Renormalization group treatment of rigidity percolation

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Renormalization group calculations are used to give exact solutions for rigidity percolation on hierarchical lattices. Algebraic scaling transformations for a simple example in two dimensions produce a transition of second order, with an unstable critical point and associated scaling laws. Values are provided for the order parameter exponent $\beta = 0.0775$ associated with the spanning rigid cluster and also for $d\nu = 3.533$ which is associated with an anomalous lattice dimension d and the divergence in the correlation length near the transition. In addition we argue that the number of floppy modes F plays the role of a free energy and hence find the exponent α and establish hyperscaling. The exact analytical procedures demonstrated on the chosen example readily generalize to wider classes of hierarchical lattice.

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In this letter we re-visit rigidity percolation on a lattice and show for the first time how renormalization group calculations can be exactly performed on particular bond-diluted hierarchical lattices in two dimensions and show that the transition is second order. This is in contrast to the only other exact solution known for the rigidity transition, on Cayley tree networks which is first order [1, 2].

Phase transitions associated with rigidity have experimental importance in the elastic behavior in chalcogenide glasses [3], in protein unfolding [4] and in jamming in granular materials [5]. Rigidity percolation is similar conceptually to the more familiar connectivity percolation [6, 7], except that instead of demanding a connected pathway across the sample, the more stringent condition that the connected pathway is also *rigid* is required.

Rigidity percolation on networks has been studied since 1984 when the concept was first introduced and a mean field description proved remarkably accurate [8, 9], except very close to the phase transition. Subsequent work has been mainly numerical [10, 11]. The associated rigidity phase transition has been most extensively investigated on the triangular network in two dimensions where numerical studies (using the pebble game algorithm outlined below) show that the transition is second order and described by critical exponents $\beta = 0.18 \pm 0.02$ and $d\nu = 2.42 \pm 0.12$ that are distinct from those of connectivity percolation ($\beta = 5/36 = 0.139$ and $d\nu = 8/3 = 2.667$).

Results in three dimensions using the pebble game algorithm [12] strongly suggest that the rigidity transition is first order on a bond diluted face centered cubic lattice, whereas if angular forces are included whenever two adjacent edges are present, the transition is second order. This is quite different from connectivity percolation where the transition is always second order in three dimension [6]. Further information comes from Cayley tree networks where connectivity percolation is second order, whereas rigidity percolation (from a rigid busbar) shows a strongly first order transition [2].

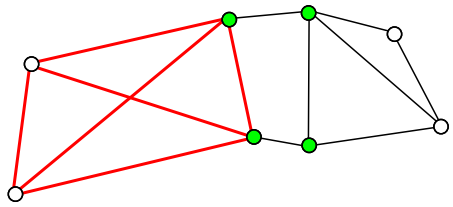

Figure 1. (Color online) An isolated piece of network is used shows an isostatic (unstressed) rigid region with edges shown by thin lines and an overconstrained (stressed) rigid region containing edges shown as thick lines. At the flexible hinges, shown as solid circles, angular motion is possible.

Some characteristics of networks related to rigidity are illustrated in Fig. 1. Such particular network realizations are elucidated by exact counting procedures [13, 14] such as the Maxwell count (used below) and in the pebble game algorithm. Both balance constraints against degrees of freedom. The latter finds the rigid clusters and the flexible joints between them and also determines redundant bonds in overconstrained regions, as illustrated in Fig. 1.

In this letter we show how exact calculations can be performed on hierarchical networks in two dimensions, taking for detailed discussion the Berker lattice [15, 16] shown in Fig. 2. When diluted, this network is one of the simplest which captures fundamental generic features for rigidity percolation. Each generation is obtained by decoration of the previous generation, creating an infinite sequence that can lead to singularities and a phase transition. An exact set of equations can be written down, relating quantities associated with generations $n+1$ and n, which can be solved at all bond concentrations p by it-

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eration. Most importantly the stable and unstable fixed points can be found and the structure of the rigidity phase transition can be described by the scaling behavior obtained by expanding about the unstable fixed point.

