Equivalent Bilevel Programming Form for the Generalized Nash Equilibrium Problem

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Abstract: Generalized Nash Equilibrium problem is widely applied but hard to solve. In this paper, we transform the generalized Nash game into a special bilevel programming with one leader and multi-followers by supposing a suppositional leader, that is an upper decision maker. The relations between their solutions are discussed. We also discuss the further simplification of the bilevel programming. Many conclusions and the further research are drawn at last..

Key words: Generalized Nash equilibrium point, Bilevel Programming, efficient solution, optimal solution.

1 Introduction

Game theory is the study of problem of conflict and cooperation among independent decision-makers. And it is a mathematical framework that describes interactions between multi-agents and allows for their incomes(Newmann and Morgenstern, 1944; Osborne and Rubinstein, 1994; Samuelson, 1997). A game defines an interaction between some agents. Each agent has a series of available strategies, where a strategy determines an action of the agent in the game. Game theory has played a substantial role in economics and has been applied to many application areas such as biology, transportation(Sun and Gao, 2006), sociology, political sciences(Schelling, 1960), psychology(Scharlemann et al., 2001), management science(Patriksson and Rockafellar, 2002), warfare and so on.

Games appear in normal form(strategic form), extensive form and coalitional form. The first two are close relatives, they constitute the basic paradigm of non-cooperative game theory. The coalitional form is the basic paradigm of cooperative game theory(Nash, 1951). Most of the game researchers pay their attentions to non-cooperative finite game with perfect information, i.e., each player in the game enjoys complete information and he/she independently selects a strategy and receives a corresponding payoff value that depends on the strategies selected by all players. Players choose their best strategies to maximize their payoffs respectively.

In this paper, we will consider the static generalized Nash equilibrium game, a kind of non-cooperative finite game with perfect information. It is also called social equilibrium game or pseudo-Nash equilibrium game(Ichiishi, 1983). In this kind of game, players affect each other when they make decisions not only on their utility functions but also on their feasible strategy sets. And it is a basic assumption that any player, when taking his decision, either does so simultaneously or without knowing the choice of the other players.

Researchers of game theory are generally aware that solving Nash equilibrium problem can be a tedious, error-prone affair, even when the game is very simple, and they also know that the need to solve a game arises with fair frequency. It is by now a well-known fact that the Nash equilibrium problem where each player solves a convex parameter program can be formulated and solved as a finite-dimensional variational inequality(Facchinei and Pang, 2003; Harker and Pang, 1990). The generalized Nash game is a Nash game in which each player's strategy set depends on the other players' strategies. The connection between the generalized Nash games and quasi-variational inequalities(QVIs) was recognized by Bensoussan (Bensoussan, 1974) as early as 1974 who studies these problems with quadratic functions in Hilbert space.

As for the generalized Nash equilibrium model, Ichiishi (Ichiishi, 1983) proved the existence of equilibrium point under the conditions that utility function u^i is quasi-concave and the mapping K^i is continuous for all $i \in \boldsymbol{I}$. A general assumption is that the utility function is concave and even is quasi-concave in most of the study. Similar to normal Nash equilibrium problem, the generalized Nash equilibrium problem can be transformed into a quasi-variational inequality problem. But the calculation of quasi-variational inequality problem is intractable. Under some given conditions, Ichiishi (Ichiishi, 1983) calculated a quasi-variational inequality problem in virtue of a general variational inequality problem equivalently. Many effective algorithms(Pang and Yang, 1998; Daniele and Maugeri, 2002; Pang, 2002)have been established to solve the general variational inequality problem such as Newton method and diagonalization method and so on.

Bi-level programming has been the focus of hierarchical system for many years and many excellent results have been founded (Vicente and Calamai, 1994; Zhu et al, 2004; Dussault et al, 2006; Shi et al, 2006).In this paper, we transform the generalized Nash equilibrium problem into a special bi-level programming with multi-follower.

The remainder of the paper is organized as follows. Section 2 introduces the basic concepts and notations, and the generalized Nash equilibrium problem is described. Section 3 deals with the transformation between the generalized Nash equilibrium game and Bi-level programming. Section 4 presents another equivalent form for GNEP, and some conclusions are drawn in the last section.

