

## On the approximation in the smoothed finite element method (SFEM)

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### Abstract

This letter aims at resolving the issues raised in the recent short communication [1] and answered by [2] by proposing a systematic approximation scheme based on non-mapped shape functions, which both allows to fully exploit the unique advantages of the smoothed finite element method (SFEM) [3, 4, 5, 6, 7, 8, 9] *and* resolve the existence, linearity and positivity deficiencies pointed out in [1].

We show that Wachspress interpolants [10] computed in the physical coordinate system are very well suited to the SFEM, especially when elements are heavily distorted (obtuse interior angles). The proposed approximation leads to results which are almost identical to those of the SFEM initially proposed in [3].

These results that the proposed approximation scheme forms a strong and rigorous basis for construction of smoothed finite element methods.

**keywords:** Smoothed Finite Element Method, boundary integration, Wachspress Interpolants, strain smoothing, rational basis finite elements, SFEM, isoparametric

## 1 INTRODUCTION

The smoothed finite element method (SFEM) was first proposed in [3]. This new numerical method, based on gradient (strain) smoothing, is rooted in meshfree stabilized conforming nodal integration [11] and was shown to provide a suite of finite elements with a range of interesting properties. Those properties depend on the number of smoothing cells employed within each finite element (see [12] for a review of recent developments and properties) and include:

- improved dual accuracy and superconvergence;
- relative insensitivity to volumetric locking;
- relative insensitivity to mesh distortion;
- softer than the FEM.

A rigorous theoretical framework was provided in [5, 8] and the method was extended to plates [9], shells [6] and coupled with the extended finite element method [12].

Table 1: Shape function value at different sites within an element (Figure (1))

Site	Node 1	Node 2	Node 3	Node 4	Description
1	1.0	0.0	0.0	0.0	Field node
2	0.0	1.0	0.0	0.0	Field node
3	0.0	0.0	1.0	0.0	Field node
4	0.0	0.0	0.0	1.0	Field node
5	0.5	0.5	0.0	0.0	Side midpoint
6	0.0	0.5	0.5	0.0	Side midpoint
7	0.0	0.0	0.5	0.5	Side midpoint
8	0.5	0.0	0.0	0.5	Side midpoint
9	0.25	0.25	0.25	0.25	Intersection of two bimedians

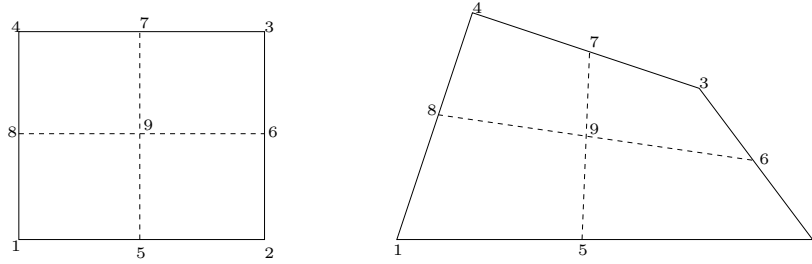


Figure 1: A four-node element divided into four smoothing cells.

The essential feature of the SFEM is that no isoparametric mapping is required, which implies that the approximation can be defined in the physical space directly, thereby providing freedom in the selection of the element geometry.

In the initial paper [3] (Eq. (22) and reproduced here for simplicity Equation (1)), non-mapped Lagrange shape functions are proposed as a possibility to calculate the shape functions at an arbitrary point within a smoothed finite element. It is then stated in the same paper (p863 last paragraph) that “unless state otherwise, we still use the averaged shape functions [3] for convenience.” These shape functions are recalled in Table 1 and Figure (1), for ease of reading.

$$N_e(\mathbf{x}_e) = \begin{bmatrix} 1 & x_e & y_e & x_e y_e \end{bmatrix} \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}^{-1} \quad (1)$$

In fact, in our work on the SFEM [4, 6, 7, 8, 9], and, to our knowledge, in all other work published to date [3, 4, 5, 6], these “averaged shape functions” have been used, with good results.

Yet, [1] provides a critique of the SFEM stating that the approximation provided by Equation (1) are inadequate because:

- they do not always exist (as described in the 1975 book [10]);
- they may not be positive everywhere in the element;
- they may not be linear everywhere in the element.

