (\text{const.})2^{-h}\,, \qquad \frac{1}{\beta L^2}\sum_{\mathbf{k}'} \|\hat g^{(h)}_\omega(\mathbf{k}')\| \le (\text{const.})2^{2h}\,, \end{aligned} \tag{4.33}$$

that is, every propagator on scale h is associated to a factor 2^{-h} and every loop integral on scale h is associated to a factor 2^{3h}. The estimate Eq.(4.32) follows from: (i) counting the number of propagators and loop integrals on each scale h for every given labelled Feynman diagram; (ii) realizing that the corresponding dimensional estimate is uniform in the Feynman diagram, for all the diagrams associated to the same GN tree; (iii) performing the sum over scale labels for every fixed GN tree; (iv) counting the number of Feynman diagrams associated to each GN tree and the total number of GN trees contributing to order N in renormalized perturbation theory. See [18, Proof of Theorem 2.1] for more details.

The bound Eq.(4.32) tells us that the N-th order contribution to the effective potential is *finite in norm*, uniformly in h. *If the effective couplings remain small in the infrared*, informations obtained from our renormalized expansion by lowest order truncations are reliable at weak coupling. The importance of having an expansion with finite coefficients should not be underestimated. The naive perturbative expansion in e the fine structure constant is plagued by *logarithmic infrared divergences* and higher orders are more and more divergent. More precisely, one can find classes of diagrams of order N in e contributing to the effective potential on scale h whose size grows like $O(|h|^n)$. Therefore, finite order truncations of the naive perturbation theory do not give a priori any reliable information on the IR behavior of the theory, not even at weak coupling.

Regarding the combinatorial factor in the r.h.s. of Eq.(4.32), we note that the $(N/2)!$ dependence is compatible with Borel summability of the theory. However, summability does not follow from our bounds. Constructive estimates on large fields for the bosonic sector (in the spirit of, e.g., [6]) combined with determinant estimates for the fermionic sector (in the spirit of, e.g., [15]) may allow a full non-perturbative construction of the theory. However, this goes beyond the scope of this paper.

Let us conclude this section by adding a comment, which will be useful for the study of the flow of the effective parameters discussed in the next sections. Consider the splitting $\hat g^{(h)}_\omega(\mathbf{k}') = \hat g^{(h)}_{D,\omega}(\mathbf{k}') + r^{(h)}_\omega(\mathbf{k}')$ mentioned in the remark after Eq.(4.30). As mentioned there, the rest $r_\omega^{(h)}$ is better behaved in the infrared than the relativistic propagator $\hat g^{(h)}_{D,\omega}$. More precisely,

$$\|r_\omega^{(h)}(\mathbf{k}')\| \le \text{const.}\,, \qquad \frac{1}{\beta L^2}\sum_{\mathbf{k}'}\|r_\omega^{(h)}(\mathbf{k}')\| \le (\text{const.})2^{3h}\,, \tag{4.34}$$

which should be compared with the second line of Eq.(4.33). It is apparent that $r_\omega^{(h)}$ is associated to a dimensional gain 2^h as compared to the leading term $\hat g^{(h)}_{D,\omega}(\mathbf{k}')$. This implies that if we decompose $W^{N;(h)}_{2n,m,p}$ as

$$\hat W^{N;(h)}_{2n,m,p} = \hat W^{N;(h),D}_{2n,m,p} + \widetilde W^{N;(h)}_{2n,m,p} \tag{4.35}$$

where $W^{N;(h),D}_{2n,m,p}$ is obtained from $W^{N;(h)}_{2n,m,p}$ by replacing all the propagators $\hat g^{(h)}_\omega(\mathbf{k}')$ by $\hat g^{(h)}_{D,\omega}(\mathbf{k}')$ (and by neglecting the contributions coming from the UV propagators on scale $h=1$), then $\widetilde W^{N;(h)}_{2n,m,p}$ is dimensionally negligible in the IR as compared to the "relativistic" contribution $\hat W^{N;(h),D}_{2n,m,p}$; i.e., $\widetilde W^{N;(h)}_{2n,m,p}$ admits a bound similar to Eq.(4.32), with an extra factor (dimensional gain) proportional to $2^{\theta h}$, with $0<\theta<1$. This follows from the improved dimensional estimate on the propagator $r_\omega^{(h)}$, Eq.(4.34), and from the fact that "long GN trees are exponentially depressed", i.e., the property referred to as "short memory property", see [18, Section 2.4].

5. WARD IDENTITIES AND THE FLOW OF THE RENORMALIZED PARAMETERS

We have seen that the effective potentials (and, similarly, the correlation functions) can be written as series in the effective charges $e_{\mu,h}$ and the effective masses $\nu_{\mu,h}$ with bounded coefficients at all orders, uniformly in the infrared cut-off, provided that the ratios Z_h/Z_{h-1} remain close to 1 and that effective Fermi velocity remains bounded away from zero along the RG flow. Of course, such expansions are useful only if the running coupling constants remain small for all values of h, a fact that we are going to prove to be true, thanks to exact lattice WIs. In this section we first study the flow of the running coupling constants $\{e_{\mu,h},\nu_{\mu,h}\}_{h\le 0}$ and next the one of the other renormalized parameters.

A. The flow of the electric charge and of the photon mass

The key idea is to get informations on the running coupling constants $\{e_{\mu,h},\nu_{\mu,h}\}_{h\le 0}$ by using the WIs Eq.(3.13) for the sequence of reference models $\mathcal{W}^{0,h^*}(\Phi,J,\lambda)$, where the scale of the bosonic IR cutoff h^* is thought of as a parameter. For each choice of the IR cutoff h^*, the generating functional $\mathcal{W}^{0,h^*}(\Phi,J,\lambda)$ is computed by a multiscale integration procedure similar to the one described in the previous section, with the important difference that after the integration of the scale h^* we are left with a purely fermionic theory, which is *super-renormalizable*: in fact, setting $m=0$ in the formula Eq.(4.17) for the scaling dimension of the kernels of the effective potentials, we see that the scaling dimension of the reference model *below* the cutoff h^* is $D=3-2n-p$. In particular, the kernels of the effective interaction (i.e., those with $p=0$) are always negative once the two-legged subdiagrams have been renormalized; as shown in [15, 20, 21], the effect of the integration of the scales $\le h^*$ is just to renormalize by a small finite amount the effective parameters $Z_{h^*}, v_{h^*}, Z^{\pm}_{K,h^*}, \lambda^K_{j,\underline{\mu},h^*}$. Let us denote by $\{e^{[h^*]}_{\mu,h},\nu^{[h^*]}_{\mu,h}\}_{h\le 0}$ the running coupling constants of the reference model with infrared cut-off on scale h^*. Of course, if $h\ge h^*$,

$$\{e^{[h^*]}_{\mu,h},\nu^{[h^*]}_{\mu,h}\}_{h^*\le h\le 0} = \{e_{\mu,h},\nu_{\mu,h}\}_{h^*\le h\le 0}\,, \tag{5.1}$$

where the constants in the r.h.s. are those of the model without IR cutoff, i.e., $e_{\mu,h} := e^{[-\infty]}_{\mu,h}$ and $\nu_{\mu,h} := \nu^{[-\infty]}_{\mu,h}$. On the other hand, as proved in [18, Appendix B], the two- and three-points correlation functions of the reference model are proportional to the inverse wave function renormalization and to the effective charges, i.e., if $\mathbf{k}'$ and $\mathbf{p}$ are such that $|\mathbf{k}'| = 2^{h^*}$ and $|\mathbf{p}| \le 2^{h^*}$, then

$$\hat{\mathcal{S}}^{0,h^*}_{2,0}(\mathbf{p}^\omega_F + \mathbf{k}') = \frac{\hat{g}^{(h^*)}_\omega(\mathbf{k}')}{Z_{h^*-1}}\big(1 + \bar{B}_{\omega,h^*}(\mathbf{k}')\big) \ , \tag{5.2}$$

$$\hat{\mathcal{S}}^{0,h^*}_{2,1;0}(\mathbf{p}^\omega_F + \mathbf{k}', \mathbf{p}) = i\frac{\hat{g}^{(h^*)}_\omega(\mathbf{k}'+\mathbf{p})}{Z_{h^*-1}}\Big(e_{0,h^*}\Gamma^0_\omega + eB^0_{\omega,h^*}(\mathbf{k}',\mathbf{p})\Big)\hat{g}^{(h^*)}_\omega(\mathbf{k}') \ , \tag{5.3}$$

$$\hat{\mathcal{S}}^{0,h^*}_{2,1;l}(\mathbf{p}^\omega_F + \mathbf{k}', \mathbf{p}) = iv_{h^*-1}\frac{\hat{g}^{(h^*)}_\omega(\mathbf{k}'+\mathbf{p})}{Z_{h^*-1}}\Big(e_{1,h^*}\Gamma^l_\omega + eB^l_{\omega,h^*}(\mathbf{k}',\mathbf{p})\Big)\hat{g}^{(h^*)}_\omega(\mathbf{k}') \ , \qquad l \in \{1,2\} \ , \tag{5.4}$$

where the correction terms $\bar{B}_{\omega,h^*}(\mathbf{k}')$ and $B^\mu_{\omega,h^*}(\mathbf{k}',\mathbf{p})$ are of order $\bar{\varepsilon}^2_{h^*}$ (recall that $\bar{\varepsilon}_h = \max\{|e_{\mu,k}|, |\nu_{\mu,k}|\}_{h\le k\le 0}$), uniformly in h^*, for all $|\mathbf{k}'| = 2^{h^*}$ and $|\mathbf{p}| \le 2^{h^*}$. For later use, let us note that $\bar{B}_{\omega,h^*}(\mathbf{k}')$ is differentiable in $\mathbf{k}'$ and its derivatives computed at $|\mathbf{k}'| = 2^{h^*}$ are dimensionally bounded by (const.)$2^{-h^*}\bar{\varepsilon}^2_{h^*}$.

Thanks to Eqs.(5.2)–(5.4), we see that informations on the mutual relations between the correlation functions of the reference model with cutoff h^* (which are provided by the WIs) imply relations between the effective charges on scale h. Regarding the photon mass, an equation similar in spirit to Eqs.(5.2)-(5.4) is valid, namely, if $|\mathbf{p}| \le 2^{h^*}$,

$$\hat{\mathcal{S}}^{0,h^*}_{0,2;(\mu,\nu)}(\mathbf{p}) = 2^{h^*}\Big(\nu_{\mu,h^*}\delta_{\mu\nu} + B^{\mu\nu}_{h^*}(\mathbf{p})\Big) \ , \tag{5.5}$$

where the correction term $B^{\mu\nu}_{h^*}(\mathbf{p})$ is of order $\bar{\varepsilon}^2_{h^*}$, uniformly in h^*, for all $|\mathbf{p}| \le 2^{h^*}$. The proof of Eq.(5.5) can be worked out along the same lines of [18, Appendix B] and is based on the following remarks: for all scales $h > h^*$, by construction, the external field J appears in the effective potential in the combination $A+J$, see Eq.(3.1). As discussed in the previous section, the corresponding kernel, $\hat{W}^{(h)}_{0,2,0;(\mu,\nu)}(\mathbf{p})$ is equal to $\hat{W}^{(h)}_{0,2,0;(\mu,\nu)}(\mathbf{p}) = 2^h\Big(\nu_{\mu,h}\delta_{\mu\nu} + \bar{B}^{\mu\nu}_h(\mathbf{p})\Big)$, with $\bar{B}^{\mu\nu}_h(\mathbf{p})$ a bounded correction, of second or higher order in the effective coupling constants. Therefore, after the integration of all the scales $h \ge h^*$, we are left with a kernel JJ equal to the r.h.s. of Eq.(5.5), with a slightly different correction term $\bar{B}^{\mu\nu}_{h^*}(\mathbf{p})$ replacing $B^{\mu\nu}_{h^*}(\mathbf{p})$. From that scale on, we are left with the super-renormalizable theory studied in [15, 21], in which the only marginal interactions are the $J\Psi^+\Psi^-$ terms, whose coefficient is renormalized by a finite amount under the RG flow from h^* to $-\infty$, see [21]. Therefore, the dominant correction terms to $\hat{\mathcal{S}}^{0,h^*}_{0,2;(\mu,\nu)}(\mathbf{p})$ coming from the integration of the scales below h^* are obtained by contracting two effective vertices $J\Psi^+\Psi^-$ on scale $h < h^*$, and then summing over h. The resulting contribution is dimensionally bounded as $\sum_{h<h^*} O(2^h\bar{\varepsilon}^2_{h^*}) = O(2^{h^*}\bar{\varepsilon}^2_{h^*})$. Higher order corrections are bounded in a similar way, using the hierarchical structure of the GN trees, see also [21].

Now, combining Eq.(5.5) with the first WI in Eq.(3.13) computed at $\mathbf{p} = p\mathbf{u}_\mu$ (with $\mathbf{u}_\mu$ the unit vector in direction μ and $p = |\mathbf{p}| \le 2^{h^*}$), we find:

$$\nu_{\mu,h^*} = -\lim_{p\to 0} B^{\mu\mu}_{h^*}(p\mathbf{u}_\mu) = O(\bar{\varepsilon}^2_{h^*}) \ , \tag{5.6}$$

which means that there is no spontaneous generation of the photon mass (i.e., the photon field remains unscreened). If read in naive (non-renormalized) perturbation theory, the above identity is equivalent to an infinite sequence of cancellations taking place at all orders among the graphs contributing to the photon mass. At lowest order, the cancellation takes place between the two graphs in Fig.3, as discussed in Appendix E 1.

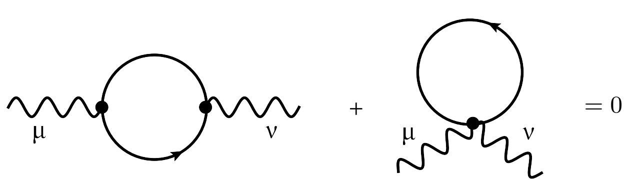

FIG. 3. The lowest order contributions to the "mass" of the field A_μ with $\mu = 1, 2$, which cancel out exactly when computed at transferred momentum $\mathbf{p} = \mathbf{0}$, as proved in Appendix E 1.

A similar argument can be applied to control the flow of the effective charge. In fact, note that by computing the second WI in Eq.(3.13) at $(\mathbf{k}', \mathbf{p}) = (2^{h^*}\mathbf{u}_\mu, p\mathbf{u}_\mu)$ and by taking the limit $p \to 0$, we find:

$$\hat{S}^{0,h^*}_{2,1;\mu}(\mathbf{p}^\omega_F + 2^{h^*}\mathbf{u}_\mu, \mathbf{0}) = -e\partial_\mu \hat{S}^{0,h^*}_{2,0}(\mathbf{p}^\omega_F + 2^{h^*}\mathbf{u}_\mu) \ . \tag{5.7}$$

By plugging Eqs.(5.2)–(5.4) into Eq.(5.7), we get

$$e_{0,h^*} = e\Big(1 + B^0_{\omega,h^*}(\mathbf{p}_F^\omega + 2^{h^*}\mathbf{u}_0, \mathbf{0}) - 2^{h^*}\partial_0 \bar{B}_{\omega,h^*}(\mathbf{p}_F^\omega + 2^{h^*}\mathbf{u}_0)\Big) =: e(1+\bar{A}_{0,h^*}) = e(1+O(\bar\varepsilon^2_{h^*}))\ , \tag{5.8}$$

$$e_{1,h^*} = e\Big(1 + B^1_{\omega,h^*}(\mathbf{p}_F^\omega + 2^{h^*}\mathbf{u}_1, \mathbf{0})\Gamma^1_\omega - 2^{h^*}\partial_1 \bar{B}_{\omega,h^*}(\mathbf{p}_F^\omega + 2^{h^*}\mathbf{u}_1)\Big) =: e(1+\bar{A}_{1,h^*}) = e(1+O(\bar\varepsilon^2_{h^*}))\ , \tag{5.9}$$

which tell us that *the effective charges e_{μ,h^*} remain close to the unperturbed value* $e_{\mu,0} = e$ at all scales $h^* \le 0$ and at all orders in renormalized perturbation theory. If read in naive perturbation theory, Eqs.(5.8)-(5.9) are equivalent to infinitely many cancellations taking place at all orders among the logarithmically divergent graphs contributing to the dressing of the electric charge. At lowest order, the cancellation takes place between the graphs in Fig.4, as discussed in Appendix E 2.

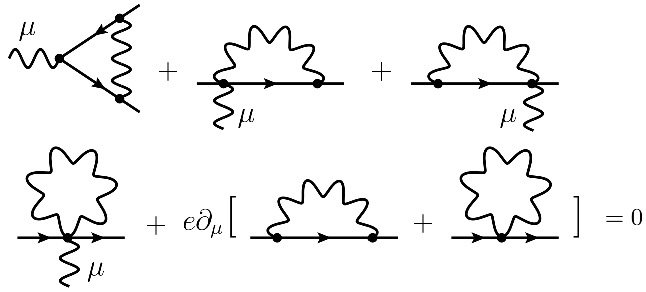

FIG. 4. The lowest order contributions to the dressing of the electric charge, which cancel out exactly when computed at the Fermi points $\mathbf{k} = \mathbf{p}_F^\omega$ and at transferred momentum $\mathbf{p} = \mathbf{0}$, as proved in Appendix E 2.

The corrections $\bar{A}_{\mu,h^*}$ in Eqs.(5.8)-(5.9) are given by sums over GN trees of order two or higher in the effective couplings; their N-th order is bounded by $(\text{const.})^N(\bar\varepsilon_{h^*})^N(N/2)!$ Moreover, $|\bar{A}_{\mu,h^*} - \bar{A}_{\mu,-\infty}| = O(\bar\varepsilon^2_{-\infty}(v_{-\infty} - v_h)) + O(\bar\varepsilon^2_{-\infty}2^{h/2})$, see [18, Section 4 and Appendix B] for a proof of these facts. Note that a priori $\bar{A}_{0,-\infty} \ne \bar{A}_{1,-\infty}$; however, the approximate Lorentz invariance of the theory combined with the flow equation for the Fermi velocity will allow us to show that $\bar{A}_{0,-\infty} = \bar{A}_{1,-\infty}$, see the next subsection for a discussion of this point.

Remark. The discussion in this section is very similar to the corresponding one for a model of Dirac fermions with electromagnetic interactions in the continuum in the presence of a fixed rotationally invariant UV cutoff, see [18, Section 3]. The key difference is that the analogues of Eqs.(5.8)-(5.9) in [18, Section 3] have an extra term in the l.h.s., i.e., they read $e_{\mu,h^*}(1-\alpha_\mu) = e(1 + A_{\mu,h^*})$, with $|A_{\mu,h^*} - A_{\mu,-\infty}| = O(\bar\varepsilon^2_{-\infty}(v_{-\infty} - v_h)) + O(\bar\varepsilon^2_{-\infty}2^{h/2})$ and $|A_{0,-\infty} - A_{1,-\infty}| \le (\text{const.})e^4$; see [18, Eqs.(4.13)-(4.14)]. The extra constants α_μ in the l.h.s. are due to the corrections to the WIs produced by the momentum cutoff used in [18], which explicitly breaks gauge invariance; as shown in [18], $\alpha_0 \ne \alpha_1$, the difference being of second order in the electric charge, which implies that Lorentz invariance is not recovered asymptotically in the IR. This has to be contrasted with the case studied here, where exact lattice gauge invariance is preserved by the RG flow, so that no correction terms appear in the l.h.s. of Eqs.(5.8)-(5.9) and Lorentz invariance spontaneously emerge in the deep IR, as proved in the next subsection.

B. The flow of the effective parameters

We are now left with studying the evolution of the effective parameters $v_h, Z_h, Z^\pm_{K,h}, \lambda^K_{j,\underline{\mu},h}$ under the RG flow. If we use the analogue of the decomposition Eq.(4.35), we can write the flow equation for these parameters as:

$$\frac{v_{h-1}}{v_h} = 1 + \beta^v_h + r^v_h\ , \qquad \frac{Z_{h-1}}{Z_h} = 1 + \beta^z_h + r^z_h\ , \tag{5.10}$$

$$\frac{Z^\pm_{K,h-1}}{Z^\pm_{K,h}} = 1 + \beta^\pm_h + r^\pm_h\ , \qquad \lambda^K_{j,\underline{\mu},h-1} = \lambda^K_{j,\underline{\mu},h} + r^\lambda_{j,\underline{\mu},h}\ , \tag{5.11}$$

where $\beta^\#_h$ are the relativistic part of the *beta function* that, by definition, are obtained by replacing all the propagators $\hat{g}^{(h)}_\omega(\mathbf{k}')$ contributing to the r.h.s. of the flow equation by their relativistic part $\hat{g}^{(h)}_{D,\omega}(\mathbf{k}')$, while $r^\#_h$ are the rests, which are smaller by a factor $2^{\theta h}$, $\theta \in (0,1)$, as compared to the corresponding dominant terms (the reason is the same as the one sketched after Eq.(4.35)). Note that both $\beta_h^\#$ and $r_h^\#$ are functions of the whole sequence of effective parameters $\{e_{\mu,k}, \nu_{\mu,k}, Z_{k-1}, v_{k-1}, Z^{\pm}_{K,k}, \lambda^K_{j,\underline{\mu},h}\}_{h\le k\le 0}$ that are bounded respectively by $O(\bar\varepsilon_h^2)$ and $O(2^{\theta h}\bar\varepsilon_h^2)$, provided that the ratios Z_k/Z_{k-1} are close to 1 and that v_k are bounded away from zero, for all $h\le k\le 0$.