2 Concepts and properties

To describe a generalized Nash equilibrium game, we need to specify three factors, that is the number of players, the set of strategies available to each player and their payoff functions which determine each player's payoff as a function of the strategies choosed by all players(Stinchcombe, 2005).

Consider a finite n-person generalized Nash game in normal form.

Let $\boldsymbol{I}$ denote the finite set of players, $\boldsymbol{I} = \{1, 2, \cdots, n\}$.

For any player $i \in \boldsymbol{I}$, its *strategy set* $S^{\mathbf{i}} = \{s_1^i, s_2^i, \cdots, s_{n_i}^i\}$, consisting of n_i possible actions called *pure strategies*. A player's *mixed strategy* is a probability distribution over his space of pure strategies. In other words, a mixed strategy consists of a random draw of a pure strategy. It can be represented by a nonnegative vector $\boldsymbol{x}^i = (x_1^i, x_2^i, \cdots, x_{n_i}^i)$, Where $\sum_{k=1}^{n_i} x_k^i = 1$. Then the mixed strategy set of player i is $\boldsymbol{X}^i = \{\boldsymbol{x}^i = (x_1^i, x_2^i, \cdots, x_{n_i}^i) | \sum_{k=1}^{n_i} x_k^i = 1,\ x_k^i \geq 0,\ k = 1, 2, \cdots, n_i\}$. In particular, for some mixed strategy $\boldsymbol{S}^i = \{s_1^i, s_2^i, \cdots, s_{n_i}^i\}$, if there exists some component $x_k^i = 1$, this strategy is just the k th pure strategy. So we can see the pure strategy is a special case of the mixed strategy and then we denote the strategy set of player i as $\boldsymbol{X}^i \subseteq \boldsymbol{R}^{n_i}$ and the feasible strategy space of the game as $\boldsymbol{X} = \prod_{i \in \boldsymbol{I}} \boldsymbol{X}^i \subseteq \boldsymbol{R}^m$, where $m = \sum_{i \in \boldsymbol{I}} n_i$. And then we indicate $\boldsymbol{X}^{-i} = \prod_{i \in \boldsymbol{I} \setminus \{i\}} \boldsymbol{X}^i \subseteq \boldsymbol{R}^{m-n_i}$ as the Descartes product of all players' strategy sets except for the strategy set of player i.

Let $K^i : \boldsymbol{X}^{-i} \to \boldsymbol{X}^i$ be a point-to-set mapping, that is, $\forall \boldsymbol{x}^{-i} \in \boldsymbol{X}^{-i}$, $K^i(\boldsymbol{x}^{-i}) \subseteq \boldsymbol{X}^i$, where $\boldsymbol{x}^{-i} = (\boldsymbol{x}^1, \cdots, \boldsymbol{x}^{i-1}, \boldsymbol{x}^{i+1}, \cdots, \boldsymbol{x}^n)$. Here the mapping may portray the influence ability of the other $n-1$ players to player i . Let $u^i : grK^i \times \boldsymbol{X}^{-i} \to \mathbf{R}$ be the *utility (or payoff) function* for player i , where grK^i is the value region of mapping K^i .

Given the above factors, we can portray the generalized Nash equilibrium game as the ternary group $\{\boldsymbol{X}^i,\ K^i,\ u^i\}_{i \in \boldsymbol{I}}$.

In game theory, what is emphasized is individual rationality (Cruz and Simaan, 2000). Every player will choose the strategy which optimize his/her utility function under the condition of other players fixed their strategies. That is, $\forall i \in \boldsymbol{I}$, if other $n-1$ players chosen their optimal strategies as $\boldsymbol{x}^{-i*} = (\boldsymbol{x}^{1*}, \cdots, \boldsymbol{x}^{(i-1)*}, \boldsymbol{x}^{(i+1)*}, \cdots, \boldsymbol{x}^{n*})$, then player i should optimize his utility function $u^i(\boldsymbol{x}^i, \boldsymbol{x}^{-i*})$ on the feasible strategy set $K^i(\boldsymbol{x}^{-i*})$. This course can be described as the following parameter programming denoted as $(EP_{(\boldsymbol{x}^{-i*})})$:

$$\begin{aligned} &\max_{\boldsymbol{x}^i} u^i(\boldsymbol{x}^i, \boldsymbol{x}^{-i*}) \\ &s.t.\ \ \boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i*}) \end{aligned} \tag{2.1}$$