Because of this [1] disqualifies the current version of the SFEM and discredits the existing results of [3, 4, 5, 6, 7, 8, 9], despite the fact (also noted in [2]) that those non-mapped Lagrange shape functions of Equation (1) were in general not used in the aforementioned papers.

In this contribution, we show that it is possible to resolve the three issues mentioned by [1] about the Lagrange non-mapped shape functions while retaining the advantageous features of the smoothed finite element method, in particular its ability to deal with extremely distorted meshes.

## 2 WACHSPRESS INTERPOLANTS

Wachspress [10] presented a rigorous formulation for generating shape functions on arbitrary polygons, based on projective geometry. The Wachspress shape functions are unconventional compared to the polynomials used in the finite element literature, as they are in general rational functions, i.e, the ratio of two polynomials [13]. These shape functions have the following essential features of interpolants (for arbitrary  $n$ -sided polygons):

- The  $N_i$  satisfy the partition of unity and Kronecker Delta properties;
- Shape function,  $N_i$  is linear on sides adjacent to node  $i$ .
- The  $N_i$  are linear complete.

These properties make Wachspress interpolants effective to build approximations on arbitrary  $n$ -gons, quadrilaterals in particular. Consider the quadrilateral (Q4), on which Wachspress interpolants will be formulated, shown in Figure (2). Let  $l_1, l_2, l_3$  and  $l_4$  be the equations of the lines corresponding to each of the four sides of the quadrilateral, (1 – 2, 2 – 3, 3 – 4 & 4 – 1) respectively, and written in parametric form as:

$$l_1(x, y) = a_1x + b_1y + c_1 = 0 \tag{2a}$$

$$l_2(x, y) = a_2x + b_2y + c_2 = 0 \tag{2b}$$

$$l_3(x, y) = a_3x + b_3y + c_3 = 0 \tag{2c}$$

$$l_4(x, y) = a_4x + b_4y + c_4 = 0 \tag{2d}$$

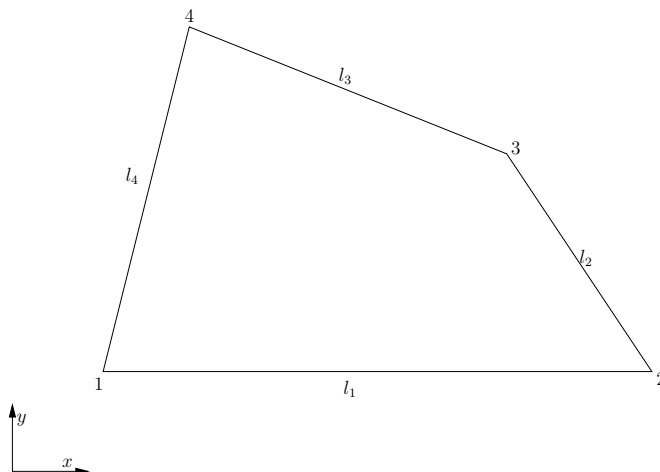


Figure 2: A sample quadrilateral.

where  $a_i, b_i$  and  $c_i$  for  $i = 1, 2, 3, 4$  are real constants. The wedge functions corresponding to each node,  $w_i$ , are defined so that they vary linearly along the edges adjacent to each node and vanish at the remaining nodes as [10]:

$$w_1(x, y) = \kappa_1 l_2(x, y) l_3(x, y) \quad (3a)$$

$$w_2(x, y) = \kappa_2 l_3(x, y) l_4(x, y) \quad (3b)$$

$$w_3(x, y) = \kappa_3 l_4(x, y) l_1(x, y) \quad (3c)$$

$$w_4(x, y) = \kappa_4 l_1(x, y) l_2(x, y) \quad (3d)$$

where the  $\kappa_i$  are constants. In order that the Wachspress interpolants  $N_i$ , satisfy the partition of unity requirement, it is defined as:

$$N_i(x, y) = \frac{w_i}{\sum w_i}(x, y) \quad (4)$$

**Remark 1** For a 4-sided polygon, the Wachspress rational basis interpolants degenerate to regular polynomials, i.e.  $\sum w_i(x, y)$  is a constant.