It is instructive to do a Maxwell count [17] on the first three generations of the Berker lattice shown in Fig. 2. The number of floppy modes F is given by the difference in the number of degrees of freedom $2V$, associated with the number of vertices V, and the number of constraints which are associated with the number E of edges. However, in general not all the edges are independent constraints and so E must be corrected by the number R of redundant edges so that

$$F = 2V - E + R. \tag{1}$$

The number of floppy modes F in Eq. (1) contains the 3 rigid body motions in two dimensions (two translations and one rotation) that become insignificant in the limit of a very large number of edges. For the top panel in Fig. 2, and ignoring R for the moment, $F-3 = 2\cdot 2-1-3 = 0$ while for the second and third panels $F-3 = 2\cdot 5-8-3 = -1$ and $F-3 = 2\cdot 29-64-3 = -9$ respectively, where the negative numbers signify that not all the bonds are independent in these rigid diagrams, and we have removed the 3 macroscopic floppy modes on the left hand side of the count. Therefore there is a single redundant edge in the second panel and 9 redundant edges in the third panel of Fig. 2 (one for each of the eight replications of the second panel, plus one new one). By removing edges randomly from the third panel (i.e. bond diluting), first the redundancy is reduced and eventually there is no rigid path between the two solid vertices, and rigidity is lost. Note that the Maxwell count for Fig. 1 gives $F-3 = 2\cdot 8-13-3 = 0$ (again removing the three floppy modes), but as there is one floppy mode associated with the solid vertices, there is also one redundant edge associated with the heavier solid edges in the left side of the diagram.

The number of edges E or more precisely bonds $N_n^b(p)$, and the number of vertices V or sites $N_n^s(p)$ in the n^{th} generation becomes, for the undiluted case ($p=1$) shown in Fig. 2 is

$$\begin{aligned} E &= N_n^b(1) = 8^n, \\ V &= N_n^s(1) = (3.8^n+11)/7. \end{aligned} \tag{2}$$

An important quantity is the mean coordination defined by $\langle r\rangle = 2E/V$ which tends to an asymptotic value $\langle r\rangle = 14/3 = 4.667$ for the undiluted lattice. It is important that this quantity be above 4, which is the mean field value of the mean coordination needed for rigidity in two dimensions [9, 11]. The number of redundant edges is $(8^n-1)/7$ so that the fraction of redundant edges for large n approaches $1/7 = 14.3\%$ in the undiluted lattice. For the triangular lattice, this fraction is even higher at $1/3 = 33.3\%$.

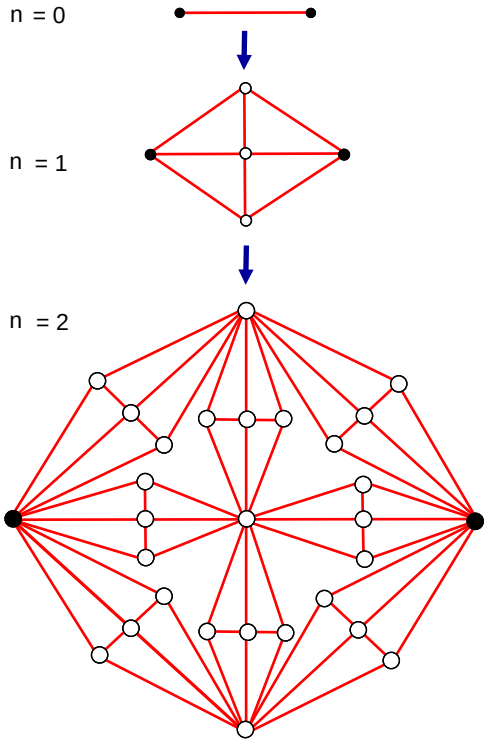

Figure 2. (Color online) Each generation of the undiluted hierarchical lattice is labeled by an index n and $n = 0, 1, 2$ shown here starting at the top.

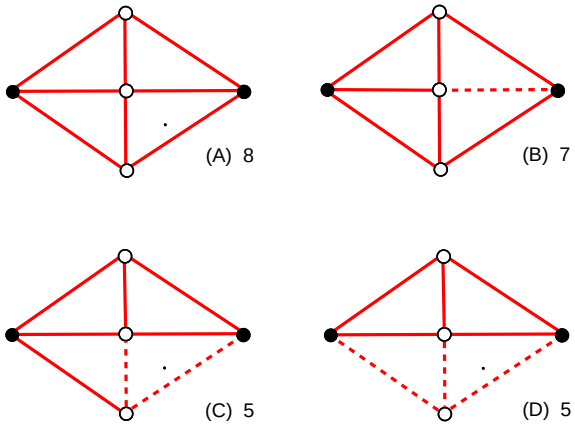

Figure 3. (Color online) Showing the four distinct types of graphs that lead to a rigid connection between the two solid circles. Edges present are shown as solid lines and missing edges as dashed lines. Here (A) has all eight bonds present and is rigid with one redundant edge and has probability p^8, (B) has any single edge missing and has probability $8p^7(1-p)$, (C) has any pair of edges missing from the three lower (shown) or the three upper ones and has probability $6p^6(1-p)^2$ and (D) has a triple of edges missing either from the lower or upper part of the graph and has probability $2p^5(1-p)^3$. The number of edges in the rigid cluster is indicated by the number under each graph. All other graphs (not shown) do not rigidly connect the two solid circles.