In the second equation in Eq.(5.11), we used the fact that the dominant part $\beta^\lambda_{j,\underline{\mu},h}$ is exactly zero for all choices of $j,\underline{\mu},h$, by a parity argument (inspection of perturbation theory immediately shows that all the contributions to $\beta^\lambda_{j,\underline{\mu},h}$ are given by integrals of odd functions of $\mathbf{k}'$ over a domain that is invariant under $\mathbf{k}'\to-\mathbf{k}'$). The bound on $r^\lambda_{j,\underline{\mu},h}$ immediately shows that $\lambda^K_{j,\underline{\mu},h}$ is uniformly bounded by $O(\bar\varepsilon_h^2)$ in the IR.

Regarding the flow of v_h, Z_h, we note that modulo the correction terms r_h^v and r_h^z, they are the same as those for Dirac fermions in the continuum, derived and written down in [18]. In particular, using [18, Eq.(3.14)], we can write:

$$\frac{v_{h-1}}{v_h} = 1+\frac{\log 2}{4\pi^2}\left[\frac{8}{5}e^2(1-v_h)(1+A_h')+\frac{4}{3}e(1+B_h')(e_{0,h}-e_{1,h})\right]+r_h^v\ , \tag{5.12}$$

where A_h' is a sum of contributions that are finite at all orders in the effective couplings, which are either of order two or more in the effective charges, or vanishing at $v_k=1$; similarly, B_h' is a sum of contributions that are finite at all orders in the effective couplings, which are of order two or more in the effective charges. The proof of Eq.(5.12) is based on the remark that $\beta_{h,v}$ would be vanishing if $v_h:=1$ and $e_{0,h}:=e_{1,h}$, by Lorentz invariance (see [18, Section 3.3] for more details); the numerical coefficients follow from an explicit computation, see [18, Appendix C]. From (5.12) it is apparent that v_h tends as $h\to-\infty$ to a limit value

$$v_{-\infty}=1+\frac{5}{6e}(e_{0,-\infty}-e_{1,-\infty})(1+C'_{-\infty}) \tag{5.13}$$

with $C'_{-\infty}$ a sum of contributions that are finite at all orders in the effective couplings, which are of order two or more in the effective charges. The fixed point (5.13) is found simply by requiring that in the limit $h\to-\infty$ the argument of the square brackets in (5.12) vanishes.

Consider now the identities Eqs.(5.8)-(5.9). In a Lorentz invariant theory (i.e., in a theory where all the propagators $\hat g_\omega^{(h)}(\mathbf{k}')$ are replaced by their Dirac approximations $\hat g^{(h)}_{D,\omega}(\mathbf{k}')$, the Fermi velocity is equal to the speed of light, $v_h:=1$, and the charges $e_{\mu,h}$ are μ-independent, $e_{0,h}=e_{1,h}$), we would get that $A_{0,h}=A_{1,h}$. Therefore, using the GN trees representation for $A_{0,h}, A_{1,h}$, their approximate Lorentz symmetry and the short memory property (see [18, Section 2.4]), we find:

$$e_{0,h-1}-e_{1,h-1}=E_{1,h}(e_{0,h}-e_{1,h})+E_{2,h}(1-v_h)+E_{3,h}\ , \tag{5.14}$$

where $E_{1,h}$ and $E_{2,h}$ are $O(\bar\varepsilon_h^2)$ and $E_{3,h}=O(\bar\varepsilon_h 2^{\theta h})$, for $\theta\in(0,1)$. If we combine Eq.(5.14) with Eq.(5.12) we get

$$e_{0,-\infty}-e_{1,-\infty}=E_{-\infty}(e_{0,-\infty}-e_{1,-\infty})\ , \tag{5.15}$$

with $E_{-\infty}=O(\bar\varepsilon^2_{-\infty})$, which implies the *spontaneous emergence of Lorentz invariance* in the deep IR, i.e.,

$$e_{0,-\infty}=e_{1,-\infty}\quad\text{and}\quad v_{-\infty}=1\ . \tag{5.16}$$

Using these informations into Eq.(5.12), we see that the approach of v_h to the speed of light is *anomalous*, i.e.,

$$1-v_h\simeq A(v)\,2^{\tilde\eta h}\ ,\qquad \tilde\eta=\frac{2e^2}{5\pi^2}+O(e^4)\ , \tag{5.17}$$

where $A(v)$ is a function of e and v that vanishes linearly at $v=1$, i.e., for v close to 1, $A(v)=(1-v)\big(1+O(1-v)+O(e^2)\big)$. The "$\simeq$" in the first equation means that the ratio of the two sides tends to 1 as $h\to-\infty$.

Regarding the flow of the other renormalization parameters, a second order computation of the dominant contribution to the beta function (see [18, Appendix C] and Appendix D 1), the fact that $e_{\mu,h}$ are close to $e_{-\infty}=e(1+O(e^2))$, asymptotically as $h\to-\infty$, together with Eq.(5.17), shows that

$$\beta_h^z=\frac{e^2}{12\pi^2}\log 2+O(e^2(1-v)2^{ce^2h})+O(e^4)\ , \tag{5.18}$$

$$\beta_h^+=\frac{e^2}{12\pi^2}\log 2+O(e^2(1-v)2^{ce^2h})+O(e^4)\ ,\qquad \beta_h^-=\frac{3e^2}{4\pi^2}\log 2+O(e^2(1-v)2^{ce^2h})+O(e^4)\ , \tag{5.19}$$

for some $c > 0$. Eqs.(5.18)-(5.19) imply that

$$Z_h \simeq B^0(v)\, 2^{-h\eta}\ , \qquad Z^+_{K,h} \simeq B^+(v)\, 2^{-h\eta^+_K}\ , \qquad Z^-_{K,h} \simeq B^-(v)\, 2^{-h\eta^-_K}\ , \tag{5.20}$$

with

$$\eta = \frac{e^2}{12\pi^2} + O(e^4)\ , \qquad \eta^+_K = \frac{e^2}{12\pi^2} + O(e^4)\ , \qquad \eta^-_K = \frac{3e^2}{4\pi^2} + O(e^4)\ , \tag{5.21}$$

and $B^\#(v) = 1 + O(1-v) + O(e^2)$, which concludes the study of the flow of the renormalized parameters.

6. THE KEKULÉ RESPONSE FUNCTION

In this section we explicitly compute the Kekulé response function (the other responses can be obtained in a similar way and will be discussed in the next section),

$$R^{(K)}_{ij}(\mathbf{x}) = \lim_{\beta\to\infty}\lim_{L\to\infty}\lim_{h^*\to-\infty} \frac{\partial^2}{\partial\Phi^K_{i,\mathbf{x}}\partial\Phi^K_{j,\mathbf{0}}} \mathcal{W}^{0,h^*}(\Phi,0,0)\Big|_{\Phi=0}\ . \tag{6.1}$$

The iterative construction of the generating functional described in Section 4 induces an explicit representation for the Kekulé response function in terms of GN trees, completely analogous to the one described in [21, Proof of Proposition 1]. In particular, using the analogue of the decomposition in Eq.(4.35), we can rewrite

$$R^{(K)}_{ij}(\mathbf{x}) = R^{(K),D}_{ij}(\mathbf{x}) + \widetilde{R}^{(K)}_{ij}(\mathbf{x})\ , \tag{6.2}$$

where $R^{(K),D}_{ij}(\mathbf{x})$ is obtained by replacing the lattice by the Dirac propagators in the expansion for $R^{(K)}_{ij}$ and $\widetilde{R}^{(K)}_{ij}(\mathbf{x})$ is the rest. Using the same strategy leading to the bounds Eq.(4.32) and, more specifically, to [21, Eqs.(2.81)–(2.84)], we find that the N-th order contribution in renormalized perturbation theory to $R^{(K),D}_{ij}(\mathbf{x})$ is bounded, for all $M \geq 0$, by

$$\big|R^{N;(K),D}_{ij}(\mathbf{x})\big| \leq |e|^N \Big(\frac{N}{2}\Big)! \sum_{h=-\infty}^{0}\sum_{\bar h=h}^{0}\sum_{\omega=\pm}\Big(\frac{Z^\omega_{K,h}}{Z_h}\Big)^2 2^h 2^{3\bar h}\frac{(C_M)^N}{1+(2^{\bar h}|\mathbf{x}|)^M}\ , \tag{6.3}$$

for a suitable constant C_M. The factors $2^h 2^{3\bar h}\big[1+(2^{\bar h}|\mathbf{x}|)^M\big]^{-1}$ represent the dimensional bound on all the labelled Feynman diagrams corresponding to the same GN tree, where: (i) h is the lowest among the scales of the propagators in the diagram; (ii) $\bar h$ is the lowest among the scales of the propagators in a path connecting the two special vertices of type $\Phi^K\Psi^+\Psi^-$; (iii) $2^h = 2^{h(3-2n-m-p)}\big|^{n=m=0}_{p=2}$ is the scaling dimension of the graph; (iv) $2^{3\bar h}$ is the dimensional gain coming from the fact that the locations $\mathbf{x}$ and $\mathbf{0}$ of the external fields Φ^K are *fixed* rather than integrated over the whole space-time domain (as it is the case for the Feynman diagrams contributing to the thermodynamic functions, where all the space-time labels of the vertices are integrated over the whole space); (v) $\big[1+(2^{\bar h}|\mathbf{x}|)^M\big]^{-1}$ is the decay factor coming from the propagators on a path connecting the two special vertices of type $\Phi^K\Psi^+\Psi^-$. Now, picking $M = 5$, exchanging the order of summations over h and $\bar h$, and summing over h gives (recall the asymptotic relations Eq.(5.20))

$$\begin{aligned}
&\big|R^{N;(K),D}_{ij}(\mathbf{x})\big| \leq (\text{const.})^N |e|^N\Big(\frac{N}{2}\Big)! \sum_{\bar h=-\infty}^{0}\frac{2^{\bar h(4+2\eta-2\eta^-_K)}}{1+(2^{\bar h}|\mathbf{x}|)^5} \quad\Rightarrow\\
\Rightarrow\quad &\big|R^{N;(K),D}_{ij}(\mathbf{x})\big| \leq (\text{const.})^N |e|^N\Big(\frac{N}{2}\Big)!\frac{1}{1+|\mathbf{x}|^{4+2\eta-2\eta^-_K}}\ ,
\end{aligned}\tag{6.4}$$

where we used the fact that $\eta^-_K > \eta^+_K$. The correction term $\widetilde{R}^{(K)}_{ij}(\mathbf{x})$ admits a similar bound, with a dimensional gain factor that implies a faster decay in real space. Using the symmetries in Appendix B together with relativistic invariance (see symmetry [18, (7)]) and proceeding as in Appendix C, we find that the symmetry structure of $R^{N;(K)}_{ij}(\mathbf{x})$ is:

$$R^{N;(K)}_{ij}(\mathbf{x}) = a^{(N)}_K e^N \frac{\cos\big(\vec p^{\,+}_F(\vec x - \vec\delta_i + \vec\delta_j)\big)}{|\mathbf{x}|^{4-\xi^{(K)}}} + \textit{faster decaying terms}, \tag{6.5}$$

where the faster decaying correction terms come from: (i) the irrelevant terms that, scale by scale, produce corrections smaller by a factor $O(2^h)$ as compared to the dominant terms; (ii) the Lorentz-symmetry breaking terms, i.e., the terms proportional to $1-v_h$ that come from a rewriting of the effective Fermi velocity v_h in the definition of $\hat{g}_{D,\omega}$ as $v_h = 1-(1-v_h)$; these produce corrections that, scale by scale, are smaller by a factor $O(e^2 2^{(\text{const.})e^2 h})$ as compared to the dominant terms. In Eq.(6.5), $\xi^{(K)} = 2\eta^-_K - 2\eta = \frac{4e^2}{3\pi^2} + O(e^4)$ and $a^{(N)}_K$ is a suitable constant, bounded in absolute value by $(\text{const.})^N \left(\frac{N}{2}\right)!$ The 0-th order constant, which gives the dominant contribution to the Kekulé response function, is given by the value of the graph in Fig.5.

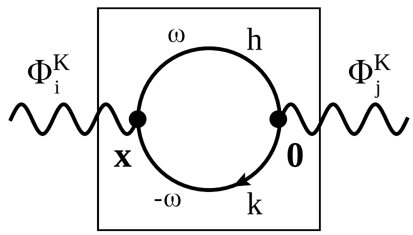

FIG. 5. Leading contribution to the Kekulé response function; a sum over $h \le 0$ is understood.

More explicitly,

$$R^{0;(K),D}_{ij}(\mathbf{x}) = -2 \sum_{h,k=-\infty}^{0} \sum_{\omega} e^{i(\mathbf{p}^\omega_F - \mathbf{p}^{-\omega}_F)\mathbf{x}} \frac{(Z^-_{K,h\vee k})^2}{Z_h Z_k} \int \frac{d\mathbf{k}}{2\pi|\mathcal{B}|} \int \frac{d\mathbf{p}}{2\pi|\mathcal{B}|} f_h(\mathbf{k}) f_k(\mathbf{p}) e^{i(\mathbf{k}-\mathbf{p})\mathbf{x}} \cdot$$

$$\cdot \mathrm{Tr}\Big\{ \frac{-ik_0\Gamma^0_\omega + i\vec{k}\cdot\vec{\Gamma}_\omega}{\mathbf{k}^2} \Gamma^-_{\omega,i} \frac{-ip_0\Gamma^0_{-\omega} + i\vec{p}\cdot\vec{\Gamma}_{-\omega}}{\mathbf{p}^2} \Gamma^-_{-\omega,j} \Big\} + O(e^2|\mathbf{x}|^{-4+\xi^{(K)}}) + \textit{faster decaying terms}\;, \tag{6.6}$$

where $h\vee k = \max\{h,k\}$. Recalling that $\Gamma^-_{\omega,j} = e^{-i\omega\theta_j}\sigma_1 = e^{i\mathbf{p}^\omega_F(\boldsymbol{\delta}_j - \boldsymbol{\delta}_1)}\sigma_1$ and using the fact that $\mathrm{Tr}\{\Gamma^\nu_\omega \sigma_1 \Gamma^\nu_{-\omega}\sigma_1\} = 2$ for all $\nu \in \{0,1,2\}$ and that $\mathrm{Tr}\{\Gamma^\mu_\omega \sigma_1 \Gamma^\nu_{-\omega}\sigma_1\} = 0$ for $\mu \neq \nu$, we can rewrite

$$R^{0;(K),D}_{ij}(\mathbf{x}) = 4 \sum_{h,k\le 0} \sum_{\omega} e^{-i\mathbf{p}^\omega_F(\mathbf{x}-\boldsymbol{\delta}_i+\boldsymbol{\delta}_j)} \frac{(Z^-_{K,h\vee k})^2}{Z_h Z_k} \int \frac{d\mathbf{k}}{2\pi|\mathcal{B}|} \int \frac{d\mathbf{p}}{2\pi|\mathcal{B}|} f_h(\mathbf{k}) f_k(\mathbf{p}) e^{i(\mathbf{k}-\mathbf{p})\mathbf{x}} \frac{\mathbf{k}\cdot\mathbf{p}}{|\mathbf{k}|^2|\mathbf{p}|^2}\;, \tag{6.7}$$

modulo corrections $O(e^2|\mathbf{x}|^{-4+\xi^{(K)}})$ or decaying faster than $|\mathbf{x}|^{-4+\xi^{(K)}}$ at infinity. Using Eq.(5.20), we can replace $\frac{(Z^-_{K,h\vee k})^2}{Z_h Z_k}$ by $\big(\frac{B^-(v)}{B^0(v)}\big)^2 2^{-2\eta^-_K(h\vee k)} 2^{\eta h + \eta k}$, modulo faster decaying corrections. Finally, rewriting, e.g., $2^{\eta h} = |\mathbf{x}|^{-\eta}\big[1 + \big((2^h|\mathbf{x}|)^\eta - 1\big)\big]$, it is easy to realize that the contributions to the response function coming from the terms like $|\mathbf{x}|^{-\eta}\big[(2^h|\mathbf{x}|)^\eta - 1\big]$ are $O(e^2|\mathbf{x}|^{-4+\xi^{(K)}})$, see [48, Section 4.3.2]. Therefore, using the identity $\sum_{h\le 0} f_h(\mathbf{k}) = \chi(\mathbf{k})$,

$$R^{0;(K),D}_{ij}(\mathbf{x}) = 4\Big(\frac{B^-(v)}{B^0(v)}\Big)^2 |\mathbf{x}|^{2(\eta^-_K - \eta)} \sum_{\omega} e^{-i\mathbf{p}^\omega_F(\mathbf{x}-\boldsymbol{\delta}_i+\boldsymbol{\delta}_j)} \int \frac{d\mathbf{k}}{2\pi|\mathcal{B}|}\chi(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}} \frac{\mathbf{k}}{|\mathbf{k}|^2} \cdot \int \frac{d\mathbf{p}}{2\pi|\mathcal{B}|}\chi(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \frac{\mathbf{p}}{|\mathbf{p}|^2}\;, \tag{6.8}$$

modulo the aforementioned corrections. Finally, using the fact that $\int \frac{d\mathbf{k}}{2\pi|\mathcal{B}|}\chi(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}} \frac{\mathbf{k}}{|\mathbf{k}|^2} = \frac{i3\sqrt{3}}{8\pi}\frac{\mathbf{x}}{|\mathbf{x}|^3} +$ *faster decaying terms*, we can rewrite:

$$R^{0;(K),D}_{ij}(\mathbf{x}) = \frac{27}{8\pi^2}\Big(\frac{B^-(v)}{B^0(v)}\Big)^2 \frac{\cos\Big(\mathbf{p}^+_F(\mathbf{x}-\boldsymbol{\delta}_i+\boldsymbol{\delta}_j)\Big)}{|\mathbf{x}|^{4-\xi^{(K)}}} + O(e^2|\mathbf{x}|^{-4+\xi^{(K)}}) + \textit{faster decaying terms}\;. \tag{6.9}$$

Using the fact that $B^{\#}(v) = 1 + O(1-v) + O(e^2)$ we finally get Eq.(2.20).

As we already remarked in the introduction, the effect of the interaction is that of *enhancing* the response function of the Kekulé distortion, but still preserving its integrability, so that $\hat{R}^{(K)}_{ij}(\mathbf{p})$ is finite. However, $\partial_\mathbf{p}\hat{R}^{(K)}_{ij}(\mathbf{p})$ is *singular* in $\mathbf{p} = \mathbf{p}^\omega_F$. In fact, the Fourier transform of Eq.(2.20) gives:

$$\hat{R}^{(K)}_{ij}(\mathbf{p}) = (\text{const.})\Big[e^{-i\mathbf{p}^+_F(\boldsymbol{\delta}_i-\boldsymbol{\delta}_j)}|\mathbf{p}-\mathbf{p}^+_F|^{1-\xi^{(K)}} + e^{-i\mathbf{p}^-_F(\boldsymbol{\delta}_i-\boldsymbol{\delta}_j)}|\mathbf{p}-\mathbf{p}^-_F|^{1-\xi^{(K)}}\Big] + \textit{more regular terms}\;, \tag{6.10}$$

which implies that $\partial_\mathbf{p}\hat{R}^{(K)}_{ij}(\mathbf{p})$ diverges like $|\mathbf{p}-\mathbf{p}^\omega_F|^{-\xi^{(K)}}$ at the Fermi points, as claimed in Section 2 C.