To illustrate the Wachspress interpolants for arbitrary quadrilaterals and to answer Zhang *et al.*, [1]'s queries on

- the existence of shape functions for arbitrary quadrilaterals;
- positivity of the shape functions;
- linearity of the shape functions,

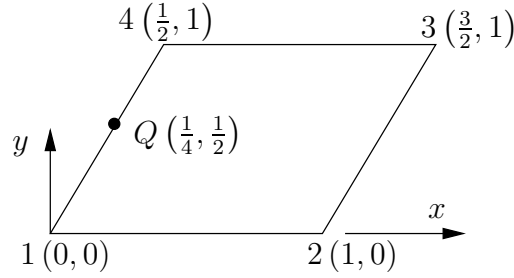


Figure 3: A parallelogram shaped element

we choose the same parallelogram element as used in [1] (cf. page 1292, Figure 2). This element is shown in Figure (3) for ease of reading.

The shape functions for the parallelogram shaped element (see Figure (3)) based on Wachspress interpolation write:

$$N_1(x, y) = (y - 1) \left( x - \frac{1}{2}y - 1 \right) \quad (5a)$$

$$N_2(x, y) = (1 - y) \left( x - \frac{1}{2}y \right) \quad (5b)$$

$$N_3(x, y) = y \left( x - \frac{1}{2}y \right) \quad (5c)$$

$$N_4(x, y) = -y \left( x - \frac{1}{2}y - 1 \right) \quad (5d)$$

From the above equations, it can be easily verified that the shape functions derived using the Wachspress approach satisfy all the required properties (positivity, Kronecker delta property, linear completeness) as shown in the book [10]. What is more, the quality of these functions is independent of the shape of the element, which make them ideal candidates for use in the context of the SFEM where highly distorted meshes are of interest given the absence of isoparametric mapping.

For the mid-point of side 1-4, i.e., the point  $Q$  with coordinates  $(\frac{1}{4}, \frac{1}{2})$  (see Figure (3)), the shape function values are:

$$\mathbf{N}(Q) = \begin{Bmatrix} N_1(Q) \\ N_2(Q) \\ N_3(Q) \\ N_4(Q) \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{Bmatrix} \quad (6)$$

as opposed to

$$\mathbf{N}(Q) = \begin{pmatrix} N_1(Q) \\ N_2(Q) \\ N_3(Q) \\ N_4(Q) \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{-1}{8} \\ \frac{5}{8} \end{pmatrix} \quad (7)$$

as described in [1] (cf. page. 1293).

**Remark 2** *By a systematic approach using rational basis functions, the shape functions on arbitrary  $n$ -gons, in particular  $Q_4$ 's can be derived in the physical coordinate system, without the need for isoparametric mapping, and without the negative side effects of non-mapped Lagrange shape functions described in [1].*

**Remark 3** *The form of the Wachspress shape functions depend on the element geometry and hence have to be recomputed for each element in the mesh, which is computationally expensive. However, an important advantage is that their quality does not deteriorate for highly distorted elements, including concave domains. Interested readers are referred to [10, 13, 14] for advances in this direction.*

## 3 NUMERICAL RESULTS

### 3.1 Patch Test

It is straightforward to check that the linear patch test is satisfied down to machine precision. The interested reader is referred to the corresponding author to obtain a MATLAB code including this and other test cases.

### 3.2 Bending of a cantilever beam

As a second example, bending of a thick cantilever subjected to a parabolic load at the free end is examined as shown in Figure (4). The geometry is: length  $L=8$ , height  $D=4$  and thickness  $t=1$ . The material properties are: Young's modulus  $E=3 \times 10^7$ , and the parabolic shear force  $P=250$ . The exact solution of this problem is available in [15].