Then the generalized Nash equilibrium problem (GNEP) can be presented by the following series of parameter programming, that is,

For $i \in I$, player i solves the parameter programming (2.2)

$$\max_{\boldsymbol{x}^i} u^i(\boldsymbol{x}^i, \boldsymbol{x}^{-i*}) \qquad (2.2)$$
$$s.t. \quad \boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i*})$$

where$\boldsymbol{x}^{-i*}$ is the optimal decisions of the players except for player i.

A Nash equilibrium point is a strategy profile such that there is no agent's interest to deviate unilaterally. So is the generalized Nash equilibrium point. We may portray it with the mathematical formulation.

Definition 2.1 *A* ***Generalized Nash Equilibrium Point*** *is defined as a point* $\boldsymbol{x}^* = (\boldsymbol{x}^{1*}, \boldsymbol{x}^{2*}, \cdots, \boldsymbol{x}^{n*})$ *such that* $\forall i \in \boldsymbol{I}$ *, the following conditions hold,* $\boldsymbol{x}^{i*} \in K^i(\boldsymbol{x}^{-i*})$*and* $u^i(\boldsymbol{x}^*) \geq u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i*})$, $\forall \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i*})$.

The generalized Nash equilibrium conditions show that no player can increase his/her expected reward by unilaterally changing his/her strategy only. Here $\boldsymbol{x}^{i*} \in K^i(\boldsymbol{x}^{-i*})$ means that the optimal strategy of every player must be in the region decided by other players at the generalized Nash equilibrium point. This is just the difference between the generalized Nash equilibrium game and the normal form game.

From the above model and definition we can see the calculation of generalized Nash equilibrium point is hard enough. Because solving the parameter programming is not easy. An effective solution is transforming the GNEP and using the known algorithm. In the next section, we transform the generalized Nash equilibrium problem into a special bi-level programming problem and discuss the relations between their solutions.

3 Equivalent Bilevel Programming Form for GNEP

In the game portrayed above, all the players is evenness but they influence each other by their decision variables. So we must resolve all of the n parameter programmes at the same time in order to solve the generalized Nash equilibrium point. We know there is another effective model to describe the complex interactive influence among all the players, that is the multi-level programming. Bilevel programming is the simplest multi-level programming and it describes the delicate hierarchical relations between the upper leader and the lower follower.

Suppose there is an upper leader in the generalized Nash equilibrium problem, and it is endowed with the corresponding decision variable, constraints and objective function. We get a bilevel programming. So we may transform the transverse relations among all the decision-makers in the game into the lengthways relationship between the upper leader and the lower follower.

Denote the upper leader's decision variable as $\boldsymbol{x} = (\boldsymbol{x}^1, \boldsymbol{x}^2, \cdots, \boldsymbol{x}^n) \in \boldsymbol{R}^m$, where $\boldsymbol{x}^i \in \boldsymbol{R}^{n_i}$, $i \in \boldsymbol{I}$ and $\sum_{i\in \boldsymbol{I}} n_i = m$ such that $\forall i \in \boldsymbol{I}$ the condition $\boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i})$ holds. His objective is to maximize the function $-(\boldsymbol{y}-\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x})$ where $\boldsymbol{y} = (\boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^n) \in \boldsymbol{R}^m$, $\boldsymbol{y}^i \in \boldsymbol{R}^{n_i}$, $i \in \boldsymbol{I}$ and $\sum_{i\in \boldsymbol{I}} n_i = m$.