7. OTHER RESPONSE FUNCTIONS

An analysis analogous to the one for the Kekulé response function can be worked out for the other responses and is sketched below. Regarding the CDW and AF responses, the symmetries of the model imply that the relevant terms of the form $\Phi^{CDW}A$ or $\Phi^{AF}A$ are vanishing, while the marginal terms of the form $\Phi^{CDW}\Psi^+\Psi^-$ and $\Phi^{AF}\Psi^+\Psi^-$ have the following structure:

$$\mathcal{L}\hat{\mathcal{B}}^{(h)}_{CDW}(\sqrt{Z_{h-1}}\Psi^{(\le h)}, A^{(\le h)}, \Phi) = \tag{7.1}$$
$$= \frac{1}{\beta^2 L^2 |\mathcal{S}_L|}\sum_{\substack{\mathbf{k}',\mathbf{p}\\ \omega,\sigma,j}}\Big[Z^+_{CDW,h}\hat{\Phi}^{CDW}_{j,\mathbf{p}}\hat{\Psi}^{(\le h)+}_{\mathbf{k}'+\mathbf{p},\sigma,\omega}\sigma_3\hat{\Psi}^{(\le h)-}_{\mathbf{k}',\sigma,\omega} + Z^-_{K,h}\hat{\Phi}^{CDW}_{j,\mathbf{p}^\omega_F-\mathbf{p}^{-\omega}_F+\mathbf{p}}\hat{\Psi}^{(\le h)+}_{\mathbf{k}'+\mathbf{p},\sigma,\omega}\big[e^{i\omega\theta_j n_-}\sigma_3\big]\hat{\Psi}^{(\le h)-}_{\mathbf{k}',\sigma,-\omega}\Big] ,$$

$$\mathcal{L}\hat{\mathcal{B}}^{(h)}_{AF}(\sqrt{Z_{h-1}}\Psi^{(\le h)}, A^{(\le h)}, \Phi) = \tag{7.2}$$
$$= \frac{1}{\beta^2 L^2 |\mathcal{S}_L|}\sum_{\substack{\mathbf{k}',\mathbf{p}\\ \omega,\sigma,j}}\sigma\Big[Z^+_{AF,h}\hat{\Phi}^{AF}_{j,\mathbf{p}}\hat{\Psi}^{(\le h)+}_{\mathbf{k}'+\mathbf{p},\sigma,\omega}\sigma_3\hat{\Psi}^{(\le h)-}_{\mathbf{k}',\sigma,\omega} + Z^-_{K,h}\hat{\Phi}^{AF}_{j,\mathbf{p}^\omega_F-\mathbf{p}^{-\omega}_F+\mathbf{p}}\hat{\Psi}^{(\le h)+}_{\mathbf{k}'+\mathbf{p},\sigma,\omega}\big[e^{i\omega\theta_j n_-}\sigma_3\big]\hat{\Psi}^{(\le h)-}_{\mathbf{k}',\sigma,-\omega}\Big] .$$

Similar expressions are valid for the other external fields Φ^a. Note that besides the marginal terms spelled out in the previous equation, there may also be marginal terms of the form ΦAA, $\Phi\Phi A$ or $p_\mu \Phi A$; the flow of the corresponding coupling constants can be easily controlled along the same lines followed to prove the boundedness of the flow of $\lambda^K_{j,\underline{\mu},h}$ in Section 5 B.

As seen in the previous section, the flow of the renormalization constants $Z^\pm_{CDW,h}$ and $Z^\pm_{AF,h}$ control the large distance decay of the CDW and AF response functions. In Appendix D 2 it is shown that

$$\frac{Z^+_{CDW,h-1}}{Z^+_{CDW,h}} = 1 + \frac{3e^2}{4\pi^2}\log 2 + O(e^4) + \cdots , \qquad \frac{Z^-_{CDW,h-1}}{Z^-_{CDW,h}} = 1 + \frac{e^2}{12\pi^2}\log 2 + O(e^4) + \cdots , \tag{7.3}$$

which implies that

$$Z^\pm_{CDW,h} \simeq B^\pm_{CDW}(v)\, 2^{-h\eta^\pm_{CDW}} , \tag{7.4}$$

with

$$\eta^+_{CDW} = \frac{3e^2}{4\pi^2} + O(e^4) , \qquad \eta^-_{CDW} = \frac{e^2}{12\pi^2} + O(e^4) , \tag{7.5}$$

and $B^\pm_{CDW}(v) = 1 + O(1-v) + O(e^2)$. The exponents for the AF are exactly the same, simply because the kernels of the marginal source terms $\Phi^{AF}\Psi^+\Psi^-$ are the same as the corresponding terms of the form $\Phi^{CDW}\Psi^+\Psi^-$, see Eqs.(7.1)-(7.2). Therefore, proceeding as in the previous section, we get, for $a = CDW, AF$,

$$R^{(a)}_{ij}(\mathbf{x}) = \frac{27}{8\pi^2}\Big(\frac{B^+_a(v)}{B^0(v)}\Big)^2\frac{1}{|\mathbf{x}|^{4-\xi^{(a)}}} + O(e^2|\mathbf{x}|^{-4+\xi^{(a)}}) + \textit{faster decaying terms} , \tag{7.6}$$

$$\xi^{(CDW)} = \xi^{(AF)} = 2(\eta^+_{CDW} - \eta) = \frac{4e^2}{3\pi^2} + O(e^4) . \tag{7.7}$$

As for the Kekulé response function, the anomalous decay in Eq.(7.6) implies the presence of a singularity in the first derivative of the corresponding Fourier transforms, in analogy with Eq.(6.10).

It is now clear how to extend such analysis to all other responses. The large distance asymptotic behavior depends on the specific values of the critical exponents, which are determined at leading order by a second order computation along the lines of Appendix D: one first identifies the structure of the local part of the effective source terms, in analogy with Eqs.(4.28),(7.1),(7.2); for each such term two renormalization constants appear, corresponding to processes with transferred momentum equal to $\mathbf{0}$ or to $\mathbf{p}^\omega_F - \mathbf{p}^{-\omega}_F$; the flow of each of them is given by a diagram like the one in Fig.6 and can be computed as in Appendix D. The resulting values of the critical exponents $\xi^{(channel)} = 2(\eta_{channel} - \eta)$ at second order in the electric charge are summarized in Table I.

The interaction removes the degeneracy of the critical exponents and, therefore, we can identify the excitations whose response functions decay slowest at infinity, which correspond to the order parameters in putative strong coupling broken phases. In particular:

channel	critical exponent
$a^+_{\omega,\sigma}b_{\omega,\sigma}+b^+_{\omega,\sigma}a_{\omega,\sigma}$	0
$a^+_{\omega,\sigma}b_{-\omega,\sigma}+b^+_{\omega,\sigma}a_{-\omega,\sigma}$	$4e^2/(3\pi^2)$
$ia^+_{\omega,\sigma}b_{\omega,\sigma}-ib^+_{\omega,\sigma}a_{\omega,\sigma}$	0
$ia^+_{\omega,\sigma}b_{-\omega,\sigma}-ib^+_{\omega,\sigma}a_{-\omega,\sigma}$	0
$a^+_{\omega,\sigma}a_{\omega,\sigma}+b^+_{\omega,\sigma}b_{\omega,\sigma}$	0
$a^+_{\omega,\sigma}a_{-\omega,\sigma}+b^+_{\omega,\sigma}b_{-\omega,\sigma}$	0
$a^+_{\omega,\sigma}a_{\omega,\sigma}-b^+_{\omega,\sigma}b_{\omega,\sigma}$	$4e^2/(3\pi^2)$
$a^+_{\omega,\sigma}a_{-\omega,\sigma}-b^+_{\omega,\sigma}b_{-\omega,\sigma}$	0

channel	critical exponent
$\sigma(a^+_{\omega,\sigma}a_{\omega,\sigma}+b^+_{\omega,\sigma}b_{\omega,\sigma})$	0
$\sigma(a^+_{\omega,\sigma}a_{-\omega,\sigma}+b^+_{\omega,\sigma}b_{-\omega,\sigma})$	0
$\sigma(a^+_{\omega,\sigma}a_{\omega,\sigma}-b^+_{\omega,\sigma}b_{\omega,\sigma})$	$4e^2/(3\pi^2)$
$\sigma(a^+_{\omega,\sigma}a_{-\omega,\sigma}-b^+_{\omega,\sigma}b_{-\omega,\sigma})$	0
$\omega(a^+_{\omega,\sigma}a_{\omega,\sigma}-b^+_{\omega,\sigma}b_{\omega,\sigma})$	$4e^2/(3\pi^2)$
$i\sigma(a^+_{\omega,\sigma}a^+_{\omega,-\sigma}+b^+_{\omega,\sigma}b^+_{\omega,-\sigma})+c.c.$	$-e^2/(3\pi^2)$
$i\sigma\omega(a^+_{\omega,\sigma}a^+_{\omega,-\sigma}+b^+_{\omega,\sigma}b^+_{\omega,-\sigma})+c.c.$	$-e^2/(3\pi^2)$
$a^+_{\omega,\sigma}b^+_{-\omega,\sigma}-b^+_{\omega,\sigma}a^+_{\omega,\sigma}+c.c.$	$-e^2/(3\pi^2)$
$i\sigma(a^+_{\omega,\sigma}b^+_{-\omega,-\sigma}-b^+_{\omega,\sigma}a^+_{-\omega,-\sigma})+c.c.$	$-e^2/(3\pi^2)$

TABLE I. Lowest order contributions to the anomalous exponents. Summation over the valley and spin indices is understood.

1. the dominant exponents correspond to the second and seventh channel in the left table and to the third and fifth channel in the right table. The first three are the K, CDW and AF local order parameters discussed in the introduction. The fifth channel in the right table was introduced in [27]. Microscopically, this order parameter may be understood as a specific pattern of circulating currents and its enhancement in the presence of interactions is in agreement with the expectation that a time-reversal broken fixed point should emerge in the strong coupling regime [30].

2. The last four channels in the right table are Majorana-type masses that correspond to inter-node and intra-node (uniform and non-uniform) Cooper pairings [37]. Their critical exponents are all negative, which means that superconducting order is unfavored at intermediate to strong coupling. There are other possible Cooper pairings besides those explicitly reported in the table and it turns out that all their exponents are either equal to or smaller than $-e^2/(3\pi^2)$ at second order.

3. The exponents of the density-density and current-current response functions at zero transferred momentum (third and fifth channel in the left table) are vanishing. Actually, using the second WI in Eq.(3.13) it can be easily proved that these two exponents are zero at all orders in renormalized perturbation theory. This is indeed a necessary prerequisite for having finite conductivity and a semi-metallic behavior also in the presence of interactions, as expected [49].

8. RENORMALIZATION GROUP ANALYSIS IN PRESENCE OF A MASS TERM

As discussed in the introduction, it is natural to investigate the effects of a mass term coupled to the local order parameters associated to the largest critical exponents. In this section we discuss how the RG construction is modified by the presence of a mass. Consider, e.g., the Hamiltonian in Eq.(2.27) (similar considerations are valid for CDW, AF masses or for the "Haldane mass", i.e., the mass associated to the local order parameter in the fifth line of the right part of Table I). The model is still invariant under the same symmetries (1)–(8) described in Section 4 B, the only novelty being that now the value of j_0 appearing in the very definition of the Kekulé mass, see Eq.(2.27), should also be changed under the symmetry transformations; i.e.,

- under (4), $j_0 \to j_0 - 1$;
- under (6.b), $j_0 \to r_v j_0$;
- j_0 is left invariant in all other cases.

In the presence of the mass term, the multiscale integration of the generating functional is performed in a way very similar to the one described in Section 4 for the massless case. After the integration of the ultraviolet degrees of freedom and the definition of the quasi-particle fields, one immediately realizes that the presence of the mass produces new local relevant terms of the form $\sum_{\omega,\sigma}\Delta^{(j_0)}_{\omega,h}a^{(\le h)+}_{\mathbf{k}',\sigma,\omega}b^{(\le h)-}_{\mathbf{k}',\sigma,-\omega}+c.c.$, which can be step by step inserted into the definition of the fermionic propagator. The symmetries of the model imply the following conditions on the complex constants $\Delta^{(j_0)}_{\omega,h}$:

1. using (4), we find that $\Delta^{(j_0)}_{\omega,h}=e^{-i\omega\frac{2\pi}{3}}\Delta^{(j_0-1)}_{\omega,h}$;

2. using (5), we find that $\Delta^{(j_0)}_{\omega,h} = \big[\Delta^{(j_0)}_{-\omega,h}\big]^*$;

3. using (6.b), we find that $\Delta^{(j_0)}_{\omega,h} = \Delta^{(r_v j_0)}_{-\omega,h}$.

This fixes the structure of the effective mass term, in the form:

$$\mathcal{L}^{(K)}_{mass}\mathcal{V}^{(h)}(\Psi,A) = \Delta^K_h \frac{1}{\beta L^2}\sum_{\mathbf{k}',\omega,\sigma} \hat{\Psi}^{(\leq h)+}_{\mathbf{k}',\omega,\sigma}\Gamma^-_{\omega,j_0}\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',-\omega,\sigma}\ , \tag{8.1}$$

with Δ^K_h a real constant such that $\Delta^K_0 = \Delta_0$ is the same as the one in Eq.(2.27). The term in Eq.(8.1) is the only extra contribution to the local part of the effective action. In fact, the localization procedure can be now modified by requiring that the localization operator extracts from the kernels with $2n+m+p=2$ (or $=3$) the first two (or the first) terms of a Taylor series in $(\underline{\mathbf{k}},\underline{\mathbf{p}},\underline{\mathbf{q}})$ *and* in $\{\Delta^K_k\}_{k\leq 0}$: this is because all the terms in perturbations theory proportional to the mass itself have a dimensional gain of $O(2^{(1-\text{const}.e^2)h})$ with respect to their "massless" bound, see [17, Sections 3.2 and 3.3] or [48, Section 5.3.2]. Therefore, with this new definition of localization, the only possible extra local term in the effective action is a quadratic bilinear in the fermionic fields as the one in Eq.(8.1). Note also that such term can be rewritten in relativistic form as:

$$\mathcal{L}^{(K)}_{mass}\mathcal{V}^{(h)}(\Psi,A) = -\Delta^K_h \frac{1}{\beta L^2}\sum_{\mathbf{k}',\sigma} \overline{\psi}^{(\leq h)+}_{\mathbf{k}',\sigma}\big[e^{i\theta_{j_0}\gamma_5}\gamma_5\big]\psi^{(\leq h)-}_{\mathbf{k}',\sigma}\ , \tag{8.2}$$

with $\theta_{j_0} = \frac{2\pi}{3}(j_0-1)$.

Remark. The masses corresponding to the CDW, AF and Haldane mass terms can be worked out along the same lines. It turns out that these effective masses have the form:

$$\mathcal{L}^{(CDW)}_{mass}\mathcal{V}^{(h)}(\Psi,A) = \Delta^{CDW}_h \frac{1}{\beta L^2}\sum_{\mathbf{k}',\omega,\sigma} \hat{\Psi}^{(\leq h)+}_{\mathbf{k}',\omega,\sigma}\sigma_3\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\omega,\sigma}\ , \tag{8.3}$$

$$\mathcal{L}^{(AF)}_{mass}\mathcal{V}^{(h)}(\Psi,A) = \Delta^{AF}_h \frac{1}{\beta L^2}\sum_{\mathbf{k}',\omega,\sigma} \sigma\hat{\Psi}^{(\leq h)+}_{\mathbf{k}',\omega,\sigma}\sigma_3\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\omega,\sigma}\ , \tag{8.4}$$

$$\mathcal{L}^{(H)}_{mass}\mathcal{V}^{(h)}(\Psi,A) = \Delta^{H}_h \frac{1}{\beta L^2}\sum_{\mathbf{k}',\omega,\sigma} \omega\hat{\Psi}^{(\leq h)+}_{\mathbf{k}',\omega,\sigma}\sigma_3\hat{\Psi}^{(\leq h)-}_{\mathbf{k}',\omega,\sigma}\ , \tag{8.5}$$

which can be rewritten in relativistic form as

$$\mathcal{L}^{(CDW)}_{mass}\mathcal{V}^{(h)}(\Psi,A) = i\Delta^{CDW}_h \frac{1}{\beta L^2}\sum_{\mathbf{k}',\sigma} \overline{\psi}^{(\leq h)+}_{\mathbf{k}',\sigma}\gamma_3\psi^{(\leq h)-}_{\mathbf{k}',\sigma}\ , \tag{8.6}$$

$$\mathcal{L}^{(AF)}_{mass}\mathcal{V}^{(h)}(\Psi,A) = i\Delta^{AF}_h \frac{1}{\beta L^2}\sum_{\mathbf{k}',\sigma} \sigma\overline{\psi}^{(\leq h)+}_{\mathbf{k}',\sigma}\gamma_3\psi^{(\leq h)-}_{\mathbf{k}',\sigma}\ , \tag{8.7}$$

$$\mathcal{L}^{(H)}_{mass}\mathcal{V}^{(h)}(\Psi,A) = i\Delta^{H}_h \frac{1}{\beta L^2}\sum_{\mathbf{k}',\sigma} \overline{\psi}^{(\leq h)+}_{\mathbf{k}',\sigma}\gamma_3\gamma_5\psi^{(\leq h)-}_{\mathbf{k}',\sigma}\ , \tag{8.8}$$

which are the same as those considered in [26, 30].

As mentioned above, the fermionic bilinear in Eq.(8.1) is inserted step by step into the definition of the fermionic propagator. As a consequence, the effective propagator at scale h acquires a mass gap of size Δ^K_h. Of course, this gap is not visible as long as $|\Delta^K_h| \ll 2^h$. Therefore, if the bare mass Δ_0 is small, the multiscale integration and the dimensional bounds remain unchanged up to a scale h_0 such that $\Delta^K_{h_0} \simeq 2^{h_0}$. At that point the propagator has a mass of size comparable with 2^{h_0} itself, and we can integrate the fermionic degrees of freedom associated to scales $\leq h_0$ in a single step. From that on, we are left with a purely bosonic theory. The symmetries of the theory can be still used to prove that the only local terms in the effective action for such a bosonic theory are photon mass terms of the form $\sum_\mu 2^h\tilde{\nu}_{\mu,h}A^{(\leq h)}_\mu A^{(\leq h)}_\mu$, $h\leq 0$, with $\tilde{\nu}_{\mu,h}$ independent of j_0, see [48, Section 5.3.2]. Since the Kekulé mass term in the Hamiltonian does not break gauge invariance, the same argument used in Section 5 A to control the flow of $\tilde{\nu}_{\mu,h}$ can be repeated here to show that $\tilde{\nu}_{\mu,h} = O(e^2)$, which allows one to safely integrate all scales up to $-\infty$.

We are left with the problem of computing the scale h_0 that separates the massless and massive regimes. In order to do this, we need to control the flow of Δ_h^K, which is driven, as usual, by a beta function equation:

$$\frac{\Delta_{h-1}^K}{\Delta_h^K} = \frac{Z_h}{Z_{h-1}}\left(1+\beta_{K,h}^{\Delta}\right) . \tag{8.9}$$

Proceeding once again as in Appendix D, we find that

$$\beta_{K,h}^{\Delta} = \frac{3e^2}{4\pi^2}\log 2 + O(e^4) + O(e^2 2^h) + O(e^2(1-v)2^{(\mathrm{const.})e^2 h}) . \tag{8.10}$$

The resulting flow is:

$$\Delta_h^K \simeq \Delta_0(v)2^{-\eta_\Delta h} , \quad \text{with} \quad \eta_\Delta = \eta_K^- - \eta = \frac{2e^2}{3\pi^2} + O(e^4) \tag{8.11}$$

and $\Delta_0(v) = \Delta_0(1+O(e^2)+O(1-v))$. Therefore, the equation for the *dressed electron mass* $\Delta = 2^{h_0}$ becomes:

$$\Delta = 2^{h_0} = \Delta_0(v)2^{-\eta_\Delta h_0} \quad \Rightarrow \quad \Delta = \left[\Delta_0(v)\right]^{1/(1+\eta_\Delta)} , \tag{8.12}$$

which proves Eq.(2.28).