In this problem, two types of mesh are considered: one is uniform and regular and the other is an irregular mesh, with the coordinates of interior nodes given by:

$$x' = x + (2r_c - 1) \alpha_{ir} \Delta x \quad (8a)$$

$$y' = y + (2r_c - 1) \alpha_{ir} \Delta y \quad (8b)$$

$$(8c)$$

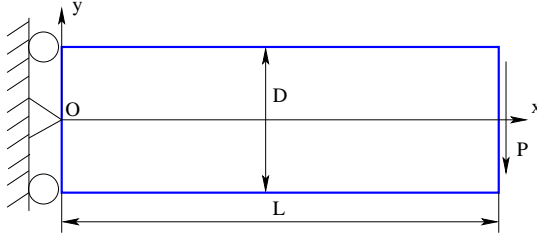


Figure 4: Geometry and loads for a cantilever beam.

where  $r_c$  is a random number between 0 and 1.0,  $\alpha_{ir} \in [0, 0.5]$  is an irregularity factor that controls the shapes of the elements and  $\Delta x$  and  $\Delta y$  are the initial regular element sizes in the  $x$ - and  $y$ - directions respectively.

The domain is discretized with quadrilateral elements and two quadrilateral subcells for each element are used for the current study, denoted by  $SCkQ4$ <sup>1</sup>. Figure (6) shows a discretization of the cantilever beam with quadrilateral elements using an irregular mesh. A value of  $\alpha_{ir} = 0.5$  is used to generate the irregular mesh. Under plane stress conditions and for a Poisson ratio  $\nu=0.3$ , the exact strain energy is 0.0398333. Figure (5) illustrates the convergence of the strain energy and the rate of convergence in the energy norm of the elements built using the Wachspress interpolants compared with those of the SFEM results ([8]) for regular quadrilateral meshes. The total strain energy and the rate of convergence for different irregular meshes ( $\alpha_{ir} = 0.2$  and  $\alpha_{ir} = 0.5$ ) are shown in Figures (7), (8) and (9).

It is seen that for regular meshes, the results are identical and for irregular meshes, there is a small difference, but the rate of convergence is not affected significantly. It is seen that with increase in mesh index<sup>2</sup>, both methods converge to the exact solution. With the use of Wachspress interpolants, the question of positivity of the shape functions and existence of the shape functions for arbitrary quadrilaterals is resolved yet preserving the true essence of the SFEM.

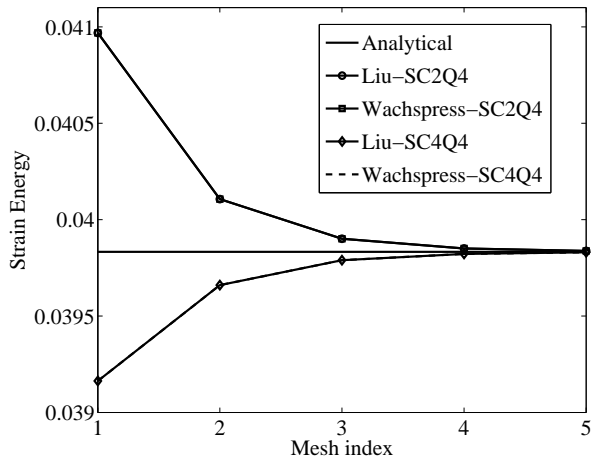
## 4 CONCLUSIONS

This letter showed that it is possible to both retain the highly desirable features of the smoothed FEM (SFEM), and its true essence, i.e. strain smoothing and boundary integration, without sacrificing a rigorous approximation where the shape functions are known explicitly at any point of the smoothed finite element.

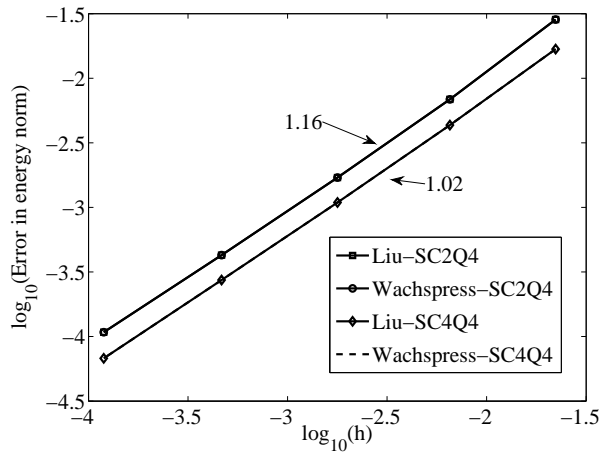
The proposed interpolation scheme, known as Wachspress interpolation, provides a suitable means to suppress the problems of definition, positivity and linearity associated with non-mapped Lagrange shape functions. It also provides a general framework to define approximation over arbitrary (possibly non-convex)

