

On the Fermi Function of Squeezed Coherent States

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Abstract

Fermi observed in 1930 that the state of a quantum system may be defined in two different (but equivalent) ways, namely by its wavefunction Ψ or by a certain function g_F on phase space canonically associated with Ψ . In this Note we study we study Fermi's function when Ψ is a squeezed coherent state. We relate it with the Wigner transform of Ψ , thus generalising a previous observation of Benenti and Strini. We show that the symplectic capacity of the phase space ellipsoid $g_F(x, p) \leq 0$ is bounded by $h/2$ and $nh/2$ (n the number of degrees of freedom).

1 Introduction

In a largely forgotten paper [4] from 1930 Enrico Fermi associates to every quantum state Ψ a certain hypersurface $g_F(x, p) = 0$ in phase space. The underlying idea is actually extremely simple, and consists in observing that any complex twice continuously differentiable function $\Psi(x) = R(x)e^{i\Phi(x)/\hbar}$

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($R(x) \geq 0$ and $\Phi(x)$ real) defined on \mathbb{R}^n satisfies the partial differential equation

$$\left[(-i\hbar\nabla_x - \nabla_x\Phi)^2 + \hbar^2 \frac{\nabla_x^2 R}{R} \right] \Psi = 0. \quad (1)$$

where ∇_x^2 is the Laplace operator in the variables x_1, \dots, x_n . Performing the gauge transformation $-i\hbar\nabla_x \rightarrow -i\hbar\nabla_x - \nabla_x\Phi$ this equation is in fact equivalent to the trivial equation

$$\left(-\hbar^2\nabla_x^2 + \hbar^2 \frac{\nabla_x^2 R}{R} \right) R = 0. \quad (2)$$

The operator

$$\widehat{g}_F = (-i\hbar\nabla_x - \nabla_x\Phi)^2 + \hbar^2 \frac{\nabla_x^2 R}{R} \quad (3)$$

appearing in the left-hand side of Eqn. (1) is the quantisation (in every reasonable physical quantisation scheme) of the real observable

$$g_F(x, p) = (p - \nabla_x\Phi)^2 + \hbar^2 \frac{\nabla_x^2 R}{R} \quad (4)$$

and the equation $g_F(x, p) = 0$ determines a hypersurface \mathcal{H}_F in phase space $\mathbb{R}_{x,p}^{2n}$ which Fermi ultimately *identifies* with the state Ψ itself.

Of course, Fermi's analysis was very heuristic and its mathematical rigour borders the unacceptable (at least by modern standards). Fermi's paper has recently been rediscovered by Benenti [1] and Benenti and Strini [2], who study its relationship with the level sets of the Wigner transform of Ψ .

The aim of the present Note is to push further the analysis in [1, 2] by considering the Fermi function of squeezed coherent states. We will also study the symplectic capacity of the corresponding set bounded by the Fermi surface \mathcal{H}_F (which is in this case an ellipsoid).

Notation 1 *The points in configuration and momentum space are written $x = (x_1, \dots, x_n)$ and $p = (p_1, \dots, p_n)$; in formulas x and p are viewed as column vectors. We will also use the collective notation $z = (x, p)$ for the phase space variable. The matrix $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ (0 and I the $n \times n$ zero and identity matrices) defines the standard symplectic form on the phase space \mathbb{R}_x^{2n} via the formula $\sigma(z, z') = Jz \cdot z' = p \cdot x' - p' \cdot x$. We write $\hbar = h/2\pi$, h being Planck's constant.*

2 The Harmonic Oscillator

As an appetizer we begin by considering the fiducial coherent state

$$\Psi_0(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} e^{-|x|^2/2\hbar} \quad (5)$$

with $|x|^2 = x \cdot x$; it is the ground state of the n -dimensional isotropic harmonic oscillator with mass and frequency equal to one:

$$\frac{1}{2}(-\hbar^2\nabla_x^2 + |x|^2)\Psi_0 = \frac{1}{2}n\hbar\Psi_0. \quad (6)$$

The operator (1) is here

$$\widehat{g}_F = -\hbar^2\nabla_x^2 + |x|^2 - n\hbar \quad (7)$$

and the relation $\widehat{g}_F\Psi_0 = 0$ is hence equivalent to Eqn. (6). The Fermi function is

$$g_F(x, p) = |p|^2 + |x|^2 - n\hbar \quad (8)$$

and the ellipsoid \mathcal{W}_F is thus the disk $|x|^2 + |p|^2 \leq n\hbar$ whose area is $n\pi\hbar = nh/2$.