9. GAP EQUATION

The variational equation corresponding to the minimization problem Eq.(2.29) is:

$$\phi_{\vec{x},j} = \frac{g^2}{\kappa}\langle \zeta_{\vec{x},j}^K\rangle^{\phi} , \tag{9.1}$$

where $\langle\cdot\rangle^{\phi}$ is the statistical average in the presence of the phonon field $\phi_{\vec{x},j}$. We now want to check that the distortion $\phi_{\vec{x},j}^{(j_0)}$ defined in Eq.(2.30) is a stationary point of the total energy, provided ϕ_0 and Δ_0 are chosen properly. The stationarity condition Eq.(9.1) with $\phi_{\vec{x},j} = \phi_{\vec{x},j}^{(j_0)}$ in the limit $\beta, L \to \infty$ is equivalent to the two following coupled self-consistent equations for ϕ_0 and Δ_0:

$$\begin{aligned}\phi_0 &= \frac{g^2}{\kappa}\sum_{n\geq 0}\frac{(ie)^n}{n!}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{k}}{2\pi|\mathcal{B}|}\langle\left[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\right]^{*n}_{\mathbf{p}}\,\hat{a}^+_{\mathbf{k}+\mathbf{p},\sigma}\hat{b}^-_{\mathbf{k},\sigma}\rangle^{(j_0)}\,e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} + c.c. ,\\ \frac{\Delta_0}{3} &= \frac{g^2}{\kappa}\sum_{n\geq 0}\frac{(ie)^n}{n!}\int\frac{d\mathbf{p}'}{(2\pi)^3}\frac{d\mathbf{k}'}{2\pi|\mathcal{B}|}\langle\left[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\right]^{*n}_{\mathbf{p}'}\,\hat{a}^+_{\mathbf{k}'+\mathbf{p}'+\mathbf{p}_F^{-\omega},\sigma}\hat{b}^-_{\mathbf{k}'+\mathbf{p}_F^{\omega},\sigma}\rangle^{(j_0)}\,e^{-i\mathbf{k}'(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}e^{-i\mathbf{p}_F^{\omega}(\boldsymbol{\delta}_{j_0}-\boldsymbol{\delta}_1)} + c.c. ,\end{aligned} \tag{9.2}$$

where $\left[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\right]^{*n}_{\mathbf{p}}$ was defined in Eq.(B.2), $\langle\cdot\rangle^{(j_0)}$ is the statistical average in the presence of the phonon field $\phi_{\vec{x},j}^{(j_0)}$ and we denoted by $\hat{a}^{\pm}_{\mathbf{k},\sigma}, \hat{b}^{\pm}_{\mathbf{k},\sigma}$ the first and second components of the spinor $\hat{\Psi}^{\pm}_{\mathbf{k},\sigma}$, respectively. These equations are well defined provided that the right hand sides of the two equations are independent of j, j_0 and ω. Using the symmetries of the model, we find that (see Appendix C 3 for a proof)

$$\begin{aligned}\langle\!\langle \hat{a}^+_{\mathbf{k},\sigma}\hat{b}^-_{\mathbf{k},\sigma}\rangle\!\rangle_j^{(j_0)} &:= \sum_{n\geq 0}\frac{(ie)^n}{n!}\int\frac{d\mathbf{p}}{(2\pi)^3}\langle\left[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\right]^{*n}_{\mathbf{p}}\,\hat{a}^+_{\mathbf{k}+\mathbf{p},\sigma}\hat{b}^-_{\mathbf{k},\sigma}\rangle^{(j_0)} = \Omega(\vec{k})A(\mathbf{k}) ,\\ \langle\!\langle \hat{a}^+_{\mathbf{k}'+\mathbf{p}_F^{-\omega},\sigma}\hat{b}^-_{\mathbf{k}'+\mathbf{p}_F^{\omega},\sigma}\rangle\!\rangle_j^{(j_0)} &:= \sum_{n\geq 0}\frac{(ie)^n}{n!}\int\frac{d\mathbf{p}'}{(2\pi)^3}\langle\left[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\right]^{*n}_{\mathbf{p}'}\,\hat{a}^+_{\mathbf{k}'+\mathbf{p}'+\mathbf{p}_F^{-\omega},\sigma}\hat{b}^-_{\mathbf{k}'+\mathbf{p}_F^{\omega},\sigma}\rangle^{(j_0)} = \Omega(\vec{k}')B(\mathbf{k}')e^{i\mathbf{p}_F^{\omega}(\boldsymbol{\delta}_{j_0}-\boldsymbol{\delta}_1)} ,\end{aligned} \tag{9.3}$$

with $A(\mathbf{k})$ and $B(\mathbf{k}')$ two functions, independent of ω and j_0, transforming as follows under the discrete symmetries of the model:

$$A(\mathbf{k}) = A(T\mathbf{k}) = A(I\mathbf{k}) = A^*(R_h\mathbf{k}) = A(R_v\mathbf{k}) , \tag{9.4}$$

$$B(\mathbf{k}') = B(T\mathbf{k}') = B(I\mathbf{k}') = B^*(R_h\mathbf{k}') = B(R_v\mathbf{k}') , \tag{9.5}$$

The dimensional bounds following from the multiscale integration, combined with an explicit lowest order computation, show that, if $|\mathbf{k}'| \simeq 2^h$ with $h \geq h_0$ (here h_0 is the scale defined in Eq.(8.12)),

$$\langle\langle \hat{a}^+_{\mathbf{k}'+\mathbf{p}^\omega_F,\sigma} \hat{b}^-_{\mathbf{k}'+\mathbf{p}^\omega_F,\sigma} \rangle\rangle^{(j_0)}_j = \frac{1}{Z_h} \frac{v_h \Omega(\vec{k}' + \vec{p}^{\,\omega}_F)}{k_0^2 + v_h^2 |\Omega(\vec{k}' + \vec{p}^{\,\omega}_F)|^2} (1 + A'(\mathbf{k}')) \;, \tag{9.6}$$

$$\langle\langle \hat{a}^+_{\mathbf{k}'+\mathbf{p}^{-\omega}_F,\sigma} \hat{b}^-_{\mathbf{k}'+\mathbf{p}^\omega_F,\sigma} \rangle\rangle^{(j_0)}_j = \frac{\Omega(\vec{k}')}{3} \frac{\Delta^K_h}{Z_h} \frac{e^{i\mathbf{p}^\omega_F(\boldsymbol{\delta}_{j_0} - \boldsymbol{\delta}_1)}}{k_0^2 + v_h^2 |\Omega(\vec{k}' + \vec{p}^{\,\omega}_F)|^2} (1 + B'(\mathbf{k}')) \;, \tag{9.7}$$

where the correction terms $A'(\mathbf{k}')$ and $B'(\mathbf{k}')$ satisfying the same symmetry properties as $A(\mathbf{k})$ and $B(\mathbf{k}')$ in Eqs.(9.4)-(9.5) and of order $O(e^2) + O(2^h) + O(\Delta_h 2^{-h})$. If we plug Eqs.(9.6)-(9.7) into Eq.(9.2), we are led to the self-consistent equations

$$\phi_0 \simeq 2\frac{g^2}{\kappa} \sum_\omega \int_{\Delta \lesssim |\mathbf{k}'| \lesssim 1} \frac{d\mathbf{k}'}{2\pi|\mathcal{B}|} \frac{1}{Z(\mathbf{k}')} \frac{v(\mathbf{k}')|\Omega(\vec{k}' + \vec{p}^{\,\omega}_F)|^2}{k_0^2 + v(\mathbf{k}')^2 |\Omega(\vec{k}' + \vec{p}^{\,\omega}_F)|^2} \;, \tag{9.8}$$

$$\Delta_0 \simeq 6\frac{g^2}{\kappa} \int_{\Delta \lesssim |\mathbf{k}'| \lesssim 1} \frac{d\mathbf{k}'}{2\pi|\mathcal{B}|} \frac{1}{Z(\mathbf{k}')} \frac{\Delta(\mathbf{k}')}{k_0^2 + v(\mathbf{k}')^2 |\Omega(\vec{k}' + \vec{p}^{\,\omega}_F)|^2} \;, \tag{9.9}$$

where the "$\simeq$" indicates that we are neglecting higher order corrections and, for $\mathbf{k}'$ small, $Z(\mathbf{k}') \sim |\mathbf{k}'|^{-\eta}$, $\Delta(\mathbf{k}') \sim \Delta_0 |\mathbf{k}'|^{-\eta_\Delta}$ and $1 - v(\mathbf{k}') \sim (1-v)|\mathbf{k}'|^{\tilde{\eta}}$. It is apparent that Eq.(9.8) can be solved by fixing ϕ_0 to be a suitable (positive) constant of order g^2.

On the other hand, Eq.(9.9) is equivalent to Eq.(2.31) and leads to Eq.(2.32) and to the conclusions spelled out after Eq.(2.32). In particular, at small coupling, the critical phonon coupling g_c above which the gap equation admits a non trivial solution scales like $g_c \sim \sqrt{v}$. This can be proved by observing that the range of ρ such that the denominator of the integrand in the r.h.s. of Eq.(2.32) is larger than (say) $\frac{3}{2}v$ is contained in the interval $[0, \rho^*]$, where $1 - (1-v)(\rho^*)^{\tilde{\eta}} = \frac{3}{2}v$, which gives, for small v, an exponentially small ρ^*, i.e., $\rho^* \sim e^{-v/(2\tilde{\eta})}$; in other words, at small coupling, the range of momenta such that the effective Fermi velocity is substantially different from the bare one is exponentially small, and gives no relevant contribution to the r.h.s. of Eq.(2.32). However, the gap equation can be naturally extrapolated at intermediate to strong coupling. As noted in the introduction, the larger the charge, the smaller g_c; possibly, g_c goes to zero for large enough values of the electric charge.

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Appendix A: Functional integral representation and gauge invariance

In this Appendix we sketch the derivation of the functional integral representation presented in Section 3 and prove its gauge invariance properties.

1. Functional integral representation

Using standard methods of many body theory [44] and the explicit form of the electron and photon propagators discussed in Section 2 A, we can write the partition function $\mathrm{Tr}e^{-\beta H_\Lambda}$ of our model in the Coulomb gauge as

$$\frac{\mathrm{Tr}\{e^{-\beta H_\Lambda}\}}{\mathrm{Tr}\{e^{-\beta(H^0_\Lambda + H^f_\Lambda)}\}} = \int P(d\Psi) P^{(C)}(d\vec{A}) e^{\overline{\mathcal{V}}(\Psi, \vec{A})} \;, \tag{A.1}$$

which should be understood as an identity between power series in the electric charge e. In Eq.(A.1), $P(d\Psi)$ is the same Grassmann gaussian integration as in Eq.(3.4), $P^{(C)}(d\vec{A})$ is the gaussian measure

$$P^{(C)}(d\vec{A}) = \frac{1}{\mathcal{N}_A} \prod_{\mathbf{p}\in\mathcal{D}^+_{\beta,L}} \prod_{i=1,2} d\mathrm{Re}A_{i,\mathbf{p}} d\mathrm{Im}A_{i,\mathbf{p}} \exp\Big\{ -\frac{1}{2\beta|\mathcal{S}_L|} \sum_{\substack{\mathbf{p}\in\mathcal{D}_{\beta,L}\\ i,j=1,2}} \hat{A}_{i,\mathbf{p}} \big[\hat{w}^{(C)}(\mathbf{p})\big]^{-1}_{ij} \hat{A}_{j,-\mathbf{p}} \Big\} \;, \tag{A.2}$$

with $\hat{w}^{(C)}_{ij}(\mathbf{p})$ the covariance matrix in Eq.(2.14) and, if and $\varphi_{\rho\rho'}(\mathbf{x})=\delta(x_0)\varphi(\vec{x}+(\rho-\rho')\vec{\delta}_1)$,

$$\overline{\mathcal{V}}(\Psi,\vec{A})=t\sum_{\substack{\sigma=\uparrow,\downarrow\\ j=1,2,3}}\int d\mathbf{x}\Big[\big(e^{ie\int_0^1 ds\,\vec{\delta}_j\vec{A}_{\mathbf{x}+s\boldsymbol{\delta}_j}}-1\big)\Psi^+_{\mathbf{x},\sigma,1}\Psi^-_{\mathbf{x}+\boldsymbol{\delta}_j-\boldsymbol{\delta}_1,\sigma,2}+\big(e^{ie\int_0^1 ds\,\vec{\delta}_j\vec{A}_{\mathbf{x}+s\boldsymbol{\delta}_j}}-1\big)\Psi^+_{\mathbf{x}+\boldsymbol{\delta}_j-\boldsymbol{\delta}_1,\sigma,2}\Psi^-_{\mathbf{x},\sigma,1}\Big]-$$

$$-\frac{e^2}{2}\sum_{\substack{\rho,\rho'=1,2\\ \sigma,\sigma'=\uparrow\downarrow}}\int d\mathbf{x}\int d\mathbf{y}\,\Psi^+_{\mathbf{x},\sigma,\rho}\Psi^-_{\mathbf{x},\sigma,\rho}\varphi_{\rho\rho'}(\mathbf{x}-\mathbf{y})\Psi^+_{\mathbf{y},\sigma',\rho'}\Psi^-_{\mathbf{y},\sigma',\rho'}\ , \tag{A.3}$$

The quartic fermionic term in $\overline{\mathcal{V}}$ can be eliminated by a Hubbard-Stratonovich transformation, at the cost of introducing an extra component $A_{0,\mathbf{x}}$ of the photon field. This yields a representation of the partition function in terms of a functional integral analogous to Eq.(3.1):

$$\int P(d\Psi)P^{(C)}(d\vec{A})e^{\overline{\mathcal{V}}(\Psi,A)}=\int P(d\psi)P^{\xi}(dA)e^{\mathcal{V}(\Psi,A)}\Big|_{\xi=1}\ , \tag{A.4}$$

where $\mathcal{V}(\Psi,A)$ is given by Eq.(3.6) and $P^\xi(dA)$ by Eq.(3.5), with $\hat{w}^{\xi,h^*}$ replaced by $\hat{w}^{\xi}:=\hat{w}^{\xi,-\infty}$. The functional integral representation for the observables can be derived along the same lines.

2. Derivation of The Ward Identities

The derivation of the Ward Identities involves manipulation of the functional integral. Therefore, it is crucial to define with care the cutoffs used to regularize the functional integral and how they are eliminated. First of all, keeping the IR cutoff at scale 2^{h^*} fixed, we introduce an ultraviolet cut-off, by replacing the fermionic and bosonic propagator by $\chi_M(k_0)\hat{S}_0(\mathbf{k})$ and $\chi_M(p_0)w^{\xi,h^*}_{\mu\nu}(\mathbf{p})$, with $\chi_M(k_0)=\chi(2^{-M}|k_0|)$. Next, we replace the compact support cut-off functions by new functions, exponentially close to them but with full support in $\mathbb{R}$; i.e., we replace $\chi(t)$ by a smooth function $\chi^\varepsilon(t)$ such that $\chi^\varepsilon(t)=\chi(t)=1$ for $0\le t\le\frac{1}{2}$ and $\chi(t)<\chi^\varepsilon(t)\le\chi(t)+\varepsilon e^{-t}$ for $t>\frac{1}{2}$. Finally, we let the Grassmann field live on a finite lattice, both in the space and time variables: pick $N\in\mathbb{N}$ and define:

$$\Lambda^{(N)}_\beta=\{x_0=\frac{\beta n}{N}:\ 0\le n<N\}\times\Lambda\ ,\qquad \mathcal{B}^{(N)}_{\beta,L}:=\{k_0=\frac{2\pi}{\beta}(n+\frac{1}{2}):\ 0\le n<N\}\times\mathcal{B}_L\ ,$$

Let $\tilde{\Psi}$ be a Grassmann field on $\Lambda^{(N)}_\beta$ with antiperiodic boundary conditions in the x_0 variable, with propagator $\tilde{S}_0(k_0,\vec{k}):=\hat{S}_0(d_N(k_0),\vec{k})$, where $d_N(k_0)=i\frac{e^{-ik_0\beta/N}-1}{\beta/N}$. We shall think of Eq.(3.1) as the $M,N\to\infty$ and $\varepsilon\to0$ limit of a regularized functional, with the limits taken in the following order:

$$e^{\mathcal{W}^{\xi,h^*}(\Phi,J,\lambda)}=\lim_{M\to\infty}\lim_{\varepsilon\to0}\lim_{N\to\infty}e^{\mathcal{W}^{\xi,h^*}_{M,N}(\Phi,J,\lambda)}\ ,\quad e^{\mathcal{W}^{\xi,h^*}_{M,N}(\Phi,J,\lambda)}=\int P_{M,N}(d\tilde{\Psi})P^{\xi,h^*}_M(dA)e^{\tilde{\mathcal{V}}(\tilde{\Psi},A+J)+\mathcal{B}(\tilde{\Psi},A+J,\Phi)+(\lambda,\tilde{\Psi})}\ , \tag{A.5}$$

where the interaction and source terms are the same as in Section 3, with the only differences that $\int d\mathbf{x}$ should be understood as a shorthand for $\frac{\beta}{N}\sum_{\mathbf{x}\in\Lambda^{(N)}_\beta}$ and the term in the second line of Eq.(3.6) should be replaced by

$$-\frac{N}{\beta}\sum_{\sigma=\uparrow\downarrow}\int d\mathbf{x}\Big[\tilde{\Psi}^+_{\mathbf{x},\sigma,1}\tilde{\Psi}^-_{(x_0+\frac{\beta}{N},\vec{x}),\sigma,1}\big(e^{ie\int_0^{\beta/N}A_{0,(x_0+s,\vec{x})}}-1\big)+\tilde{\Psi}^+_{\mathbf{x},\sigma,2}\tilde{\Psi}^-_{(x_0+\frac{\beta}{N},\vec{x}),\sigma,2}\big(e^{ie\int_0^{\beta/N}A_{0,(x_0+s,\vec{x}+\vec{\delta}_1)}}-1\big)\Big]\ ,$$

Now, the regularized functional $e^{\mathcal{W}^{\xi,h^*}_{M,N}(\Phi,J,\lambda)}$ is a non-singular finite dimensional integral. Therefore, we can freely perform unitary change of variables of the fields $\tilde{\Psi}$, without affecting the value of the integral. In particular, by performing the unitary *phase transformation* $\tilde{\Psi}^\pm_{\mathbf{x},\sigma,\rho}\to\tilde{\Psi}^\pm_{\mathbf{x},\sigma,\rho}e^{\pm ie\alpha(\mathbf{x}+(\rho-1)\boldsymbol{\delta}_1)}$, we find:

$$0=\frac{\partial}{\partial\hat{\alpha}_{\mathbf{p}}}\mathcal{W}^{\xi,h^*}_{M,N}(\Phi,J+\partial\alpha,\lambda e^{ie\alpha})\Big|_{\alpha=0}+\Delta^\varepsilon_{M,N}(\mathbf{p};\Phi,J,\lambda)\ , \tag{A.6}$$

where

$$\Delta^\varepsilon_{M,N}(\mathbf{p};\Phi,J,\lambda)=\frac{\partial}{\partial\phi_{\mathbf{p}}}\log\int P_{M,N}(d\tilde{\Psi})P^{\xi,h^*}_M(dA)e^{\tilde{\mathcal{V}}(\tilde{\Psi},A+J)+\mathcal{B}(\tilde{\Psi},A+J,\Phi)+(\lambda,\tilde{\Psi})+\mathcal{C}(\tilde{\Psi},\phi)}\Big|_{\phi=0}\ , \tag{A.7}$$

and

$$\mathcal{C}(\Psi,\phi) := \frac{-ie}{\beta^2 L^2|\mathcal{S}_L|}\sum_{\substack{\mathbf{k}\in\mathcal{B}^{(N)}_{\beta,L}\\ \mathbf{p}\in\mathcal{D}_{\beta,L}}}\phi_{\mathbf{p}}\hat\Psi^+_{\mathbf{k}+\mathbf{p},\sigma}C^\varepsilon_{M,N}(\mathbf{k},\mathbf{p})\hat\Psi^-_{\mathbf{k},\sigma}\ ,$$

$$C^\varepsilon_{M,N}(\mathbf{k},\mathbf{p}) = \Big[\big(\chi^\varepsilon_M(k_0+p_0)\big)^{-1}-1\Big]B_N(\mathbf{k}+\mathbf{p})\Gamma_0(\vec p) - \Big[\big(\chi^\varepsilon_M(k_0)\big)^{-1}-1\Big]\Gamma_0(\vec p)B_N(\mathbf{k})\ , \tag{A.8}$$

where $B_N(\mathbf{k}) = [\hat S_0(d_N(k_0),\vec k)]^{-1}$. The correction term $\Delta^\varepsilon_{M,N}$ is due to the presence of the imaginary time UV cut-off in $P_{M,N}(d\tilde\Psi)$, which slightly breaks gauge invariance. It can be studied by a multiscale analysis of the functional integral in Eq.(A.7), using ideas similar to (but much simpler than) those of [18, Appendix D] (using methods first developed in [4, 5]), where the correction to the WIs due to a fixed fermionic UV cutoff at momenta of order 1 were studied. Here the main difference is that the UV cutoff is only on the imaginary time coordinates and it cuts off momenta of scale 2^M. By the support properties of the kernel $C^\varepsilon_{M,N}(\mathbf{k},\mathbf{p})$ all the contributions to $\Delta^\varepsilon_{M,N}(\mathbf{k},\mathbf{p})$ have at least one loop momentum flowing at scale 2^M. Moreover, since the scaling dimensions in the UV theory for the time variables are all negative, see [15, Appendix C] and [48], it is easy to show that all such contributions go exponentially to zero as $M,N\to\infty$:

$$\lim_{M\to\infty}\lim_{\varepsilon\to 0}\lim_{N\to\infty}\Delta^\varepsilon_{M,N}(\mathbf{p};\Phi,J,\lambda) = 0\ , \tag{A.9}$$

for all fixed $\mathbf{p}$, which proves Eq.(3.11). More details on the proof of Eq.(A.9) can be found in [48].

3. Independence of the functional integral on the choice of ξ

In this subsection we show that the averages of gauge covariant operators do not depend on the choice of the gauge fixing parameter ξ appearing in the bosonic integration measure. This allows us to work in the technically convenient *Feynman gauge*, corresponding to the choice $\xi = 0$.

Let $F(\Psi,A)$ be a gauge invariant function, i.e., $F(\Psi,A) = F(\Psi e^{ie\alpha}, A+\partial\alpha)$. We want to show that

$$\partial_\xi\frac{\int P(d\Psi)P^{\xi,h^*}(dA)e^{\mathcal{V}(\Psi,A)}F(\Psi,A)}{\int P(d\Psi)P^{\xi,h^*}(dA)e^{\mathcal{V}(\Psi,A)}} = 0\ . \tag{A.10}$$

As in the derivation of the WIs, the proof requires a number of manipulation of the functional integral. In principle, we should proceed as in the previous subsection, by first introducing proper UV cutoffs in the time variable, then keep track of the possible correction terms produced by the manipulations and finally discuss the vanishing of the corrections in the limit where the UV cutoffs are removed. However, the result is that, once again, these correction terms vanish in the limit, so here we will neglect them from the very first moment.