<sup>1</sup>SCkQ4 implies k quadrilateral subcells for 4 noded quadrilateral elements

<sup>2</sup>Mesh index =  $\frac{\text{number of elements in the } x\text{-direction}}{\text{length of the cantilever beam}}$



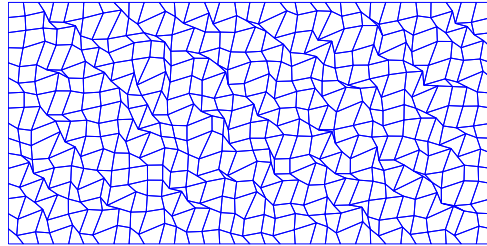
(a)



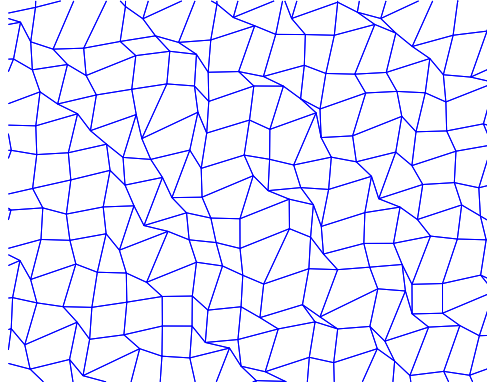
(b)

Figure 5: The convergence of the numerical energy to the exact energy and convergence rate in the energy norm with regular meshes for the cantilever beam problem: (a) strain energy and (b) the convergence rate.





(a)



(b)

Figure 6: Cantilever beam:(a) irregular mesh with extremely distorted elements and (b) zoomed view of the mesh.

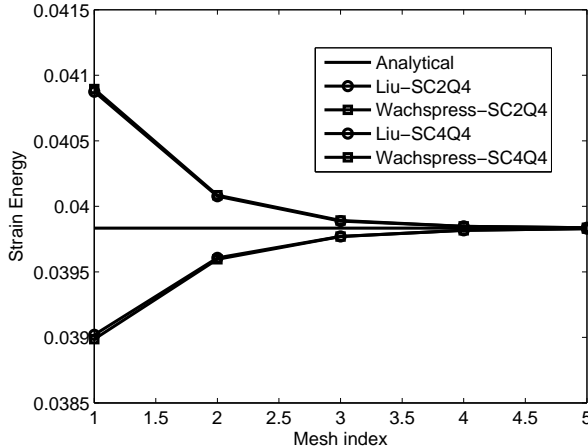


Figure 7: The convergence of the numerical energy to the exact energy with irregular meshes ( $\alpha_{ir} = 0.2$ ) for the cantilever beam problem.

polytopes including curved edges or surfaces.

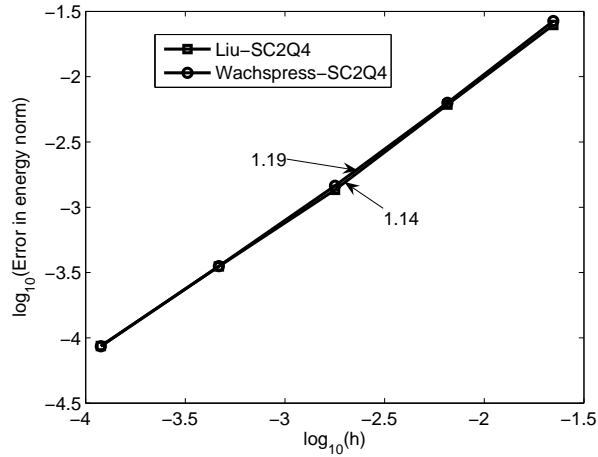
The Wachspress approximation was conclusively tested within the context of the smoothed FEM, showing that it yields a method which passes the patch test and provides, even for very high element distortion, accurate and optimally convergent results in linear elasticity.