Then we get the following bilevel programming:

$$\left(BP_1\right) \qquad \begin{aligned} &\max_{(\boldsymbol{x},\boldsymbol{y})} \ f(\boldsymbol{x},\boldsymbol{y}) = -(\boldsymbol{y}-\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x}) \\ &s.t. \begin{cases} \boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i}) \quad i \in \boldsymbol{I} \\ \text{and } \boldsymbol{y} \text{ is an efficient solution of the programming } V(\boldsymbol{x})\text{:} \\ \mathbf{V}(\boldsymbol{x}) \qquad \max_{\boldsymbol{y}} \left(u^1(\boldsymbol{y}^1, \boldsymbol{x}^{-1}), \cdots, u^n(\boldsymbol{y}^n, \boldsymbol{x}^{-n})\right) \\ \qquad\qquad s.t. \ \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i}) \ \ \forall i \in \boldsymbol{I} \end{cases} \end{aligned}$$

The relations between the generalized Nash equilibrium point of GNEP and the optimal solutions of (BP_1) is satisfied. We have the following conclusions to describe the equivalence.

Lemma 3.1 *For the given* $\mathbf{x}^* \in \mathbf{R}^m$*, the multi-objective programming* $V(\mathbf{x}^*)$ *has efficient solution if and only if* $\forall i \in \mathbf{I}$, $EP_{(\mathbf{x}^{-i*})}$ *has optimal solution.*

Proof:

If for some given $\boldsymbol{x}^* \in \boldsymbol{R}^m$, the multi-objective programming $V(\boldsymbol{x}^*)$ has efficient solution, there exists some $j \in \boldsymbol{I}$, such that the correspond $EP_{(\boldsymbol{x}^{-j*})}$ has no optimal solution.

Because the programming $V(\boldsymbol{x}^*)$ has efficient solution, we can see the feasible region $EP_{(x^{-j*})}$, that is the set $K^j(\boldsymbol{x}^{-j*})$, is not empty. Since $EP_{(\boldsymbol{x}^{-j*})}$ has no optimal solution on $K^j(\boldsymbol{x}^{-j*})$, we can know that the function $u^j(y^j, x^{-j*})$ is not upper-bounded on the set $K^j(\boldsymbol{x}^{-j*})$.

Further, we suppose $\boldsymbol{y}^* = (\boldsymbol{y}^{1*}, \boldsymbol{y}^{2*}, \cdots, \boldsymbol{y}^{n*})$ is an efficient solution of $V(\boldsymbol{x}^*)$. It is right to know that $\boldsymbol{y}^{j*} \in K^j(\boldsymbol{x}^{-j*})$. Because the function u^j is unbounded on set $K^j(\boldsymbol{x}^{-j*})$, we know there must exist some $\bar{\boldsymbol{y}}^j \in K^j(\boldsymbol{x}^{-j*})$ such that $u^j(\bar{\boldsymbol{y}}^j, \boldsymbol{x}^{-j*}) > u^j(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*})$.

Let $\bar{\boldsymbol{y}} = (\boldsymbol{y}^{1*}, \cdots, \boldsymbol{y}^{(j-1)*}, \bar{\boldsymbol{y}}^j, \boldsymbol{y}^{(j+1)*}, \cdots, \boldsymbol{y}^{n*})$. It is easy to know that $\bar{\boldsymbol{y}}$ is a feasible solution to $V(\boldsymbol{x}^*)$ and we have the following inequalities:

$$u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}) = u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}),\ \forall i \in \boldsymbol{I},\ i \neq j,\ \ and\ \ u^j(\bar{\boldsymbol{y}}^j, \boldsymbol{x}^{-j*}) > u^j(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*})$$

This is a contradiction to the existence of the efficient solution to $V(\boldsymbol{x}^*)$. So $\forall i \in \boldsymbol{I}$, $EP_{(\boldsymbol{x}^{-i*})}$ has optimal solution.