With obvious notation, let us denote the gaussian measure in Eq.(3.5) by

$$P^{\xi,h^*}(dA) = \frac{\mathcal{D}A\, e^{-\frac12(A,(w^{\xi,h^*})^{-1}A)}}{\int\mathcal{D}A\, e^{-\frac12(A,(w^{\xi,h^*})^{-1}A)}}\ , \tag{A.11}$$

and, given a function of Ψ, A, let us define

$$\mathbb{E}^\xi_A(\mathcal{O}(\Psi,A)) = \frac{\int\mathcal{D}A\, e^{-\frac12(A,(w^{\xi,h^*})^{-1}A)}\mathcal{O}(\Psi,A)}{\int\mathcal{D}A\, e^{-\frac12(A,(w^{\xi,h^*})^{-1}A)}}\ . \tag{A.12}$$

Note that, letting $\mathcal{O}_1 := e^{\mathcal{V}}F$ and $\mathcal{O}_2 := e^{\mathcal{V}}$, Eq.(A.10) can be rewritten as

$$\partial_\xi\frac{\int P(d\Psi)\mathbb{E}^\xi_A(\mathcal{O}_1(\Psi,A))}{\int P(d\Psi)\mathbb{E}^\xi_A(\mathcal{O}_2(\Psi,A))} = 0\ , \tag{A.13}$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are both gauge invariant functions. So in order to prove Eq.(A.10) it is enough to prove that $\int P(d\Psi)\partial_\xi\mathbb{E}^\xi_A(\mathcal{O}(\Psi,A)) = 0$, with $\mathcal{O}$ a gauge invariant function. To this purpose, note that

$$\partial_\xi\mathbb{E}^\xi_A(\mathcal{O}(\Psi,A)) = -\frac{1}{2\beta|\mathcal{S}_L|}\sum_{\substack{\mathbf{p}\in\mathcal{D}_{\beta,L}\\ \mu,\nu=0,1,2}}\partial_\xi[\hat w^{\xi,h^*}(\mathbf{p})]^{-1}_{\mu\nu}\Big[\mathbb{E}^\xi_A(\hat A_{\mu,\mathbf{p}}\hat A_{\nu,-\mathbf{p}}\mathcal{O}(\Psi,A)) - \hat w^{\xi,h^*}_{\mu\nu}(\mathbf{p})\mathbb{E}^\xi_A(\mathcal{O}(\Psi,A))\Big]\ . \tag{A.14}$$

Integrating by parts the first term in square brackets, we find:

$$\partial_\xi \mathbb{E}^\xi_A(\mathcal{O}(\Psi,A)) = -\frac{1}{2\beta|\mathcal{S}_L|}\sum_{\substack{\mathbf{p}\in\mathcal{D}_{\beta,L}\\ \mu,\nu,\nu_1,\nu_2=0,1,2}} \hat{w}^{\xi,h^*}_{\nu_1\mu}(\mathbf{p})\partial_\xi[\hat{w}^{\xi,h^*}(\mathbf{p})]^{-1}_{\mu\nu}\hat{w}^{\xi,h^*}_{\nu\nu_2}(\mathbf{p})\mathbb{E}^\xi_A\big(\frac{\partial^2\mathcal{O}(\Psi,A)}{\partial\hat{A}_{\nu_1,\mathbf{p}}\partial\hat{A}_{\nu_2,-\mathbf{p}}}\big)$$

$$= \frac{1}{2\beta|\mathcal{S}_L|}\sum_{\substack{\mathbf{p}\in\mathcal{D}_{\beta,L}\\ \mu,\nu=0,1,2}} \partial_\xi\hat{w}^{\xi,h^*}_{\mu\nu}(\mathbf{p})\mathbb{E}^\xi_A\big(\frac{\partial^2\mathcal{O}(\Psi,A)}{\partial\hat{A}_{\mu,\mathbf{p}}\partial\hat{A}_{\nu,-\mathbf{p}}}\big)\ . \tag{A.15}$$

Using the explicit form of $\hat{w}^{\xi,h^*}_{\mu\nu}(\mathbf{p})$, Eq.(3.3), the last expression can be further rewritten as

$$\partial_\xi \mathbb{E}^\xi_A(\mathcal{O}(\Psi,A)) = -\frac{1}{2\beta|\mathcal{S}_L|}\sum_{\substack{\mathbf{p}\in\mathcal{D}_{\beta,L}\\ \mu,\nu=0,1,2}}\int_{\mathbb{R}}\frac{dp_3}{2\pi}\frac{\chi(|p|)-\chi(2^{-h^*}|\mathbf{p}|)}{(\mathbf{p}^2+p_3^2)(|\vec{p}|^2+p_3^2)}\cdot \tag{A.16}$$

$$\cdot\big[p_\mu p_\nu - p_0(p_\mu n_\nu + p_\nu n_\mu)\big]\frac{\partial^2}{\partial\hat{J}_{\mu,\mathbf{p}}\partial\hat{J}_{\nu,-\mathbf{p}}}\mathbb{E}^\xi_A\big(\mathcal{O}(\Psi,A+J)\big)\Big|_{J=0}\ .$$

On the other hand, by the gauge invariance of $\mathcal{O}$ and performing the unitary phase transformation $\Psi^\pm_{\mathbf{x},\sigma,\rho}\to\Psi^\pm_{\mathbf{x},\sigma,\rho}e^{\pm ie\alpha(\mathbf{x}+(\rho-1)\boldsymbol{\delta}_1)}$, as in the previous subsection, we find

$$\int P(d\Psi)\mathbb{E}^\xi_A\big(\mathcal{O}(\Psi,A)\big) = \int P(d\Psi)\mathbb{E}^\xi_A\big(\mathcal{O}(\Psi,A+\partial\alpha)\big)\ ,$$

which implies

$$\frac{\partial}{\partial\hat{\alpha}_\mathbf{p}}\int P(d\Psi)\mathbb{E}^\xi_A\big(\mathcal{O}(\Psi,A+\partial\alpha)\big)\Big|_{\alpha=0} = -i\sum_{\mu=0,1,2}\int P(d\Psi)\,p_\mu\frac{\partial}{\partial\hat{J}_{\mu,\mathbf{p}}}\mathbb{E}^\xi_A\big(\mathcal{O}(\Psi,A+J)\big)\Big|_{J=0} = 0\ . \tag{A.17}$$

Integrating Eq.(A.16) with respect to $\int P(d\Psi)$ and using Eq.(A.17) finally gives the desired cancellation,

$$\int P(d\Psi)\partial_\xi\mathbb{E}^\xi_A(\mathcal{O}(\Psi,A)) = 0\ .$$

Appendix B: Symmetry transformations

In this Appendix we prove that both the gaussian integrations and the effective potentials are invariant under the symmetries (1)–(8) listed in Section 4 B. As already done in Section 4, we restrict for simplicity to the case that $J=\lambda=0$ and that all the external fields Φ^a but the one with $a=K$ are set to 0. The invariance of the effective potentials on scale h and of the single-scale integrations under the stated transformations follows from the fact that the bare interaction and source terms, Eqs.(3.6)-(3.7), and all the single scale integrations $P(d\Psi^{(h)})P(dA^{(h)})$ are separately invariant under the same transformations. Moreover, since the single scale integrations are obtained from the bare integration $P(d\Psi)P^{0,h^*}(dA)$ by recursively including the local terms $\mathcal{L}_\psi\mathcal{V}^{(h)}$ (which is invariant under the symmetries (1)–(8), see Eq.(4.21) and [18, Proof of Lemma 1]), one can immediately convince oneself that the desired invariance properties of the effective potentials at all scales follow from the invariance of the bare interaction, bare source term and bare integration under the analogue of the symmetries (1)–(8) written in terms of the fields $\hat{\Psi}^\pm_{\mathbf{k},\sigma}$ (rather then in terms of the quasi-particle fields $\hat{\Psi}^\pm_{\mathbf{k}',\sigma,\omega}$, as done in Section 4 B). The symmetries (1)–(8), if rewritten in terms of $\hat{\Psi}^\pm_{\mathbf{k},\sigma}$, read as follows.

(1) Spin flip: $\hat{\Psi}^\varepsilon_{\mathbf{k},\sigma}\to\hat{\Psi}^\varepsilon_{\mathbf{k},-\sigma}$, and $A_{\mu,\mathbf{p}}$, $\hat{\Phi}^K_{j,\mathbf{p}}$ are left invariant;

(2) Global $U(1)$: $\hat{\Psi}^\varepsilon_{\mathbf{k},\sigma}\to e^{i\varepsilon\alpha_\sigma}\hat{\Psi}^\varepsilon_{\mathbf{k},\sigma}$, with $\alpha_\sigma\in\mathbb{R}$ independent of $\mathbf{k}$, and $A_{\mu,\mathbf{p}}$, $\hat{\Phi}^K_{j,\mathbf{p}}$ are left invariant;

(3) Spin SO(2): $\begin{pmatrix}\hat{\Psi}^\varepsilon_{\mathbf{k},\uparrow,\cdot}\\ \hat{\Psi}^\varepsilon_{\mathbf{k},\downarrow,\cdot}\end{pmatrix}\to e^{i\theta\sigma_2}\begin{pmatrix}\hat{\Psi}^\varepsilon_{\mathbf{k},\uparrow,\cdot}\\ \hat{\Psi}^\varepsilon_{\mathbf{k},\downarrow,\cdot}\end{pmatrix}$, with θ independent of $\mathbf{k}$, and $A_{\mu,\mathbf{p}}$, $\hat{\Phi}^K_{j,\mathbf{p}}$ are left invariant;

(4) Discrete spatial rotations: if $T\mathbf{k}=(k_0,e^{-i\frac{2\pi}{3}\sigma_2}\vec{k})$ and $n_-=(1-\sigma_3)/2$,

$$\hat{\Psi}^-_{\mathbf{k},\sigma}\to e^{i\mathbf{k}(\boldsymbol{\delta}_3-\boldsymbol{\delta}_1)n_-}\hat{\Psi}^-_{T\mathbf{k},\sigma}\ ,\quad \hat{\Psi}^+_{\mathbf{k},\sigma}\to\hat{\Psi}^+_{T\mathbf{k},\sigma}e^{-i\mathbf{k}(\boldsymbol{\delta}_3-\boldsymbol{\delta}_1)n_-}\ ,\quad \hat{A}_{\mathbf{p}}\to T^{-1}\hat{A}_{T\mathbf{p}}\ ,\quad \hat{\Phi}^K_{j,\mathbf{p}}\to\hat{\Phi}^K_{j+1,T\mathbf{p}}\ ;$$

(5) Complex conjugation: if c is a generic constant appearing in the effective potentials or in the gaussian integrations:

$$c\to c^*\ ,\qquad \hat{\Psi}^\varepsilon_{\mathbf{k},\sigma}\to\hat{\Psi}^\varepsilon_{-\mathbf{k},\sigma}\ ,\qquad \hat{A}_{\mathbf{p}}\to-\hat{A}_{-\mathbf{p}}\ ,\qquad \hat{\Phi}^K_{j,\mathbf{p}}\to\hat{\Phi}^K_{j,-\mathbf{p}}\ ;$$

(6.a) Horizontal reflections: if $R_h\mathbf{k}=(k_0,-k_1,k_2)$ and $r_h1=1$, $r_h2=3$, $r_h3=2$,

$$\hat{\Psi}^-_{\mathbf{k},\sigma}\to\sigma_1\hat{\Psi}^-_{R_h\mathbf{k},\sigma}\ ,\qquad \hat{\Psi}^+_{\mathbf{k},\sigma}\to\hat{\Psi}^+_{R_h\mathbf{k},\sigma}\sigma_1\ ,\qquad \hat{A}_{\mathbf{p}}\to R_h\hat{A}_{R_h\mathbf{p}}e^{i\mathbf{p}\boldsymbol{\delta}_1}\ ,\qquad \hat{\Phi}^K_{j,\mathbf{p}}\to\hat{\Phi}^K_{r_hj,R_h\mathbf{p}}e^{-i\mathbf{p}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\ ,$$

(6.b) Vertical reflections: if $R_v\mathbf{k}=(k_0,k_1,-k_2)$ and $r_v1=1$, $r_v2=3$, $r_v3=2$,

$$\hat{\Psi}^\varepsilon_{\mathbf{k},\sigma}\to\hat{\Psi}^\varepsilon_{R_v\mathbf{k},\sigma}\ ,\qquad \hat{A}_{\mathbf{p}}\to R_v\hat{A}_{R_v\mathbf{p}}\ ,\qquad \hat{\Phi}^K_{j,\mathbf{p}}\to\hat{\Phi}^K_{r_vj,R_v\mathbf{p}}\ ;$$

(7) Particle-hole: if $P\mathbf{k}=(k_0,-\vec{k})$,

$$\hat{\Psi}^\varepsilon_{\mathbf{k},\sigma}\to i\hat{\Psi}^{-\varepsilon}_{P\mathbf{k},\sigma}\ ,\qquad \hat{A}_{\mathbf{p}}\to P\hat{A}_{-P\mathbf{p}}\ ,\qquad \hat{\Phi}^K_{j,\mathbf{p}}\to\hat{\Phi}^K_{j,-P\mathbf{p}}\ ,$$

(8) Time-reversal: if $I\mathbf{k}=(-k_0,\vec{k})$,

$$\hat{\Psi}^-_{\mathbf{k},\sigma}\to-i\sigma_3\hat{\Psi}^-_{I\mathbf{k},\sigma}\ ,\qquad \hat{\Psi}^+_{\mathbf{k},\sigma}\to-i\hat{\Psi}^+_{I\mathbf{k},\sigma}\sigma_3\ ,\qquad \hat{A}_{\mathbf{p}}\to I\hat{A}_{I\mathbf{p}}\ ,\qquad \hat{\Phi}^K_{j,\mathbf{p}}\to\hat{\Phi}^K_{j,I\mathbf{p}}\ .$$

Now, the proof of the fact that $P(d\Psi)$ is invariant under (1)–(8) has already been discussed in [15, Section 3.1]. The invariance of $P^{0,h^*}(dA)$ under (1)–(8) is obvious, and so is the invariance of $\mathcal{V}(\Psi,A)$ and $\mathcal{B}(\Psi,A,\Phi)$ under (1)–(3). Therefore, we are left with proving the invariance of $\mathcal{V}(\Psi,A)$ and $\mathcal{B}(\Psi,A,\Phi)$ under (4)–(8). As a preliminary step, let us rewrite the interaction and source term in momentum space:

$$\begin{aligned}\mathcal{V}(\Psi,A)=\frac{1}{\beta^2L^2|\mathcal{S}_L|}\sum_{\mathbf{k},\mathbf{p},\sigma}\Big\{t\sum_{j=1}^3\sum_{n\geq1}\frac{(ie)^n}{n!}\big[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\big]^{*n}_{\mathbf{p}}\hat{\Psi}^+_{\mathbf{k}+\mathbf{p},\sigma}\begin{pmatrix}0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^ne^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{\mathbf{k},\sigma}\\ -ie\hat{A}_{0,\mathbf{p}}\hat{\Psi}^+_{\mathbf{k}+\mathbf{p},\sigma}\Gamma_0(\mathbf{p})\hat{\Psi}^-_{\mathbf{k},\sigma}\Big\}\ ,\end{aligned}\tag{B.1}$$

$$\mathcal{B}(\Psi,A,\Phi)=\frac{1}{\beta^3L^2|\mathcal{S}_L|^2}\sum_{\mathbf{k},\mathbf{p},\mathbf{q},\sigma}\sum_{j=1}^3\hat{\Phi}^K_{j,\mathbf{q}}\sum_{n\geq0}\frac{(ie)^n}{n!}\big[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\big]^{*n}_{\mathbf{p}}\hat{\Psi}^+_{\mathbf{k}+\mathbf{p}+\mathbf{q},\sigma}\begin{pmatrix}0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^ne^{i(\mathbf{k}+\mathbf{p}+\mathbf{q})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{\mathbf{k},\sigma}\ ,$$

where, if $\eta^j_{\mathbf{p}}=(1-e^{-i\mathbf{p}\boldsymbol{\delta}_j})/(i\mathbf{p}\boldsymbol{\delta}_j)$,

$$\big[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\big]^{*n}_{\mathbf{p}}:=\frac{1}{(\beta|\mathcal{S}_L|)^{n-1}}\sum_{\mathbf{p}_1+\cdots+\mathbf{p}_n=\mathbf{p}}\eta^j_{\mathbf{p}_1}(\vec{\delta}_j\cdot\vec{A}_{\mathbf{p}_1})\cdots\eta^j_{\mathbf{p}_n}(\vec{\delta}_j\cdot\vec{A}_{\mathbf{p}_n})\ .\tag{B.2}$$

and we recall that $\Gamma_0(\vec{p})=\begin{pmatrix}1 & 0\\ 0 & e^{-i\mathbf{p}\boldsymbol{\delta}_1}\end{pmatrix}$. In the second of Eq.(B.1), $\big[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\big]^{*0}_{\mathbf{p}}$ should be interpreted as equal to $\beta|\mathcal{S}_L|\delta_{\mathbf{p},\mathbf{0}}$. Let us neglect the spin index, which plays no role in the following, and let us denote by

$$(*):=t\sum_{\mathbf{k},\mathbf{p}}\sum_{j=1}^3\sum_{n\geq1}\frac{(ie)^n}{n!}\big[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\big]^{*n}_{\mathbf{p}}\hat{\Psi}^+_{\mathbf{k}+\mathbf{p}}\begin{pmatrix}0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^ne^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{\mathbf{k}}\ ,\tag{B.3}$$

the term in the first line of Eq.(B.1) and, similarly,

$$(**):=-ie\sum_{\mathbf{k},\mathbf{p}}\hat{A}_{0,\mathbf{p}}\hat{\Psi}^+_{\mathbf{k}+\mathbf{p}}\Gamma_0(\mathbf{p})\hat{\Psi}^-_{\mathbf{k}}\ ,\tag{B.4}$$

$$(K):=\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}\sum_{j=1}^3\hat{\Phi}^K_{j,\mathbf{q}}\sum_{n\geq0}\frac{(ie)^n}{n!}\big[\eta^j_\cdot(\vec{\delta}_j\cdot\vec{A}_\cdot)\big]^{*n}_{\mathbf{p}}\hat{\Psi}^+_{\mathbf{k}+\mathbf{p}+\mathbf{q}}\begin{pmatrix}0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^ne^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{\mathbf{k}}\ .\tag{B.5}$$

Let us prove that these terms are separately invariant under the symmetries (4)–(8).

Symmetry (4). The term $(*)$ is changed under (4) as

$$(*) \to t \sum_{\mathbf{k},\mathbf{p}} \sum_{j=1}^{3} \sum_{n\geq 1} \frac{(ie)^n}{n!} \left[\eta_{\cdot}^{j+1}(\vec{\delta}_j \cdot (T^{-1}\vec{A}_{\cdot}))\right]^{*n}_{T\mathbf{p}} \hat{\Psi}^{+}_{T(\mathbf{k}+\mathbf{p})} \begin{pmatrix} 0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_3)} \\ (-1)^n e^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_3)} & 0 \end{pmatrix} \hat{\Psi}^{-}_{T\mathbf{k}} , \tag{B.6}$$

where we used that $\eta^j_{\mathbf{p}} = \eta^{j+1}_{T\mathbf{p}}$. Using the fact that $\left[\eta_{\cdot}^{j+1}(\vec{\delta}_j \cdot (T^{-1}\vec{A}_{\cdot}))\right]^{*n}_{T\mathbf{p}} = \left[\eta_{\cdot}^{j+1}(\vec{\delta}_{j+1} \cdot \vec{A}_{\cdot})\right]^{*n}_{T\mathbf{p}}$ and $\mathbf{p}\cdot\boldsymbol{\delta}_j = (T\mathbf{p})\cdot\boldsymbol{\delta}_{j+1}$, we see that the r.h.s. of Eq.(B.6) can be rewritten as

$$t \sum_{\mathbf{k},\mathbf{p}} \sum_{j=1}^{3} \sum_{n\geq 1} \frac{(ie)^n}{n!} \left[\eta_{\cdot}^{j+1}(\vec{\delta}_{j+1} \cdot \vec{A}_{\cdot})\right]^{*n}_{T\mathbf{p}} \hat{\Psi}^{+}_{T(\mathbf{k}+\mathbf{p})} \begin{pmatrix} 0 & e^{-iT\mathbf{k}(\boldsymbol{\delta}_{j+1}-\boldsymbol{\delta}_1)} \\ (-1)^n e^{iT(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_{j+1}-\boldsymbol{\delta}_1)} & 0 \end{pmatrix} \hat{\Psi}^{-}_{T\mathbf{k}} , \tag{B.7}$$

which is the same as $(*)$, as apparent by the change of variables $(T\mathbf{k}, T\mathbf{p}) \to (\mathbf{k}, \mathbf{p})$ in the sum. The proof of the invariance of the term (K) under (4) is exactly the same. Regarding the term $(**)$, it is changed as

$$(**) \to -ie \sum_{\mathbf{k},\mathbf{p}} \hat{A}_{0,T\mathbf{p}} \hat{\Psi}^{+}_{T(\mathbf{k}+\mathbf{p})} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\mathbf{p}\boldsymbol{\delta}_3} \end{pmatrix} \hat{\Psi}^{-}_{T\mathbf{k},\sigma} , \tag{B.8}$$

which is the same as $(**)$, simply because $\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\mathbf{p}\boldsymbol{\delta}_3} \end{pmatrix} = \Gamma_0(T\mathbf{p})$ and we can perform the change of variables $(T\mathbf{k}, T\mathbf{p}) \to (\mathbf{k}, \mathbf{p})$ in the sum.