For the diluted case an bond is present with probability p (concentration) and absent with probability $1-p$, so the probability of the two solid dots being rigidly connected in the second panel of Fig. 2 using the weights from Fig. 3 is $p' = p^8 + 8p^7(1-p) + 6p^6(1-p)^2 + 2p^5(1-p)^3 = 2p^5 + 2p^7 - 3p^8$. This leads to the relationship between the probabilities p_{n+1}, p_n of rigidity percolating in successive generations $n+1, n$:

$$p_{n+1} = 2p_n^5 + 2p_n^7 - 3p_n^8 \tag{3}$$

(with $p_0 = p$). The fixed points p^* satisfying $p_{n+1} = p_n = p^*$ are the trivial stable fixed points at $p^* = 0$ and $p^* = 1$ and the non-trivial unstable fixed point $p^* = 0.9446 = p_c$. Close to this latter fixed point, Eq. (3) can be linearized by differentiating to give $(p_{n+1} - p_c) = \lambda_1(p_n - p_c)$ where $\lambda_1 = 10p_c^4 + 14p_c^6 - 24p_c^7 = 1.802$.

Using the cluster probabilities and also the number of bonds in each rigid spanning cluster from Fig. 3, we find from the mean number of bonds that the probability $P_{n+1}(p)$ of a bond belong to the percolating rigid cluster is given by the recurrence relation

$$P_{n+1}(p) = \frac{1}{4}[5p_n^4 + 13p_n^6 - 14p_n^7]P_n(p) \tag{4}$$

(with $P_0(p) = p$). Near the unstable fixed point, $P_{n+1}(p) = \lambda_2 P_n(p)$ where $\lambda_2 = \frac{1}{4}[5p_c^4 + 13p_c^6 - 14p_c^7] = 0.9554$, showing that the probability of an bond being in the percolating cluster renormalizes to zero at the phase transition as expected for a second order phase transition. From Fig. 4 we can see how the singular behavior at the phase transition develops as n increases: $n = 12$ appears very close to giving the full singularity. Near $p = 1$, $P(p) = p[1 - 69(1-p)^2/4 + ...]$ where the first term is just the probability that an bond is present and the second term that at least 2 bonds must be removed to produce an bond that is present but not part of the rigid backbone, as indicated for example in panel (C) of Fig. 3.

Using the eigenvalues λ_1, λ_2, and $\lambda_3 (= 8)$ of the linearized scaling relationships for p, $P(p)$, and N_n^b respectively we obtain exponents ν, β, and *fractal* dimensionality d, from $\lambda_1 = b^{1/\nu}$, $\lambda_2 = b^{-\beta/\nu}$, and $\lambda_3 = b^d$, where b is the dilatation (length scaling) factor between successive generations of the hierarchical lattice. However, as is typical for such lattices, b is ambiguous [16]; so we quote only the values of exponents independent of b. These are $\beta = 0.0775$, which describes how the order parameter $P(p)$ goes to zero at the critical point, and the product $d\nu = 3.533$, which plays a role in hyperscaling which does apply here.

The question of hyperscaling involves the critical exponent α that describes the fluctuations associated with the *specific heat* in the system near the phase transition. The exponent α is most easily calculated by differentiating the free energy twice with respect to the the bond concentration p, and hyperscaling also relates (when it applies) to the free energy. But the question arises as to what is an appropriate free energy as rigidity percolation is not a system described by a Hamiltonian. There is strong evidence, outlined below, that the number of floppy modes given in Eq. (1) serves as the appropriate free energy for it. It can be shown that the second derivative with respect to p is positive definite. For connectivity percolation, the free energy can be found as the $s \to 1$ limit of the s-state Potts model [18] and in that case is equivalent to an appropriate version of Eq. (1) in which redundancy refers to loops or multiple pathways between two vertices and the factor 2 is omitted. In this case a single floppy mode is associated with an isolated cluster, so the free energy is just the total number of isolated clusters and of course is an extensive quantity. Finally for connectivity and rigidity, these forms of F have been used as a free energy for percolation from a busbar onto a Cayley tree network [2].