Conversely, if for every $i \in \boldsymbol{I}$, the programming $EP_{(\boldsymbol{x}^{-i*})}$ has optimal solution, denoted as $\boldsymbol{y}^{i*}$. Let $\boldsymbol{y}^* = (\boldsymbol{y}^{1*}, \boldsymbol{y}^{2*}, \cdots, \boldsymbol{y}^{n*})$. Then we have, $\forall i \in \boldsymbol{I}$, $\boldsymbol{y}^{i*} \in K^i(\boldsymbol{x}^{-i*})$. So the vector $\boldsymbol{y}^*$ is a feasible solution to $V(\boldsymbol{x}^*)$.

Now we will prove it is an efficient solution to $V(\boldsymbol{x}^*)$.

In fact, for every feasible solution $\boldsymbol{y} = (\boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^n)$ to $V(\boldsymbol{x}^*)$, because $\boldsymbol{y}^{i*}$ is the optimal solution to $EP_{(\boldsymbol{x}^{-i*})}$, $i \in \boldsymbol{I}$. So ,it is obviously that $\forall i \in \boldsymbol{I}$, $\boldsymbol{y}^i$ is a feasible solution to $EP_{(\boldsymbol{x}^{-i*})}$ and the following inequalities hold:

$$u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}) \geq u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i*}),\ \forall i \in \boldsymbol{I}$$

So we can see $\boldsymbol{y}^*$ is not only an effective solution but also an optimal solution to $V(\boldsymbol{x}^*)$.

Theorem 3.2 $\mathbf{y}^* \in \mathbf{R}^m$ *is a generalized Nash equilibrium point to GNEP if and only if there exists an* $\mathbf{x}^* \in \mathbf{R}^m$ *such that* $(\mathbf{x}^*, \mathbf{y}^*)$ *solves* (BP_1) *and the optimal value is zero.*

Proof:

First, we must notice the fact that $\boldsymbol{y}^*$ is a generalized Nash equilibrium point to GNEP shows that: $\forall i \in \boldsymbol{I}$, $\boldsymbol{y}^{i*}$ is an optimal solution to $EP_{(\boldsymbol{y}^{-i*})}$.

Let $\boldsymbol{x}^* = \boldsymbol{y}^*$, then we have:

$$\boldsymbol{y}^{i*} \in K^i(\boldsymbol{x}^{-i*}), \quad \forall i \in \boldsymbol{I}$$

To prove the conclusion, we must prove $\boldsymbol{y}^*$is an effective solution to $V(\boldsymbol{x}^*)$ firstly.

If it is not right, then there exists some $\bar{\boldsymbol{y}} \in \boldsymbol{R}^m$ such that

$$\bar{\boldsymbol{y}}^i \in K^i(\boldsymbol{x}^{-i*})\ and\ u^i(\bar{\boldsymbol{y}}^i, \boldsymbol{x}^{-i*}) \geq u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}), \quad \forall i \in \boldsymbol{I}$$

and there at least one inequality is strictly, w.l.o.g., suppose the i_0's inequality is strictly.

It is easy to know that $\bar{\boldsymbol{y}}^{i_0}$ is a feasible solution to $EP_{(\boldsymbol{y}^{-i_0*})}$. Moreover, according the above supposition, we have $u^{i_0}(\bar{\boldsymbol{y}}^{i_0}, \boldsymbol{y}^{-{i_0}^*}) > u^{i_0}(\boldsymbol{y}^{{i_0}^*}, \boldsymbol{y}^{-{i_0}^*})$ It is a contradiction to $\boldsymbol{y}^{{i_0}^*}$ is the optimal solution to $EP_{(\boldsymbol{y}^{-{i_0}^*})}$.

So the above supposition is not right and we can say $\boldsymbol{y}^*$ is an efficient solution to $V(\boldsymbol{x}^*)$.

Because of $\boldsymbol{x}^* = \boldsymbol{y}^*$, the fact $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a feasible solution to (BP_1) is obvious.

So $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an optimal solution to (BP_1), and the optimal value is zero, because of $f(\boldsymbol{x}, \boldsymbol{y}) \leq 0$, $\forall (\boldsymbol{x}, \boldsymbol{y})$ and $f(\boldsymbol{x}^*, \boldsymbol{y}^*) = 0$.