Symmetry (5). The term $(*)$ is changed as

$$(*) \to t \sum_{\mathbf{k},\mathbf{p}} \sum_{j=1}^{3} \sum_{n\geq 1} \frac{(-ie)^n}{n!} \left[\eta_{\cdot}^{j}(\vec{\delta}_j \cdot (-\vec{A}_{\cdot}))\right]^{*n}_{-\mathbf{p}} \hat{\Psi}^{+}_{-\mathbf{k}-\mathbf{p}} \begin{pmatrix} 0 & e^{+i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} \\ (-1)^n e^{-i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0 \end{pmatrix} \hat{\Psi}^{-}_{-\mathbf{k}} , \tag{B.9}$$

where we used that $(\eta^j_{\mathbf{p}})^* = \eta^j_{-\mathbf{p}}$. The r.h.s. of Eq.(B.9) is the same as $(*)$, as apparent by the change of variables $(\mathbf{k}, \mathbf{p}) \to (-\mathbf{k}, -\mathbf{p})$. The proof of the invariance of the term (K) under (5) is exactly the same. Moreover, the term $(**)$ is changed as

$$(**) \to +ie \sum_{\mathbf{k},\mathbf{p}} (-\hat{A}_{0,-\mathbf{p}}) \hat{\Psi}^{+}_{-\mathbf{k}-\mathbf{p}} \Gamma^*_0(\mathbf{p}) \hat{\Psi}^{-}_{-\mathbf{k}} , \tag{B.10}$$

which is the same as $(**)$, because $\Gamma^*_0(\mathbf{p}) = \Gamma_0(-\mathbf{p})$ and we can perform the change of variables $(\mathbf{k}, \mathbf{p}) \to (-\mathbf{k}, -\mathbf{p})$ in the sum.

Symmetry (6.a). Using the fact that $R_h\boldsymbol{\delta}_j = -\boldsymbol{\delta}_{r_h j}$ and $\eta^j_{\mathbf{p}} = e^{-i\mathbf{p}\boldsymbol{\delta}_j}\eta^{r_h j}_{R_h\mathbf{p}}$, the term $(*)$ is changed as

$$(*) \to t \sum_{\mathbf{k},\mathbf{p}} \sum_{j=1}^{3} \sum_{n\geq 1} \frac{(ie)^n}{n!} \left[\eta_{\cdot}^{r_h j}(-\vec{\delta}_{r_h j} \cdot \vec{A}_{\cdot})\right]^{*n}_{R_h\mathbf{p}} e^{-i\mathbf{p}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} \hat{\Psi}^{+}_{R_h(\mathbf{k}+\mathbf{p})} \sigma_1 \begin{pmatrix} 0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} \\ (-1)^n e^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0 \end{pmatrix} \sigma_1 \hat{\Psi}^{-}_{R_h\mathbf{k}} , \tag{B.11}$$

which can be rewritten as

$$t \sum_{\mathbf{k},\mathbf{p}} \sum_{j=1}^{3} \sum_{n\geq 1} \frac{(ie)^n}{n!} \left[\eta_{\cdot}^{r_h j}(\vec{\delta}_{r_h j} \cdot \vec{A}_{\cdot})\right]^{*n}_{R_h\mathbf{p}} \hat{\Psi}^{+}_{R_h(\mathbf{k}+\mathbf{p})} \begin{pmatrix} 0 & e^{i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} \\ (-1)^n e^{-i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0 \end{pmatrix} \hat{\Psi}^{-}_{R_h\mathbf{k}} , \tag{B.12}$$

which is the same as $(*)$, simply because $e^{i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} = e^{-iR_h\mathbf{k}(\boldsymbol{\delta}_{r_h j}-\boldsymbol{\delta}_1)}$ and we can perform the change of variables $(R_h\mathbf{k}, R_h\mathbf{p}) \to (\mathbf{k}, \mathbf{p})$ in the sum. The proof of the invariance of the term (K) under (5) is exactly the same. Moreover, the term $(**)$ is changed as

$$(**) \to -ie \sum_{\mathbf{k},\mathbf{p}} \hat{A}_{0,R_h\mathbf{p}} e^{i\mathbf{p}\boldsymbol{\delta}_1} \hat{\Psi}^{+}_{R_h(\mathbf{k}+\mathbf{p})} \sigma_1 \Gamma_0(\mathbf{p}) \sigma_1 \hat{\Psi}^{-}_{R_h\mathbf{k}} , \tag{B.13}$$

which is the same as $(**)$, because $\sigma_1\Gamma_0(\mathbf{p})\sigma_1 = e^{-i\mathbf{p}\boldsymbol{\delta}_1}\Gamma_0(R_h\mathbf{p})$ and we can perform the change of variables $(R_h\mathbf{k}, R_h\mathbf{p}) \to (\mathbf{k}, \mathbf{p})$ in the sum.

Symmetry (6.b). Using the fact that $R_v\boldsymbol{\delta}_j = \boldsymbol{\delta}_{r_v j}$ and $\eta^j_{\mathbf{p}} = \eta^{r_v j}_{R_v\mathbf{p}}$, the term $(*)$ is changed as

$$(*) \to t\sum_{\mathbf{k},\mathbf{p}}\sum_{j=1}^{3}\sum_{n\geq 1}\frac{(ie)^n}{n!}\left[\eta^{r_v j}_{\cdot}(\vec{\delta}_{r_v j}\cdot\vec{A}_{\cdot})\right]^{*n}_{R_v\mathbf{p}}\hat{\Psi}^+_{R_v(\mathbf{k}+\mathbf{p})}\begin{pmatrix}0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^n e^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{R_v\mathbf{k}}\,, \tag{B.14}$$

which is the same as $(*)$, simply because $e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} = e^{-iR_v\mathbf{k}(\boldsymbol{\delta}_{r_v j}-\boldsymbol{\delta}_1)}$ and we can perform the change of variables $(R_v\mathbf{k}, R_v\mathbf{p}) \to (\mathbf{k},\mathbf{p})$ in the sum. The proof of the invariance of the term (K) under (5) is exactly the same. Moreover, the term $(**)$ is changed as

$$-ie\sum_{\mathbf{k},\mathbf{p}}\hat{A}_{0,R_v\mathbf{p}}\hat{\Psi}^+_{R_v(\mathbf{k}+\mathbf{p})}\Gamma_0(\mathbf{p})\hat{\Psi}^-_{R_v\mathbf{k}}\,, \tag{B.15}$$

which is the same as $(**)$, because $\Gamma_0(\mathbf{p}) = \Gamma_0(R_v\mathbf{p})$ and we can perform the change of variables $(R_v\mathbf{k}, R_v\mathbf{p}) \to (\mathbf{k},\mathbf{p})$ in the sum.

Symmetry (7). The term $(*)$ changes as

$$\begin{aligned}(*) \to\ & t\sum_{\mathbf{k},\mathbf{p}}\sum_{j=1}^{3}\sum_{n\geq 1}\frac{(ie)^n}{n!}\left[\eta^j_{\cdot}(-\vec{\delta}_j\cdot\vec{A}_{\cdot})\right]^{*n}_{-P\mathbf{p}}\hat{\Psi}^+_{P\mathbf{k}}\begin{pmatrix}0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^n e^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}^T\hat{\Psi}^-_{P(\mathbf{k}+\mathbf{p})} =\\ =\ & t\sum_{\mathbf{k},\mathbf{p}}\sum_{j=1}^{3}\sum_{n\geq 1}\frac{(ie)^n}{n!}\left[\eta^j_{\cdot}(\vec{\delta}_j\cdot\vec{A}_{\cdot})\right]^{*n}_{-P\mathbf{p}}\hat{\Psi}^+_{P\mathbf{k}}\begin{pmatrix}0 & e^{-iP(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^n e^{iP\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{P(\mathbf{k}+\mathbf{p})}\,,\end{aligned} \tag{B.16}$$

which is equal to $(*)$, as apparent after the change of variables $\big(P(\mathbf{k}+\mathbf{p}), -P\mathbf{p}\big) \to (\mathbf{k},\mathbf{p})$ in the sum. The proof of the invariance of the term (K) under (5) is exactly the same. Moreover, the term $(**)$ is changed as

$$(**) \to -ie\sum_{\mathbf{k},\mathbf{p}}\hat{A}_{0,-P\mathbf{p}}\hat{\Psi}^+_{P\mathbf{k}}\Gamma_0^T(\mathbf{p})\hat{\Psi}^-_{P(\mathbf{k}+\mathbf{p})}\,, \tag{B.17}$$

which is the same as $(**)$, because $\Gamma_0^T(\mathbf{p}) = \Gamma_0(\mathbf{p}) = \Gamma_0(-P\mathbf{p})$ and we can perform the change of variables $\big(P(\mathbf{k}+\mathbf{p}), -P\mathbf{p}\big) \to (\mathbf{k},\mathbf{p})$ in the sum.

Symmetry (8). The term $(*)$ changes as

$$\begin{aligned}(*) \to & -t\sum_{\mathbf{k},\mathbf{p}}\sum_{j=1}^{3}\sum_{n\geq 1}\frac{(ie)^n}{n!}\left[\eta^j_{\cdot}(\vec{\delta}_j\cdot\vec{A}_{\cdot})\right]^{*n}_{I\mathbf{p}}\hat{\Psi}^+_{I(\mathbf{k}+\mathbf{p})}\sigma_3\begin{pmatrix}0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^n e^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\sigma_3\hat{\Psi}^-_{I\mathbf{k}} =\\ = &\ t\sum_{\mathbf{k},\mathbf{p}}\sum_{j=1}^{3}\sum_{n\geq 1}\frac{(ie)^n}{n!}\left[\eta^j_{\cdot}(\vec{\delta}_j\cdot\vec{A}_{\cdot})\right]^{*n}_{I\mathbf{p}}\hat{\Psi}^+_{I(\mathbf{k}+\mathbf{p})}\begin{pmatrix}0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\\ (-1)^n e^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{I\mathbf{k}}\,,\end{aligned} \tag{B.18}$$

which is equal to $(*)$, because $e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} = e^{-iI\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}$ and we can perform the change of variables $(I\mathbf{k}, I\mathbf{p}) \to (\mathbf{k},\mathbf{p})$ in the sum. The proof of the invariance of the term (K) under (5) is exactly the same. Moreover, the term $(**)$ is changed as

$$(**) \to -ie\sum_{\mathbf{k},\mathbf{p}}\hat{A}_{0,I\mathbf{p}}\hat{\Psi}^+_{I(\mathbf{k}+\mathbf{p})}\sigma_3\Gamma_0(\mathbf{p})\sigma_3\hat{\Psi}^-_{I\mathbf{k}}\,, \tag{B.19}$$

which is the same as $(**)$, because $\sigma_3\Gamma_0(\mathbf{p})\sigma_3 = \Gamma_0(\mathbf{p}) = \Gamma_0(I\mathbf{p})$ and we can perform the change of variables $(I\mathbf{k}, I\mathbf{p}) \to (\mathbf{k},\mathbf{p})$ in the sum. This concludes the proof of the invariance properties stated in Section 4 B.

Appendix C: Symmetry properties of the kernels

In this Appendix we exploit the lattice symmetries (1)–(8) listed in Section 4 B to prove the invariance properties Eqs.(4.20)-(4.21) of the local terms in the effective action stated in Section 4 C. We will start with the "relevant" terms with $2n+m+p=2$ (i.e., the terms of the form AA, $\Phi^K A$ or $\Psi^+\Psi^-$) and we will then proceed with the "marginal" terms with $2n+m+p=3$ (i.e., the terms of the form $A\Psi^+\Psi^-$, $\Phi^K\Psi^+\Psi^-$, AAA, $\Phi^K AA$ or $\Phi^K\Phi^K A$). In the following, we shall drop all the unnecessary labels (including the scale and spin labels), to avoid an overwhelming notation. We will think the operators T, R_v, R_h, P, I appearing in the symmetry transformations as 3×3 matrices acting on the μ-indices, with

$$T=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3}\\ 0 & \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3}\end{pmatrix}\;,\qquad R_h=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\;,\qquad R_v=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\;,\qquad P=-I=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\;. \tag{C.1}$$

1. The "relevant" terms

The structure of the local terms of the form $\Psi^+\Psi^-$ has already been studied in [15, Lemma 3], where the first equation of Eq.(4.21) was proved. Let us then look at the terms of the form AA or $\Phi^K A$.

The AA kernels. The contribution to the effective potential quadratic in the A field is proportional to

$$\sum_{\mathbf{p}} \hat{A}_{\mathbf{p}}\hat{W}(\mathbf{p})\hat{A}_{-\mathbf{p}}\;, \tag{C.2}$$

for a suitable 3×3 matrix-valued kernel $\hat{W}(\mathbf{p})$. Imposing the invariance of Eq.(C.2) under symmetries (4)–(8), we find:

$$\hat{W}(\mathbf{p}) \overset{(4)}{=} T\hat{W}(T^{-1}\mathbf{p})T^{-1} \overset{(5)}{=} [\hat{W}(-\mathbf{p})]^* \overset{(6.a)}{=} R_h\hat{W}(R_h\mathbf{p})R_h \overset{(6.b)}{=} R_v\hat{W}(R_v\mathbf{p})R_v \overset{(8)}{=} I\hat{W}(I\mathbf{p})I\;. \tag{C.3}$$

Let $\hat{W}_{\mu\nu}(\mathbf{0}) =: \nu_{\mu\nu}$ and $\partial_\alpha \hat{W}_{\mu\nu}(\mathbf{0}) =: \nu'_{\mu\nu\alpha}$; the properties (6.a), (6.b) and (8) in Eq.(C.3) imply that $\nu_{\mu\nu}=\delta_{\mu\nu}\nu_\mu$, $\nu'_{\mu\nu\alpha}=0$, while (5) in Eq.(C.3) gives $\nu_\mu=\nu_\mu^*$; finally, the property (4) in Eq.(C.3) implies that $\nu_1=\nu_2$. This proves the second equation in Eq.(4.21).

The $\Phi^K A$ kernels. The contribution to the effective potential of the form $\Phi^K A$ is proportional to

$$\sum_j\sum_{\mathbf{p}} \hat{\Phi}^K_{j,\mathbf{p}}\hat{W}^K_j(\mathbf{p})\hat{A}_{-\mathbf{p}}\;, \tag{C.4}$$

for a suitable vector-valued kernel $\hat{W}^K_j(\mathbf{p}) = \big(\hat{W}^K_{j,0}(\mathbf{p}),\, \hat{W}^K_{j,1}(\mathbf{p}),\, \hat{W}^K_{j,2}(\mathbf{p})\big)$. Using the symmetries (7) and (8) we get:

$$\hat{W}^K_j(\mathbf{p}) \overset{(7)}{=} \hat{W}^K_j(-P\mathbf{p})P \overset{(8)}{=} \hat{W}^K_j(I\mathbf{p})I\;, \tag{C.5}$$

which implies that $\hat{W}^K_j(\mathbf{p})=0$, because $P=-I$.

2. The "marginal" terms

In this subsection we study the structure of the terms of the form $A\Psi^+\Psi^-$, $\Phi^K\Psi^+\Psi^-$, AAA, $\Phi^K AA$ or $\Phi^K\Phi^K A$.

The $A\Psi^+\Psi^-$ kernels. Since A has an UV cutoff that suppresses modes $\hat{A}_{\mathbf{p}}$ with $\mathbf{p}$ close to $\pm(\mathbf{p}_F^+-\mathbf{p}_F^-)$ (and to its images over Λ^*), the only non zero terms of the form $A\Psi^+\Psi^-$ are those with the two fermi fields associated to the same ω index. [Of course, if we were interested in studying the theory without the UV cutoff on the photon field, then terms of the form $A\Psi^+_\omega\Psi^-_{-\omega}$ would be allowed, and their structure could be investigated by the same methods used here.] The contribution to the effective potential of the form $A\Psi^+_\omega\Psi^-_\omega$ is proportional to

$$\sum_{\mathbf{k}',\mathbf{p},\omega} \hat{\Psi}^+_{\mathbf{k}'+\mathbf{p},\omega}\big[\hat{W}_\omega(\mathbf{k}',\mathbf{p})\cdot\hat{A}_{\mathbf{p}}\big]\hat{\Psi}^-_{\mathbf{k}',\omega}\;, \tag{C.6}$$

for a suitable tensor-valued kernel $\hat{W}_\omega(\mathbf{k}',\mathbf{p}) = (\hat{W}_{0,\omega}(\mathbf{k}',\mathbf{p}), \hat{W}_{1,\omega}(\mathbf{k}',\mathbf{p}), \hat{W}_{2,\omega}(\mathbf{k}',\mathbf{p})$ (each component $\hat{W}_{\mu,\omega}(\mathbf{k}',\mathbf{p})$ is a 2×2 matrix, acting on the ρ indices of the grassmann fields). The invariance under the symmetries (4)–(8) implies:

$$\begin{aligned}\hat{W}_\omega(\mathbf{k}',\mathbf{p}) &\overset{(4)}{=} e^{-i(\mathbf{p}_F^\omega+\mathbf{k}'+\mathbf{p})(\boldsymbol{\delta}_1-\boldsymbol{\delta}_2)n_-}[\hat{W}_\omega(T^{-1}\mathbf{k}',T^{-1}\mathbf{p})T^{-1}]e^{i(\mathbf{p}_F^\omega+\mathbf{k}')(\boldsymbol{\delta}_1-\boldsymbol{\delta}_2)n_-} \overset{(5)}{=} -\hat{W}^*_{-\omega}(-\mathbf{k}',-\mathbf{p}) \overset{(6.a)}{=} \\ &\overset{(6.a)}{=} e^{-i\mathbf{p}\boldsymbol{\delta}_1}\sigma_1[\hat{W}_\omega(R_h\mathbf{k}',R_h\mathbf{p})R_h]\sigma_1 \overset{(6.b)}{=} \hat{W}_{-\omega}(R_v\mathbf{k}',R_v\mathbf{p})R_v \overset{(7)}{=} \hat{W}^T_{-\omega}(P(\mathbf{k}'+\mathbf{p}),-P\mathbf{p})P \overset{(8)}{=} -\sigma_3[\hat{W}_\omega(I\mathbf{k}',I\mathbf{p})I]\sigma_3\;.\end{aligned} \tag{C.7}$$

Let us define

$$\hat{W}_{\mu,\omega}(\mathbf{0},\mathbf{0}) = \sum_{\nu=0}^{3} a^{\nu}_{\mu,\omega}\sigma_\nu \ , \qquad \mathbf{a}^{\nu}_{\omega} = (a^{\nu}_{0,\omega},\, a^{\nu}_{1,\omega},\, a^{\nu}_{2,\omega}) \ , \tag{C.8}$$

with $\sigma_0 = \mathbb{1}$. The properties (C.7) imply that (summation over repeated indices is implied):

$$\begin{aligned}
\mathbf{a}^{\nu}_{\omega}\sigma_\nu &\overset{(4)}{=} [\mathbf{a}^{\nu}_{\omega}T^{-1}]e^{-i\mathbf{p}^{\omega}_F(\boldsymbol{\delta}_1-\boldsymbol{\delta}_2)n_-}\sigma_\nu e^{i\mathbf{p}^{\omega}_F(\boldsymbol{\delta}_1-\boldsymbol{\delta}_2)n_-} \ , && \text{(C.9)}\\
\mathbf{a}^{\nu}_{\omega}\sigma_\nu &\overset{(5)}{=} -[\mathbf{a}^{0}_{-\omega}]^*\sigma_0 - [\mathbf{a}^{1}_{-\omega}]^*\sigma_1 + [\mathbf{a}^{2}_{-\omega}]^*\sigma_2 - [\mathbf{a}^{3}_{-\omega}]^*\sigma_3 \ , && \text{(C.10)}\\
\mathbf{a}^{\nu}_{\omega}\sigma_\nu &\overset{(6.a)}{=} [\mathbf{a}^{0}_{\omega}R_h]\sigma_0 + [\mathbf{a}^{1}_{\omega}R_h]\sigma_1 - [\mathbf{a}^{2}_{\omega}R_h]\sigma_2 - [\mathbf{a}^{3}_{\omega}R_h]\sigma_3 \ , && \text{(C.11)}\\
\mathbf{a}^{\nu}_{\omega}\sigma_\nu &\overset{(6.b)}{=} [\mathbf{a}^{\nu}_{-\omega}R_v]\sigma_\nu \ , && \text{(C.12)}\\
\mathbf{a}^{\nu}_{\omega}\sigma_\nu &\overset{(7)}{=} [\mathbf{a}^{0}_{-\omega}P]\sigma_0 + [\mathbf{a}^{1}_{-\omega}P]\sigma_1 - [\mathbf{a}^{2}_{-\omega}P]\sigma_2 + [\mathbf{a}^{3}_{-\omega}P]\sigma_3 \ . && \text{(C.13)}\\
\mathbf{a}^{\nu}_{\omega}\sigma_\nu &\overset{(8)}{=} -[\mathbf{a}^{0}_{\omega}I]\sigma_0 + [\mathbf{a}^{1}_{\omega}I]\sigma_1 + [\mathbf{a}^{2}_{\omega}I]\sigma_2 - [\mathbf{a}^{3}_{\omega}I]\sigma_3 \ . && \text{(C.14)}
\end{aligned}$$