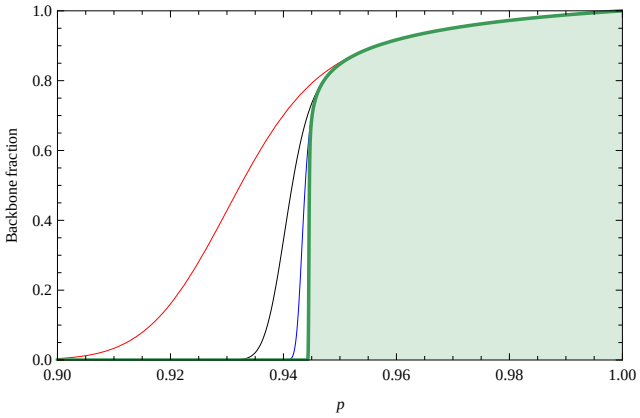

Figure 4. (Color online) Showing the probability $P(p)$ that a bond that is present is also part of the rigid backbone as a function of the probability p that an bond is present. The four curves shown are for the result of iterating Eq. (4) out to $n = 4, 6, 8, 12$ terms respectively as the curve steepens. The result for $n = 12$ is shown by the heavier line, and shows convergence on the scale of this plot to the singularity at $p_c = 0.9446$.

Rather than calculate the number F of floppy modes directly, it is easier to calculate R in Eq. (1) and hence determine F. If the number of redundant bonds at generation n is $R_{n+1}(p)$, then

$$R_{n+1}(p) = 8R_n(p) + p_n^8 \tag{5}$$

(with $R_0(p) = 0$). The factor 8 in Eq. (5) comes from the eight fold replication of any redundant bond from the previous generation (e.g. going from $n = 1$ to $n = 2$ in Fig. 2). The factor p_n^8 comes from additional redundancy if all 8 pieces of the graph are rigid (but not necessarily redundant). Eq. (5) together with (2) provides an iterative equation for the free energy $F_n(p)$ resulting from (1)

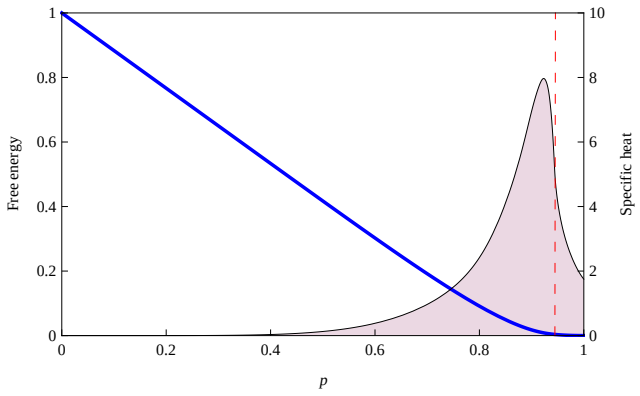

Figure 5. (Color online) Showing the number of floppy modes $f(p)$ (heavy line and left scale) and also its second derivative with respect to p (thin line and right scale) which is the *specific heat*. The vertical dashed line marks the location of $p_c = 0.9446$ which is *not* at peak of the *specific heat*.

$$F_n(p) = 2N_n^s(p) - N_n^b(p) + R_n(p). \qquad (6)$$

From the eigenvalue $\lambda_F(= 8)$ for the linearized scaling of $F_n(p)$ at p_c and large n, we find that the exponent α is negative (signifying a cusp) and also establish that the hyperscaling relationship to $d\nu$ is satisfied: $2 - \alpha = d\nu = \ln \lambda_F / \ln \lambda_1 = 3.533$.

It is convenient to define the number of floppy modes per degree of freedom as $f_n(p) = F_n(p)/[2N_n^s(p)]$ so that $0 < f < 1$. Here $f(p)$ is the thermodynamic limit as $n \to 1$ of $f_n(p)$. In Fig. 5, we show both f and its second derivative with respect to p. Solving Eqs. (6) at the critical point gives $f(p_c) = 1 - 7p_c/6 + p_c^8/6 = 0.00361$ and at small p, we have $f(p) = 1 - 7p/6 + 7p^8/48 + ...$ where the term in p^8 is the leading correction due to redundancy or the onset of dependent constraints.

In the above treatment of the rigidity percolation problem it has only been necessary to consider averages, of such things as numbers of stress-carrying bonds, redundant ones, floppy modes, etc., governed by additive composition rules. Such additivity is absent for processes such as percolation conductivity [7] (or elasticity), where probability distributions have to be rescaled.

For the additive variables of rigidity percolation, probability distributions could have been found simply (from algebraic recurrence relations for their Laplace transforms). These can provide further useful information e.g. for distinguishing the situations with/without central limit simplicity away from/near the transition.

To summarize, we have shown for the first time how renormalization group procedures can be used to describe second order phase transitions involving rigidity percolation when rigidity percolates on the Berker lattice considered here.

Outstanding questions include more rigorous approaches to establish that the number of floppy modes F is the appropriate free energy for this problem. In addition much insight would be gained by widening the scope of the lattices covered.

In that connection it should be mentioned that the Berker lattice discussed here is the simplest member of several families for which analytic results have been derived (all showing continuous transitions) which space precludes presenting here. Work continues towards finding such lattices with a first order rigidity transition, and possibly a parameter to tune the transition through a tricritical point.

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