Consider next the N -th eigenstate Ψ_N ; assume first $n = 1$. We have

$$\frac{1}{2}\left(-\hbar^2\frac{d^2}{dx^2} + x^2\right)\Psi_N = \left(N + \frac{1}{2}\right)\hbar\Psi_N$$

and the eigenfunction Ψ_N is the (unnormalized) Hermite function

$$\Psi_N(x) = e^{-|x|^2/2\hbar} H_N(x/\sqrt{\hbar}) \quad (9)$$

where

$$H_N(x) = (-1)^n e^{x^2} \frac{d^N}{dx^N} e^{-x^2}$$

is the N -th Hermite polynomial. Since Ψ_N is real, the corresponding Fermi function is

$$g_F(x, p) = p^2 + x^2 - (2N + 1)\hbar \quad (10)$$

and the Fermi set $g_F(x, p)$ is the circle

$$p^2 + x^2 = (2N + 1)\hbar \quad (11)$$

whose area is $(2N + 1)\pi\hbar = (N + \frac{1}{2})h$. In the case of an arbitrary number n of degrees of freedom the eigenstate Ψ_N is the tensor product of n Hermite functions (9) and one finds that

$$g_{\text{F}}(x, p) = |p|^2 + |x|^2 - (2N + 1)\hbar$$

hence the Fermi set is this time the ball

$$|p|^2 + |x|^2 = (2N + 1)\hbar. \quad (12)$$

3 Squeezed Coherent States

We next consider arbitrary (normalized) squeezed coherent states

$$\Psi_{X,Y}(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} (\det X)^{1/4} \exp\left[-\frac{1}{2\hbar}(X + iY)x \cdot x\right] \quad (13)$$

where X and Y are real symmetric $n \times n$ matrices, and X is positive definite. Setting $\Phi(x) = -\frac{1}{2}Yx \cdot x$ and $R(x) = \exp(-\frac{1}{2\hbar}Xx \cdot x)$ we have

$$\nabla_x \Phi(x) = -Yx \quad , \quad \frac{\nabla_x^2 R(x)}{R(x)} = -\frac{1}{\hbar} \text{Tr } X + \frac{1}{\hbar^2} X^2 x \cdot x \quad (14)$$

hence the Fermi function of $\Psi_{X,Y}$ is the quadratic form

$$g_{\text{F}}(x, p) = (p + Yx)^2 + X^2 x \cdot x - \hbar \text{Tr } X. \quad (15)$$

We can rewrite this formula as

$$g_{\text{F}}(x, p) = [x \ , \ p] M_{\text{F}} \begin{bmatrix} x \\ p \end{bmatrix} - \hbar \text{Tr } X$$

where M_{F} is the symmetric matrix

$$M_{\text{F}} = \begin{bmatrix} X^2 + Y^2 & Y \\ Y & I \end{bmatrix}. \quad (16)$$

A straightforward calculation shows that

$$M_{\text{F}} = S^T \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} S \quad (17)$$

where S is the *symplectic* matrix

$$S = \begin{bmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{bmatrix}. \quad (18)$$

4 Relation With the Wigner Function

It turns out –and this is really a striking fact!– that the matrix (17) is closely related to the Wigner transform

$$W\Psi_{X,Y}(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p \cdot y} \Psi_{X,Y}(x + \frac{1}{2}y) \Psi_{X,Y}^*(x - \frac{1}{2}y) dy \quad (19)$$

of the state $\Psi_{X,Y}$ because we have

$$W\Psi_{X,Y}(z) = \left(\frac{1}{\pi\hbar}\right)^n \exp\left(-\frac{1}{\hbar}Gz \cdot z\right) \quad (20)$$

where G is the symplectic matrix

$$G = S^T S = \begin{bmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{bmatrix} \quad (21)$$