We recall that T, R_h, R_v, P and I are the matrices in Eq.(C.1); the notation, e.g., $[\mathbf{a}T^{-1}]$ indicates the row 3-vector obtained by matrix multiplication of the row 3-vector $\mathbf{a}$ times the 3×3 matrix T^{-1}. Properties Eqs.(C.11) and (C.14) imply that

$$\mathbf{a}^{0}_{\omega} = (a^{0}_{0,\omega},\, 0,\, 0) \ , \ \ \mathbf{a}^{1}_{\omega} = (0,\, 0,\, a^{1}_{2,\omega}) \ , \ \ \mathbf{a}^{2}_{\omega} = (0,\, a^{2}_{1,\omega},\, 0) \ , \ \ \mathbf{a}^{3}_{\omega} = (0,\, 0,\, 0) \ , \tag{C.15}$$

while from Eq.(C.12) we get that

$$a^{0}_{0,\omega} = a^{0}_{0,-\omega} =: -i\lambda_0 \ , \quad a^{1}_{2,\omega} = -a^{1}_{2,-\omega} =: -\omega\lambda_2 \ , \quad a^{2}_{1,\omega} = a^{2}_{1,-\omega} =: -\lambda_1 \ , \tag{C.16}$$

with $\lambda_\mu \in \mathbb{R}$, thanks to Eq.(C.10). Therefore,

$$\hat{W}_{0,\omega}(\mathbf{0},\mathbf{0}) = -i\lambda_0\sigma_0 = i\lambda_0\Gamma^0_\omega \ , \qquad \hat{W}_{1,\omega}(\mathbf{0},\mathbf{0}) = -\lambda_1\sigma_2 = i\lambda_1\Gamma^1_\omega \ , \qquad \hat{W}_{2,\omega}(\mathbf{0},\mathbf{0}) = -\lambda_2\omega\sigma_1 = i\lambda_2\Gamma^2_\omega \ . \tag{C.17}$$

Moreover, thanks to Eq.(C.9):

$$\mathbf{a}^{1}_{\omega}\sigma_1 + \mathbf{a}^{2}_{\omega}\sigma_2 = [\mathbf{a}^{1}_{\omega}T^{-1}]\begin{pmatrix} 0 & e^{+i\frac{2\pi}{3}\omega} \\ e^{-i\frac{2\pi}{3}\omega} & 0\end{pmatrix} + [\mathbf{a}^{2}_{\omega}T^{-1}]\begin{pmatrix} 0 & -ie^{+i\frac{2\pi}{3}\omega} \\ ie^{-i\frac{2\pi}{3}\omega} & 0\end{pmatrix} \ , \tag{C.18}$$

which gives $a^{1}_{2,\omega} = \omega a^{2}_{1,\omega}$, i.e., $\lambda_1 = \lambda_2$. This proves the equation in the first line of Eq.(4.20).

The $\Phi^K\Psi^+\Psi^-$ kernels. The contribution to the effective potential of the form $\Phi^K\Psi^+_\omega\Psi^-_{\omega'}$ is proportional to

$$\sum_{\substack{\mathbf{k}',\mathbf{p}\\ j,\omega,\omega'}} \hat{\Phi}^K_{j,\mathbf{p}'+\mathbf{p}^{\omega}_F-\mathbf{p}^{\omega'}_F}\Psi^+_{\mathbf{k}'+\mathbf{p}',\omega}\hat{W}^K_{j,(\omega,\omega')}(\mathbf{k}',\mathbf{p}')\Psi^-_{\mathbf{k}',\omega'} \ , \tag{C.19}$$

for a suitable 2×2 matrix-valued potential $\hat{W}^K_{j,\underline{\omega}}(\mathbf{k}',\mathbf{p}')$. The symmetry properties (4)–(8) imply that:

$$\begin{aligned}
\hat{W}^K_{j,\underline{\omega}}(\mathbf{k}',\mathbf{p}') &\overset{(4)}{=} e^{-i(\mathbf{p}^{\omega}_F+\mathbf{k}'+\mathbf{p}')(\boldsymbol{\delta}_1-\boldsymbol{\delta}_2)n_-}\hat{W}^K_{j-1,\underline{\omega}}(T^{-1}\mathbf{k}',T^{-1}\mathbf{p}')e^{i(\mathbf{p}^{\omega'}_F+\mathbf{k}')(\boldsymbol{\delta}_1-\boldsymbol{\delta}_2)n_-} \overset{(5)}{=} [\hat{W}^K_{j,-\underline{\omega}}(-\mathbf{k}',-\mathbf{p}')]^* \overset{(6.a)}{=}\\
&\overset{(6.a)}{=} e^{i(\mathbf{p}^{\omega}_F-\mathbf{p}^{\omega'}_F+\mathbf{p}')(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)}\sigma_1\hat{W}^K_{r_hj,\underline{\omega}}(R_h\mathbf{k}',R_h\mathbf{p}')\sigma_1 \overset{(6.b)}{=} \hat{W}^K_{r_vj,-\underline{\omega}}(R_v\mathbf{k}',R_v\mathbf{p}') \overset{(7)}{=}\\
&\overset{(7)}{=} \left[\hat{W}^K_{j,(-\omega',-\omega)}(P(\mathbf{k}'+\mathbf{p}'),-P\mathbf{p}')\right]^T \overset{(8)}{=} -\sigma_3\hat{W}^K_{j,\underline{\omega}}(I\mathbf{k}',I\mathbf{p}')\sigma_3 \ ,
\end{aligned} \tag{C.20}$$

Let us define $\hat{W}^K_{j,\underline{\omega}}(\mathbf{0},\mathbf{0}) = \sum_{\nu=0}^{3} b^{\nu}_{j,\underline{\omega}}\sigma_\nu$. Then, property (8) in the third line of Eq.(C.20) implies that $b^0_{j,\underline{\omega}} = b^3_{j,\underline{\omega}} = 0$. Moreover, using properties (6.b) and (7) in the second and third lines of Eq.(C.20),

$$\begin{aligned}
b^1_{j,\underline{\omega}} &\overset{(6.b)}{=} b^1_{r_vj,-\underline{\omega}} \ , \qquad & b^2_{j,\underline{\omega}} &\overset{(6.b)}{=} b^2_{r_vj,-\underline{\omega}} \ ,\\
b^1_{j,\underline{\omega}} &\overset{(7)}{=} b^1_{j,(-\omega',-\omega)} \ , \qquad & b^2_{j,\underline{\omega}} &\overset{(7)}{=} -b^2_{j,(-\omega',-\omega)} \ ,
\end{aligned}$$

which implies, in particular, that $b^2_{1,\underline{\omega}} = 0$ and $b^1_{1,(\omega,\pm\omega)} = b_\pm$, for two suitable ($\omega$-independent) constants $b_\pm$. Using property (5) in Eq.(C.20), $b^1_{j,\underline{\omega}} \overset{(5)}{=} (b^1_{j,-\underline{\omega}})^*$, we see that both these constants are real, $b_\pm \in \mathbb{R}$. Property (4) in Eq.(C.20) implies that, if $\underline{\omega} = (\omega, \omega')$,

$$b^1_{j,\underline{\omega}}\sigma_1 + b^2_{j,\underline{\omega}}\sigma_2 \overset{(4)}{=} \begin{pmatrix} 0 & e^{i\omega'\frac{2\pi}{3}}(b^1_{j-1,\underline{\omega}} - ib^2_{j-1,\underline{\omega}}) \\ e^{-i\omega\frac{2\pi}{3}}(b^1_{j-1,\underline{\omega}} + ib^2_{j-1,\underline{\omega}}) & 0 \end{pmatrix},$$

so that, finally,

$$\hat{W}^K_{j,(\omega,\pm\omega)}(\mathbf{0},\mathbf{0}) = b_\pm \begin{pmatrix} 0 & e^{\pm i\omega\frac{2\pi}{3}(j-1)} \\ e^{-i\omega\frac{2\pi}{3}(j-1)} & 0 \end{pmatrix}, \tag{C.21}$$

which proves the second of Eq.(4.20).

The AAA, $\Phi^K AA$ and $\Phi^K\Phi^K A$ kernels. The contribution to the effective potential of the form AAA or $\Phi^K\Phi^K A$ are proportional to, respectively,

$$\sum_{\substack{\mathbf{p}_1,\mathbf{p}_2\\ \mu_1,\mu_2,\mu_3}} \hat{A}_{\mu_1,\mathbf{p}_1}\hat{A}_{\mu_2,\mathbf{p}_2}\hat{A}_{\mu_3,-\mathbf{p}_1-\mathbf{p}_2}\hat{W}_{(\mu_1,\mu_2,\mu_3)}(\mathbf{p}_1,\mathbf{p}_2) \qquad \text{or} \qquad \sum_{\substack{\mathbf{p}_1,\mathbf{p}_2\\ \mu,j_1,j_2}} \hat{\Phi}^K_{j_1,\mathbf{p}_1}\hat{\Phi}^K_{j_2,\mathbf{p}_2}\hat{A}_{\mu,-\mathbf{p}_1-\mathbf{p}_2}\hat{W}^{KK}_{(j_1,j_2),\mu}(\mathbf{p}_1,\mathbf{p}_2),$$

for suitable kernels $\hat{W}_{(\mu_1,\mu_2,\mu_3)}(\mathbf{p}_1,\mathbf{p}_2)$, $\hat{W}^{KK}_{(j_1,j_2),\mu}(\mathbf{p}_1,\mathbf{p}_2)$. Using the invariance under the symmetry (7)+(8) we find that $\hat{W}_{(\mu_1,\mu_2,\mu_3)}(\mathbf{p}_1,\mathbf{p}_2) = -\hat{W}_{(\mu_1,\mu_2,\mu_3)}(\mathbf{p}_1,\mathbf{p}_2)$ and $\hat{W}^{KK}_{(j_1,j_2),\mu}(\mathbf{p}_1,\mathbf{p}_2) = -\hat{W}^{KK}_{(j_1,j_2),\mu}(\mathbf{p}_1,\mathbf{p}_2)$, that is, they are both identically zero. On the contrary, the contribution to the effective potential of the form $\Phi^K AA$ is proportional to

$$\sum_{\substack{\mathbf{p}_1,\mathbf{p}_2\\ j,\mu_1,\mu_2}} \hat{\Phi}^K_{j,\mathbf{p}_1}\hat{A}_{\mu_1,\mathbf{p}_2}\hat{A}_{\mu_2,-\mathbf{p}_1-\mathbf{p}_2}\hat{W}^K_{j,(\mu_1,\mu_2)}(\mathbf{p}_1,\mathbf{p}_2),$$

for a suitable kernel $\hat{W}^K_{j,(\mu_1,\mu_2)}(\mathbf{p}_1,\mathbf{p}_2)$. Its local part $\hat{W}^K_{j,(\mu_1,\mu_2)}(\mathbf{0},\mathbf{0}) =: \lambda^K_{j,(\mu_1,\mu_2)}$ satisfies:

$$\lambda^K_j \overset{(4)}{=} T\lambda^K_{j-1}T^{-1} \overset{(5)}{=} \left[\lambda^K_j\right]^* \overset{(6.a)}{=} R_h\lambda^K_{r_h j}R_h \overset{(6.b)}{=} R_v\lambda^K_{r_v j}R_v ,$$

which imply

$$\lambda^K_1 = \begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c \end{pmatrix} \qquad \lambda^K_2 = \begin{pmatrix} a & 0 & 0\\ 0 & \frac{1}{4}(b+3c) & \frac{\sqrt{3}}{4}(c-b)\\ 0 & \frac{\sqrt{3}}{4}(c-b) & \frac{1}{4}(c+3b) \end{pmatrix}, \qquad \lambda^K_j = \begin{pmatrix} a & 0 & 0\\ 0 & \frac{1}{4}(b+3c) & \frac{\sqrt{3}}{4}(b-c)\\ 0 & \frac{\sqrt{3}}{4}(b-c) & \frac{1}{4}(c+3b) \end{pmatrix},$$

for some real constants a, b, c. This the general symmetry structure of the terms in the first line of Eq.(4.28).

3. Symmetry structure of the kernels in the presence of the phonon field

In this Appendix we prove Eqs.(9.3),(9.4),(9.5). Let us define, for all $\mathbf{k},\mathbf{k}' \neq \mathbf{p}^\pm_F$,

$$A^{(j_0)}_j(\mathbf{k}) := \frac{\langle\!\langle \hat{a}^+_{\mathbf{k},\sigma}\hat{b}^-_{\mathbf{k},\sigma}\rangle\!\rangle^{(j_0)}_j}{\Omega(\vec{k})}, \qquad B^{(j_0)}_{\omega,j}(\mathbf{k}') := e^{-i\mathbf{p}^\omega_F(\boldsymbol{\delta}_{j_0}-\boldsymbol{\delta}_1)}\frac{\langle\!\langle \hat{a}^+_{\mathbf{k}'+\mathbf{p}^{-\omega}_F}\hat{b}^-_{\mathbf{k}'+\mathbf{p}^\omega_F}\rangle\!\rangle^{(j_0)}_j}{\Omega(\vec{k}')}. \tag{C.22}$$

Using the symmetries listed in Section 4 B and in Appendix B and proceeding as in Appendix C, we find:

$$A^{(j_0)}_j(\mathbf{k}) \overset{(4)}{=} A^{(j_0+1)}_{j+1}(T\mathbf{k}) \overset{(5)}{=} \left[A^{(j_0)}_j(-\mathbf{k})\right]^* \overset{(6.a)}{=} \left[A^{(j_0)}_{r_h j}(R_h\mathbf{k})\right]^* \overset{(6.b)}{=} A^{(r_v j_0)}_{r_v j}(R_v\mathbf{k}) \overset{(7)}{=} \left[A^{(j_0)}_j(P\mathbf{k})\right]^* \overset{(8)}{=} A^{(j_0)}_j(I\mathbf{k}),$$

$$B^{(j_0)}_{\omega,j}(\mathbf{k}') \overset{(4)}{=} B^{(j_0+1)}_{\omega,j+1}(T\mathbf{k}') \overset{(5)}{=} \left[B^{(j_0)}_{-\omega,j}(-\mathbf{k}')\right]^* \overset{(6.a)}{=} \left[B^{(j_0)}_{-\omega,r_h j}(R_h\mathbf{k}')\right]^* \overset{(6.b)}{=} B^{(r_v j_0)}_{-\omega,r_v j}(R_v\mathbf{k}) \overset{(7)}{=} \left[B^{(j_0)}_{-\omega,j}(P\mathbf{k})\right]^* \overset{(8)}{=} B^{(j_0)}_{\omega,j}(I\mathbf{k})$$

Using the properties (6.a)+(6.b)+(7) in the latter equation, we find: $A^{(j_0)}_j(\mathbf{k}) = A^{(r_v j_0)}_j(\mathbf{k})$ and $B^{(j_0)}_{\omega,j}(\mathbf{k}) = B^{(r_v j_0)}_{-\omega,j}(\mathbf{k})$. Combining this with property (4) we get:

$$A^{(1)}_1(\mathbf{k}) = A^{(2)}_2(T\mathbf{k}) = A^{(3)}_2(T\mathbf{k}) = A^{(3)}_3(T^2\mathbf{k}) = A^{(2)}_3(T^2\mathbf{k}),$$
$$A^{(2)}_1(\mathbf{k}) = A^{(3)}_1(\mathbf{k}) = A^{(3)}_2(T\mathbf{k}) = A^{(2)}_2(T\mathbf{k}) = A^{(1)}_3(T^2\mathbf{k}), \qquad A^{(3)}_1(\mathbf{k}) = A^{(1)}_2(T\mathbf{k}) = A^{(2)}_3(T^2\mathbf{k}),$$

which implies that $A_j^{(j_0)}(\mathbf{k}) = A_j(\mathbf{k})$. The analogous equations for B imply that $B_{\omega,j}^{(j_0)}(\mathbf{k}) = B_j(\mathbf{k})$. Now, using properties (4)+(6.b) gives $A_1(\mathbf{k}) = A_2(T\mathbf{k}) = A_3(TR_v\mathbf{k}) = A_1(T^2R_v\mathbf{k}) = A_1(T^2\mathbf{k})$, which implies that $A_1(\mathbf{k}) = A_1(T\mathbf{k})$; therefore, $A_j(\mathbf{k})$ is independent of j and transforms as in Eq.(9.4). The same argument holds for B. Finally, it is straightforward to check that the previous symmetries also imply that, if $\mathbf{k} = \mathbf{p}_F^{\pm}$, then $\langle\!\langle \hat{a}^+_{\mathbf{k},\sigma}\hat{b}^-_{\mathbf{k},\sigma}\rangle\!\rangle_j^{(j_0)} = \langle\!\langle \hat{a}^+_{\mathbf{k}'+\mathbf{p}_F^{-\omega}}\hat{b}^-_{\mathbf{k}'+\mathbf{p}_F^{\omega}}\rangle\!\rangle_j^{(j_0)} = 0$, which concludes the proof of Eq.(9.3) for all $\mathbf{k},\mathbf{k}'$.

Appendix D: Lowest order computations

1. The beta function for $Z^{\pm}_{K,h}$

Here we prove the two identities in Eq.(5.19) (the identity in Eq.(5.18) has been proved in [18, Appendix C]). The relativistic part of the second order beta function is given by the sum over h_1, h_2, h_3 of the graph in Fig.6, with the constraint that $\min\{h_1,h_2,h_3\} = h$ and that $h_i - h \in \{0,1\}$. In the graph, the external wavy line with transferred momentum $\mathbf{p}_F^{\omega_1} - \mathbf{p}_F^{\omega_2}$ represents $\hat{\Phi}^K_{j,\mathbf{p}_F^{\omega_1}-\mathbf{p}_F^{\omega_2}}$ and the straight internal lines correspond to Dirac propagators on scales h_1, h_2; if $\omega_1 = \omega$ and $\omega_2 = \pm\omega$, then the graph represents a contribution to $\beta_h^+\Gamma^+_{\omega,j}$ or $\beta_h^-\Gamma^-_{\omega,j}$, where $\Gamma^{\pm}_{\omega,j}$ was defined in Eq.(4.20).

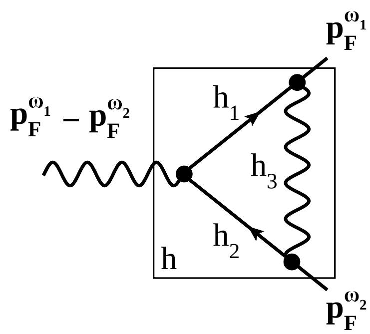

FIG. 6. The lowest order contribution to the beta function for $Z^{\omega_1\omega_2}_{K,h}$.