The results also showed that utilizing the proposed Wachspress shape functions to build the approximation in the smoothed FEM leads to no significant difference in results compared to the widely used technique of “averaged shape functions” preconized in the original paper [3] and used in all the smoothed FEM work published to date [4, 6, 7, 8, 9, 5, 6].

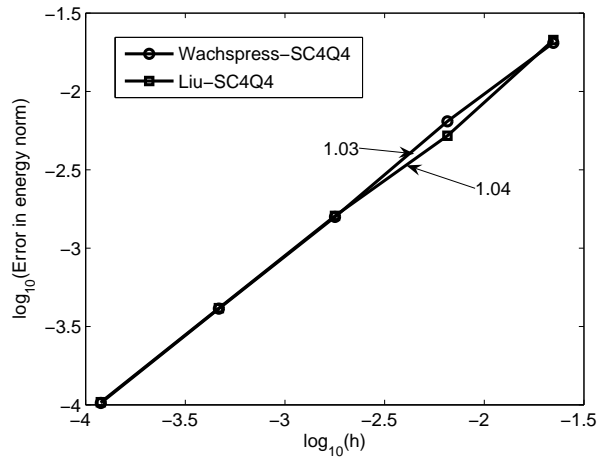
This finding reinforces the claim made in [2] that the approximation need not be known explicitly to build an optimal SFEM and dispels, also, the misconception of [1], whose detraction of the smoothed FEM rests on the misunderstanding that this method requires the use of non-mapped Lagrange shape functions.

It will be most interesting to investigate the behavior of the smoothed FEM, especially the choice of approximation, in the context of higher-order methods as well as non-polynomial enrichments [12].

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(a)



(b)

Figure 8: The convergence rate in the energy norm with irregular meshes ( $\alpha_{ir} = 0.2$ ) for the cantilever beam problem: (a) SC2Q4 and (b) SC4Q4.

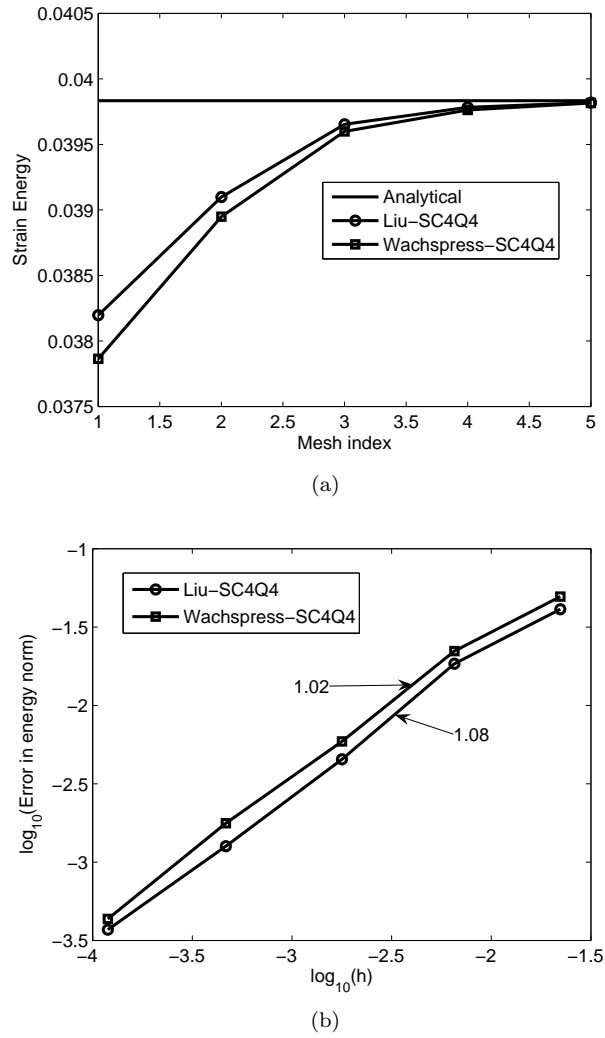


Figure 9: The convergence of the numerical energy to the exact energy and convergence rate in the energy norm with irregular meshes ( $\alpha_{ir} = 0.5$ ) for the cantilever beam problem: (a) strain energy and (b) the convergence rate.

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