(see e.g. [11, 17]). It follows from Eqn. (21) that

$$W\Psi_{X,Y}(z) = \left(\frac{1}{\pi\hbar}\right)^n e^{-\text{Tr } X} \exp\left[-\frac{1}{\hbar}g_{\mathbb{F}}(S^{-1}D^{-1/2}Sz)\right]. \quad (22)$$

with $D = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$. In particular, when $n = 1$ and $\Psi_{X,Y}(x) = \Psi_0(x)$ the fiducial coherent state (5) we have $S^{-1}D^{-1/2}S = I$ and $\text{Tr } X = 1$ hence the formula

$$W\Psi_0(z) = \left(\frac{1}{\pi\hbar}\right)^{1/4} \frac{1}{e} \exp\left[-\frac{1}{\hbar}M_{\mathbb{F}}z \cdot z\right]$$

which was already observed by Benenti and Strini [2].

5 Geometric Interpretation

In [7, 8, 9, 10, 11, 12, 13] (also see de Gosson and Luef [14]) we have applied the topological notion of symplectic capacity [15, 16, 18] to the uncertainty principle. Recall (ibid.) that the symplectic capacity $c(\mathcal{W})$ of an ellipsoid $Mz \cdot z \leq 1$ (M a symmetric positive definite $2n \times 2n$ matrix) is calculated as follows: Consider the matrix product JM (J the standard symplectic matrix); because M is positive definite JM is equivalent to the antisymmetric

matrix $M^{1/2}JM^{1/2}$ hence its $2n$ eigenvalues are of the type $\pm i\lambda_1^\sigma, \dots, \pm i\lambda_n^\sigma$ where $\lambda_j^\sigma > 0$. The positive numbers $\lambda_1^\sigma, \dots, \lambda_n^\sigma$ are called the *symplectic eigenvalues* of the matrix M and we have

$$c(\mathcal{W}) = \pi/\lambda_{\max}^\sigma \quad (23)$$

where $\lambda_{\max}^\sigma = \max\{\lambda_1^\sigma, \dots, \lambda_n^\sigma\}$. The symplectic capacity of a subset of phase space is invariant under canonical transformations (linear or not).

We denote by \mathcal{W}_F the ellipsoid $M_F z \cdot z \leq \hbar \text{Tr } X$ bounded by the Fermi hypersurface \mathcal{H}_F corresponding to the squeezed coherent state $\Psi_{X,Y}$. Let us perform the symplectic change of variables $z' = Sz$; in the new coordinates the ellipsoid \mathcal{W}_F is represented by the inequality

$$Xx' \cdot x' + Xp' \cdot p' \leq \hbar \text{Tr } X \quad (24)$$

hence $c(\mathcal{W}_F)$ is the symplectic capacity of the ellipsoid (24). Applying the rule above we thus have to find the symplectic eigenvalues $D = \begin{bmatrix} 0 & X \\ -X & 0 \end{bmatrix}$; a straightforward calculation shows that these are just the eigenvalues $\lambda_1, \dots, \lambda_n$ of X and hence

$$c(\mathcal{W}_F) = \frac{\pi \text{Tr } X}{\lambda_{\max}} \hbar \quad (25)$$

where λ_{\max} is the largest eigenvalue of X . Since $\lambda_{\max} \leq \text{Tr } X \leq n\lambda_{\max}$ it follows that we have the double inequality

$$\frac{1}{2}h \leq c(\mathcal{W}_F) \leq \frac{nh}{2}. \quad (26)$$

A consequence of this is that the Fermi ellipsoid \mathcal{W}_F of a squeezed coherent state always contains a “quantum blob”, the image of a phase space ball $B^{2n}(z_0, \sqrt{\hbar}) : |z - z_0| \leq \sqrt{\hbar}$ by a linear canonical transformation (identified with a symplectic matrix S). A quantum blob is a phase space ellipsoid with symplectic capacity $\pi\hbar = h/2$. The interest of quantum blobs come from the fact that they provide us with a coarse-graining of phase space different from the usual coarse graining by cubes with volume $\sim h^n$ commonly used in statistical mechanics.