If we neglect corrections of order $O(e^4)$ and $O(e^2(1-v)2^{ce^2h})$ for some $c>0$, in the computation of the graph we can replace $e_{\mu,h}$ by e, $\tilde{f}_h(\mathbf{k}')$ by $f_h(\mathbf{k}')$ (see Eq.(4.30) for a definition of $\tilde{f}_h$) and v_h by 1. As a result, we find:

$$\beta_h^{\pm}\Gamma^{\pm}_{\omega,j} = -e^2\sum_{\nu=0}^{2}\sum_{\substack{h_i\ge h\\ \min\{h_i\}=h}}\int\frac{d\mathbf{p}}{(2\pi)^3}f_{h_1}(\mathbf{p})f_{h_2}(\mathbf{p})f_{h_3}(\mathbf{p})\frac{1}{2|\mathbf{p}|}\Gamma^{\nu}_{\omega}\frac{1}{ip_0\Gamma^0_{\omega}+i\vec{p}\cdot\vec{\Gamma}_{\omega}}\Gamma^{\pm}_{\omega,j}\frac{1}{ip_0\Gamma^0_{\pm\omega}+i\vec{p}\cdot\vec{\Gamma}_{\pm\omega}}\Gamma^{\nu}_{\pm\omega}\ , \tag{D.1}$$

modulo corrections $O(e^4)$ and $O(e^2(1-v)2^{(\text{const.})e^2h})$. It is convenient to pass to spherical coordinates. Note that, if $F_h(|\mathbf{p}|) := f_h(\mathbf{p})^3 + 3f_h(\mathbf{p})^2 f_{h+1}(\mathbf{p}) + 3f_h(\mathbf{p})f_{h+1}(\mathbf{p})^2$, then $\int_0^{+\infty}d\rho\,\rho^{-1}F_h(\rho) = \log 2$. Therefore, after having integrated the radial coordinates we are left with:

$$\beta_h^{\pm}\Gamma^{\pm}_{\omega,j} = -\frac{e^2\log 2}{2(2\pi)^3}\sum_{\nu=0}^{2}\int_{-1}^{1}d\cos\theta\int_0^{2\pi}d\phi\,\cdot \tag{D.2}$$
$$\cdot\Gamma^{\nu}_{\omega}\big(-i\Gamma^0_{\omega}\cos\theta + i\Gamma^1_{\omega}\sin\theta\cos\phi + i\Gamma^2_{\omega}\sin\theta\sin\phi\big)\Gamma^{\pm}_{\omega,j}\big(-i\Gamma^0_{\pm\omega}\cos\theta + i\Gamma^1_{\pm\omega}\sin\theta\cos\phi + i\Gamma^2_{\pm\omega}\sin\theta\sin\phi\big)\Gamma^{\nu}_{\pm\omega}\ ,$$

which implies

$$\beta_h^{\pm} = \frac{e^2\log 2}{12\pi^2}\big[\Gamma^{\pm}_{\omega,j}\big]^{-1}\sum_{\nu=0}^{2}\Gamma^{\nu}_{\omega}\Big(\Gamma^0_{\omega}\Gamma^{\pm}_{\omega,j}\Gamma^0_{\pm\omega} + \Gamma^1_{\omega}\Gamma^{\pm}_{\omega,j}\Gamma^1_{\pm\omega} + \Gamma^2_{\omega}\Gamma^{\pm}_{\omega,j}\Gamma^2_{\pm\omega}\Big)\Gamma^{\nu}_{\omega}\ . \tag{D.3}$$

Now, if we pick the plus sign, $\Gamma^+_{\omega,j} = \sigma_1\cos\theta_j - \omega\sigma_2\sin\theta_j$, where $\theta_j = \frac{2\pi}{3}(j-1)$. Therefore, using the definitions of Γ^{μ}_{ω} in Eq.(4.11),

$$\Gamma^0_{\omega}\Gamma^+_{\omega,j}\Gamma^0_{\omega} = \Gamma^+_{\omega,j}\ ,\qquad \Gamma^1_{\omega}\Gamma^+_{\omega,j}\Gamma^1_{\omega} = \Gamma^+_{-\omega,j}\ ,\qquad \Gamma^2_{\omega}\Gamma^+_{\omega,j}\Gamma^2_{\omega} = -\Gamma^+_{-\omega,j}\ . \tag{D.4}$$

Plugging Eq.(D.4) into Eq.(D.3) we find $\beta_h^+ = \frac{e^2}{12\pi^2}\log 2$, as desired.

Similarly, picking the minus sign in Eq.(D.3), $\Gamma^-_{\omega,j} = e^{-i\omega\theta_j}\sigma_1$, so that

$$\Gamma^0_\omega \Gamma^-_{\omega,j}\Gamma^0_{-\omega} = \Gamma^-_{\omega,j}\ , \qquad \Gamma^1_\omega \Gamma^-_{\omega,j}\Gamma^1_{-\omega} = \Gamma^-_{\omega,j}\ , \qquad \Gamma^2_\omega \Gamma^-_{\omega,j}\Gamma^2_{-\omega} = \Gamma^-_{\omega,j}\ . \tag{D.5}$$

Plugging Eq.(D.5) into Eq.(D.3) we find $\beta_h^- = \frac{3e^2}{4\pi^2}\log 2$. This completes the proof of Eq.(5.19).

2. The beta function for the other renormalization constants

A similar computation can be performed for the beta function controlling the flow of the other renormalization constants. In particular, the beta functions for $Z^+_{CDW,h}$, $Z^-_{CDW,h}$ and $Z^+_{AF,h}$ and $Z^-_{AF,h}$ are given by expressions analogous to Eq.(D.3) with $\Gamma^\mu_{\omega,j}$ replaced by σ_3 and $e^{i\omega\theta_j n_-}\sigma_3$, respectively. Using the fact that

$$\Gamma^0_\omega\sigma_3\Gamma^0_\omega = \sigma_3\ , \qquad \Gamma^1_\omega\sigma_3\Gamma^1_\omega = \sigma_3\ , \qquad \Gamma^2_\omega\sigma_3\Gamma^2_\omega = \sigma_3\ ,$$

we find that $\beta^+_{CDW,h} = \frac{3e^2}{4\pi^2}\log 2$, modulo subdominant corrections. Similarly,

$$\Gamma^0_\omega e^{i\omega\theta_j n_-}\sigma_3\Gamma^0_{-\omega} = e^{i\omega\theta_j n_-}\sigma_3\ , \qquad \Gamma^1_\omega e^{i\omega\theta_j n_-} e^{i\omega\theta_j n_-}\sigma_3\Gamma^1_{-\omega} = e^{i\omega\theta_j n_+}\sigma_3\ , \qquad \Gamma^2_\omega e^{i\omega\theta_j n_-}\sigma_3\Gamma^2_{-\omega} = -e^{i\omega\theta_j n_+}\sigma_3\ ,$$

implies that $\beta^-_{CDW,h} = \frac{e^2}{12\pi^2}\log 2$, modulo subdominant corrections. This proves Eqs.(7.3)–(7.5). The exponents in Table I are computed analogously and we will not belabor the details here.

Appendix E: Lowest order check of the Ward Identities

In this section we check at lowest order in non-renormalized perturbation theory the validity of the WIs that we used in Section 5 to prove the infrared stability of the flows of the effective charge and of the photon mass.

1. Ward identity for the photon mass

We start by checking the WI Eq.(5.6) for the photon mass. At lowest order, the graphs contributing to the dressed photon mass are depicted in Fig.3. As we are going to show here, the sum of the two graphs computed at zero transferred momentum is *exactly vanishing*, for all choices of μ,ν.

The first and second order interactions (of the form $A\Psi^+\Psi^-$ and $AA\Psi^+\Psi^-$) involved in the computation of the two graphs are obtained by expanding the interaction $\mathcal{V}(\Psi, A)$ in Eqs.(3.6)-(B.1) up to second order in the electric charge:

$$\mathcal{V}(\Psi,A) = \frac{ie}{\beta^2L^2|\mathcal{S}_L|}\sum_{\mathbf{k},\mathbf{p},\sigma}\Big\{-\hat{A}_{0,\mathbf{p}}\hat{\Psi}^+_{\mathbf{k}+\mathbf{p},\sigma}\Gamma_0(\mathbf{p})\hat{\Psi}^-_{\mathbf{k},\sigma} + t\sum_{j=1}^3 \eta^j_{\mathbf{p}}(\vec{\delta}_j\vec{A}_{\mathbf{p}})\hat{\Psi}^+_{\mathbf{k}+\mathbf{p},\sigma}\begin{pmatrix} 0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} \\ -e^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{\mathbf{k},\sigma}\Big\}$$
$$-\frac{1}{2}\frac{te^2}{\beta^3L^2|\mathcal{S}_L|^2}\sum_{\substack{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2\\ \sigma}}\sum_{j=1}^3 \eta^j_{\mathbf{p}_1}\eta^j_{\mathbf{p}_2}(\vec{\delta}_j\vec{A}_{\mathbf{p}_1})(\vec{\delta}_j\vec{A}_{\mathbf{p}_2})\hat{\Psi}^+_{\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2,\sigma}\begin{pmatrix} 0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} \\ e^{i(\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2)(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\hat{\Psi}^-_{\mathbf{k},\sigma} + O(e^3)\ .$$

Defining

$$\vec{\Gamma}(\mathbf{k},\mathbf{p}) := \frac{2}{3}\sum_{j=1}^3 \eta^j_{\mathbf{p}}\vec{\delta}_j\begin{pmatrix} 0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} \\ -e^{i(\mathbf{k}+\mathbf{p})(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\ ,$$
$$\Gamma_{lm}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2) = \frac{2}{3}\sum_{j=1}^3 \eta^j_{\mathbf{p}_1}\eta^j_{\mathbf{p}_2}(\vec{\delta}_j)_l(\vec{\delta}_j)_m\begin{pmatrix} 0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} \\ e^{i(\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2)(\boldsymbol{\delta}_j-\boldsymbol{\delta}_1)} & 0\end{pmatrix}\ ,$$

and recalling that $v = \frac{3}{2}t$, we can rewrite the interaction as $\mathcal{V}(\Psi, A) = \mathcal{V}_2(\Psi, A) + O(e^3)$, with

$$\mathcal{V}_2(\Psi, A) = \frac{ie}{\beta^2 L^2 |\mathcal{S}_L|} \sum_{\mathbf{k},\mathbf{p},\sigma} \Big\{ -\hat{A}_{0,\mathbf{p}} \hat{\Psi}^+_{\mathbf{k}+\mathbf{p},\sigma} \Gamma_0(\mathbf{p}) \hat{\Psi}^-_{\mathbf{k},\sigma} + v \hat{\Psi}^+_{\mathbf{k}+\mathbf{p},\sigma} \Big(\vec{\Gamma}(\mathbf{k},\mathbf{p}) \cdot \vec{A}_{\mathbf{p}}\Big) \hat{\Psi}^-_{\mathbf{k},\sigma} \Big\}$$
$$- \frac{1}{2} \frac{ve^2}{\beta^3 L^2 |\mathcal{S}_L|^2} \sum_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2} \sum_{\substack{l,m=1,2 \\ \sigma=\uparrow\downarrow}} A_{l,\mathbf{p}_1} A_{m,\mathbf{p}_2} \hat{\Psi}^+_{\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2,\sigma} \Gamma_{lm}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2) \hat{\Psi}^-_{\mathbf{k},\sigma} \ . \tag{E.1}$$

Since the term in the last line involves only the spatial components of the photon field, the second graph in Fig.3 gives a non zero contribution only if both μ and ν are different from zero.

If $\mu = \nu = 0$, only the first graph survives, which in the $\beta, L \to \infty$ limit, if computed at $\mathbf{p} = \mathbf{0}$, gives (using the fact that $\Gamma_0(\mathbf{0}) = \mathbb{1}$)

$$+\frac{e^2}{2} \int \frac{d\mathbf{k}}{2\pi|\mathcal{B}|} \mathrm{Tr}\Big\{ \hat{S}_0(\mathbf{k}) \hat{S}_0(\mathbf{k}) \Big\} \ , \tag{E.2}$$

where $\hat{S}_0(\mathbf{k})$ was defined in Eq.(2.11) and $|\mathcal{B}| = 8\pi^2/(3\sqrt{3})$ is the area of the first Brillouin zone. Using the fact that

$$\hat{S}_0(\mathbf{k}) \hat{S}_0(\mathbf{k}) = -i\partial_0 \hat{S}_0(\mathbf{k}) \ , \tag{E.3}$$

we see that Eq.(E.2) is the integral of a total derivative, which is zero.

Let us now consider the case where $\mu = \nu = 2$. The sum of the two graphs in the $\beta, L \to \infty$ limit, if computed at $\mathbf{p} = \mathbf{0}$, gives:

$$\frac{v^2 e^2}{2} \int \frac{d\mathbf{k}}{2\pi|\mathcal{B}|} \mathrm{Tr}\Big\{ \hat{S}_0(\mathbf{k}) \Gamma_2(\mathbf{k},\mathbf{0}) \hat{S}_0(\mathbf{k}) \Gamma_2(\mathbf{k},\mathbf{0}) \Big\} + \frac{ve^2}{2} \int \frac{d\mathbf{k}}{2\pi|\mathcal{B}|} \mathrm{Tr}\Big\{ \hat{S}_0(\mathbf{k}) \Gamma_{22}(\mathbf{k},\mathbf{0},\mathbf{0}) \Big\} \ . \tag{E.4}$$

Using the fact that

$$\hat{S}_0(\mathbf{k}) v \Gamma_2(\mathbf{k},\mathbf{0}) \hat{S}_0(\mathbf{k}) = i\partial_2 \hat{S}_0(\mathbf{k}) \ , \tag{E.5}$$

and integrating by parts, we can rewrite Eq.(E.4) as

$$\frac{ve^2}{2} \int \frac{d\mathbf{k}}{2\pi|\mathcal{B}|} \mathrm{Tr}\Big\{ \hat{S}_0(\mathbf{k}) \big[-i\partial_2 \Gamma_2(\mathbf{k},\mathbf{0}) + \Gamma_{22}(\mathbf{k},\mathbf{0},\mathbf{0}) \big] \Big\} \ , \tag{E.6}$$

which is zero, simply because the matrix in square brackets is identically zero. This proves that the graphs in Fig.3 with $\mu = \nu = 2$ cancel out exactly. Using the symmetry (4), we find that the diagonal terms with $\mu = \nu = 1$ cancel out, too. The non-diagonal terms can be treated analogously.

2. Ward identity for the effective charge

Let us check at lowest order in non-renormalized perturbation theory the WI for the effective charge, Eqs.(5.8)-(5.9). This amounts to check the cancellation of the graphs depicted in Fig.4. In order to compute these graphs we use the bare fermionic propagator $\hat{S}_0(\mathbf{k})$ and the photon propagator $\hat{w}^{0,h^*}_{\mu\nu}(\mathbf{p}) =: \hat{w}^{(\geq h^*)}(\mathbf{p})\delta_{\mu\nu}$ with IR cutoff on scale h^*, see Eq.(3.3). As we are going to show, the sum of these six graphs is exactly vanishing; therefore, the dressed charge is equal to the bare one at lowest order. Remarkably, this cancellation does not depend on the presence of the infrared cutoff on the photons; this fact has been exploited in Section 5 to derive a WI for the effective charge on all IR scales. We shall only consider the cases $\mu = 0$ and $\mu = 2$; the renormalization of the charge corresponding to $\mu = 1$ is equal to the case $\mu = 2$, thanks to the discrete rotational symmetry (4).

In order to compute the graphs in Fig.4 we need the interaction $\mathcal{V}(\Psi, A)$ up to third order in e:

$$\mathcal{V}(\Psi, A) = \mathcal{V}_2(\Psi, A) + \frac{1}{6} \frac{v(ie)^3}{\beta^4 L^2 |\mathcal{S}_L|^3} \sum_{\substack{\mathbf{k},\mathbf{p}_1 \\ \mathbf{p}_2,\mathbf{p}_3}} \sum_{\substack{l,m,k=1,2 \\ \sigma=\uparrow\downarrow}} A_{l,\mathbf{p}_1} A_{m,\mathbf{p}_2} A_{k,\mathbf{p}_3} \hat{\Psi}^+_{\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3,\sigma} \Gamma_{lmk}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) \hat{\Psi}^-_{\mathbf{k},\sigma} + O(e^4) \ , \tag{E.7}$$

where $\mathcal{V}_2(\Psi, A)$ was defined in Eq.(E.1) and

$$\Gamma_{lmk}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = \frac{2}{3} \sum_{j=1}^{3} \eta^j_{\mathbf{p}_1} \eta^j_{\mathbf{p}_2} \eta^j_{\mathbf{p}_3} (\vec{\delta}_j)_l (\vec{\delta}_j)_m (\vec{\delta}_j)_k \begin{pmatrix} 0 & e^{-i\mathbf{k}(\boldsymbol{\delta}_j - \boldsymbol{\delta}_1)} \\ -e^{i(\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)(\boldsymbol{\delta}_j - \boldsymbol{\delta}_1)} & 0 \end{pmatrix} . \tag{E.8}$$

a. Case $\mu = 0$. In this case only the first and fifth graphs in Fig.4 are non-vanishing. Their value in the $\beta, L \to \infty$ limit computed at the Fermi points $\mathbf{k} = \mathbf{p}_F^\omega$ and at transferred momentum $\mathbf{p} = \mathbf{0}$ is

$$ie \sum_{\nu=0}^{2} (i\bar{e}_\nu)^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \hat{w}^{(\geq h^*)}(\mathbf{p}) \Big\{ \Gamma_\nu(\mathbf{p}_F^\omega + \mathbf{p}, -\mathbf{p}) \hat{S}_0(\mathbf{p}_F^\omega + \mathbf{p}) \Gamma_0(\mathbf{p}_F^\omega + \mathbf{p}, \mathbf{0}) \hat{S}_0(\mathbf{p}_F^\omega + \mathbf{p}) \Gamma_\nu(\mathbf{p}_F^\omega, \mathbf{p}) \Big\} +$$
$$+ e \sum_{\nu=0}^{2} (i\bar{e}_\nu)^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \hat{w}^{(\geq h^*)}(\mathbf{p}) \partial_{k_0} \Big\{ \Gamma_\nu(\mathbf{k} + \mathbf{p}, -\mathbf{p}) \hat{S}_0(\mathbf{k} + \mathbf{p}) \Gamma_\nu(\mathbf{k}, \mathbf{p}) \Big\}_{\mathbf{k} = \mathbf{p}_F^\omega} , \tag{E.9}$$

where we defined $\bar{e}_0 = e$, $\bar{e}_l = ve$ for $l \in \{1, 2\}$ and $\Gamma_0(\mathbf{k}, \mathbf{p}) := -\Gamma_0(\mathbf{p})$. Now, in the first line we can rewrite:

$$\hat{S}_0(\mathbf{p}_F^\omega + \mathbf{p}) \Gamma_0(\mathbf{p}_F^\omega + \mathbf{p}, \mathbf{0}) \hat{S}_0(\mathbf{p}_F^\omega + \mathbf{p}) = i\partial_0 \hat{S}_0(\mathbf{p}_F^\omega + \mathbf{p}) ,$$

which partially cancel with the second line. We are only left with the terms in the second line where the derivative ∂_{k_0} acts on the kernels Γ_ν; however, these terms are identically zero, simply because the kernels Γ_ν are by definition independent of k_0.

b. Case $\mu = 2$. Here the situation is more complicated, because of the presence of the second, third, fourth and sixth graph in Fig.4, and because the ∂_{k_2} derivative can now act on the kernels Γ_ν. However:

(i) repeating the same argument used in the case $\mu = 0$ and using Eq.(E.5), we see that the sum of the first and fifth graphs is equal to

$$e \sum_{\nu=0}^{2} (i\bar{e}_\nu)^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \hat{w}^{(\geq h^*)}(\mathbf{p}) \Big\{ \partial_{k_2} \Gamma_\nu(\mathbf{k} + \mathbf{p}, -\mathbf{p}) \hat{S}_0(\mathbf{k} + \mathbf{p}) \Gamma_\nu(\mathbf{k}, \mathbf{p}) + \Gamma_\nu(\mathbf{k} + \mathbf{p}, -\mathbf{p}) \hat{S}_0(\mathbf{k} + \mathbf{p}) \partial_{k_2} \Gamma_\nu(\mathbf{k}, \mathbf{p}) \Big\}_{\mathbf{k} = \mathbf{p}_F^\omega} \tag{E.10}$$

(ii) combining Eq.(E.10) with the contributions from the second and third graphs, we get

$$e \sum_{\nu=0}^{2} (i\bar{e}_\nu)^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \hat{w}^{(\geq h^*)}(\mathbf{p}) \Big\{ \big[\partial_{k_2} \Gamma_\nu(\mathbf{k} + \mathbf{p}, -\mathbf{p}) + i\Gamma_{2\nu}(\mathbf{k} + \mathbf{p}, \mathbf{0}, -\mathbf{p}) \big] \hat{S}_0(\mathbf{k} + \mathbf{p}) \Gamma_\nu(\mathbf{k}, \mathbf{p}) +$$
$$+ \Gamma_\nu(\mathbf{k} + \mathbf{p}, -\mathbf{p}) \hat{S}_0(\mathbf{k} + \mathbf{p}) \big[\partial_{k_2} \Gamma_\nu(\mathbf{k}, \mathbf{p}) + i\Gamma_{2\nu}(\mathbf{k}, \mathbf{0}, \mathbf{p}) \big] \Big\}_{\mathbf{k} = \mathbf{p}_F^\omega} , \tag{E.11}$$

which is zero, simply because the terms in square brackets are identically zero, as one can easily check.

(iii) the sum of the fourth and sixth graphs in Fig.4 gives

$$-\frac{1}{2} ve^3 \sum_{l=1,2} \int \frac{d\mathbf{p}}{(2\pi)^3} \hat{w}^{(\geq h^*)}(\mathbf{p}) \big[\partial_{k_2} \Gamma_{ll}(\mathbf{k}, \mathbf{p}, -\mathbf{p}) + i\Gamma_{2ll}(\mathbf{k}, \mathbf{0}, \mathbf{p}, -\mathbf{p}) \big]_{\mathbf{p} = \mathbf{p}_F^\omega} , \tag{E.12}$$

which is zero, simply because the term in square brackets are identically zero, as one can easily check.

The case $\mu = 1$ can be obtained from the case $\mu = 2$ by the discrete rotational symmetry (4), so this concludes our lowest order check of the Ward identities.

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