6 Comments

Benenti and Strini [2] have given first order approximations comparisons between of the level sets of the Wigner transform and the equation $g_F(x, p) \leq 0$ for sharply-peaked non-Gaussian quantum states; they claim that the Fermi function can be used with profit for a semiclassical study of such states. It would be very interesting to push their analysis further (and in an arbitrary number of degrees of freedom). This can possibly be achieved using known semiclassical approximations for the Wigner transform.

The quantity $\hbar^2 \nabla_x^2 R/R$ appearing in Eqns. (3) and (4) in the definitions of \hat{g}_F and g_F is a variant of the the quantum potential

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla_x^2 R}{R} \quad (27)$$

appearing in Bohmian mechanics (Bohm and Hiley [3]). It would be interesting to interpret Fermi's function in terms of this popular variant of quantum mechanics.

References

- [1] Benenti, G.: Gaussian wave packets in phase space: The Fermi g_F function, *Am. J. Phys.* **77**(6), 546–551 (2009)
- [2] Benenti, G., Strini, G.: Quantum mechanics in phase space: first order comparison between the Wigner and the Fermi function, *Eur. Phys. J. D* **57**, 117–121 (2010)
- [3] Bohm, D., Hiley, B.: *The Undivided Universe: An Ontological Interpretation of Quantum Theory*, London & New York: Routledge (1993)
- [4] Fermi, E.: *Rend. Lincei* **11**, 980 (1930); reprinted in *Nuovo Cimento* **7**, 361 (1930)
- [5] Folland, G.B.: *Harmonic Analysis in Phase space*, *Annals of Mathematics studies*, Princeton University Press, Princeton, N.J. (1981)
- [6] de Gosson, M.: *The Principles of Newtonian and Quantum Mechanics: The need for Planck's constant, h . With a foreword by Basil Hiley*. Imperial College Press, London (2001)

- [7] de Gosson, M.: The “symplectic camel principle” and semiclassical mechanics. *J. Phys. A: Math. Gen.* **35**(32), 6825–6851 (2002)
- [8] de Gosson, M.: Phase Space Quantization and the Uncertainty Principle. *Phys. Lett. A*, **317**/5-6 365–369 (2003)
- [9] de Gosson, M.: The optimal pure Gaussian state canonically associated to a Gaussian quantum state. *Phys. Lett. A*, **330**:3–4, 161–167 (2004)
- [10] de Gosson, M.: Cellules quantiques symplectiques et fonctions de Husimi–Wigner. *Bull. Sci. Math.* **129** 211–226 (2005)
- [11] de Gosson, M.: *Symplectic Geometry and Quantum Mechanics*, Birkhäuser, Basel, series “Operator Theory: Advances and Applications” (subseries: “Advances in Partial Differential Equations”), Vol. **166** (2006)
- [12] de Gosson, M.: The Symplectic Camel and the Uncertainty Principle: The Tip of an Iceberg? *Found. Phys.* **99**, 194–214 (2009)
- [13] de Gosson, M.: On the use of minimum volume ellipsoids and symplectic capacities for studying classical uncertainties for joint position–momentum measurements, *J. Stat. Mech.* (2010) P11005, DOI: 10.1088/1742-5468/2010/11/P11005
- [14] de Gosson, M., Luef, F.: Symplectic Capacities and the Geometry of Uncertainty: the Irruption of Symplectic Topology in Classical and Quantum Mechanics. *Physics Reports* **484**, 131–179 (2009), DOI 10.1016/j.physrep.2009.08.001
- [15] Gromov, M.: Pseudoholomorphic curves in symplectic manifolds, *Invent. Math.*, **82**, 307–347 (1985)
- [16] Hofer H. and Zehnder E.: *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Advanced texts, (Basler Lehrbücher, Birkhäuser Verlag, (1994)
- [17] Littlejohn, R.G.: The semiclassical evolution of wave packets, *Physics Reports* **138**(4–5), 193–291 (1986)
- [18] Polterovich, L.: *The Geometry of the Group of Symplectic Diffeomorphisms*, Lectures in Mathematics, Birkhäuser, (2001)