Conversely, if $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an optimal solution to (BP_1), and the optimal value is zero, we can say $\boldsymbol{x}^* = \boldsymbol{y}^*$ right now because of the special upper objective function f of (BP_1). Also we have $\boldsymbol{x}^{i*} \in K^i(\boldsymbol{x}^{-i*})$, $\forall i \in \boldsymbol{I}$ and $\boldsymbol{y}^*$ is an efficient solution to $V(\boldsymbol{x}^*)$.

Now we will prove $\boldsymbol{y}^*$ is an generalized Nash equilibrium point to GNEP.

If it is not so, that means there exists an $i \in \boldsymbol{I}$, such that $\boldsymbol{y}^{i*}$ is not an optimal solution to $EP_{(\boldsymbol{y}^{-i*})}$. According to Lemma 3.1, we know $EP_{(\boldsymbol{y}^{-i*})}$ has optimal solution, we denote it as $\bar{\boldsymbol{y}}^i$. Then we get the following conclusions, that is:

$$\bar{\boldsymbol{y}}^i \in K^i(\boldsymbol{x}^{-i*})\ and\ u^i(\bar{\boldsymbol{y}}^i, \boldsymbol{x}^{-i*}) > u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*})$$

Let $\bar{\boldsymbol{y}} = (\boldsymbol{y}^{1*}, \cdots, \boldsymbol{y}^{(i-1)*}, \bar{\boldsymbol{y}}^i, \boldsymbol{y}^{(i+1)*}, \cdots, \boldsymbol{y}^{n*})$. It is easy to know that $\bar{\boldsymbol{y}}$ is a feasible solution to $V(\boldsymbol{x}^*)$ and we have the following inequalities:

$$u^j(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*}) = u^i(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*}),\ \forall j \in \boldsymbol{I},\ j \neq i,\ \ and\ \ u^i(\bar{\boldsymbol{y}}^i, \boldsymbol{x}^{-i*}) > u^j(\boldsymbol{y}^{j*}, \boldsymbol{x}^{-j*})$$

This is a contradiction to $\boldsymbol{y}^{i*}$ is an efficient solution to $V(\boldsymbol{x}^*)$. So the conclusion is right.

4 Other equivalent forms for GNEP

Notice that the lower multi-objective programming probe $V(\boldsymbol{x})$ of the bilevel programming problem (BP_1) is separable, that is, the decision variables of every programming are independent each other not only in objective functions but also in the constrains (Chen and Craven, 1994; Mehrdad,1996). So we know the effective solution to $V(\boldsymbol{x})$ is just the aggregate of the optimal solution to every single-objective programming, that is,

$$\max_{\boldsymbol{y}^i} u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i})$$

$$s.t. \quad \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i})$$

Then, we can transform the problem $V(\boldsymbol{x})$ into a single-objective programming problem equivalently by combination the multi-objective programmes linearly. Without loss of generality, let all the weighting coefficient be one (Ashry, 2006; Balbas and Guerrs, 1996). We get the transformed single-objective programming $P(\boldsymbol{x})$ as following:

$$P(\boldsymbol{x}) \qquad \begin{array}{l} \max\limits_{\boldsymbol{y}} \sum\limits_{i=1}^{n} u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i}) \\ \\ s.t. \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i}), \forall i \in \boldsymbol{I} \end{array}$$

And the corresponding bilevel programming problem (BP_1) is transformed into the following problem (BP_2) correspondingly:

$$\Big(BP_2\Big) \qquad \begin{array}{l} \max\limits_{(\boldsymbol{x},\boldsymbol{y})} \ f(\boldsymbol{x}, \boldsymbol{y}) = -(\boldsymbol{y}-\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x}) \\ s.t. \left\{ \begin{array}{l} \boldsymbol{x}^i \in K^i(\boldsymbol{x}^{-i}) \quad i \in \boldsymbol{I} \\ \text{and } \boldsymbol{y} \text{ is an optiaml solution of the programming } P(\boldsymbol{x})\text{:} \\ \mathbf{P}(\boldsymbol{x}) \qquad \begin{array}{l} \max\limits_{\boldsymbol{y}} \Big(u^1(\boldsymbol{y}^1, \boldsymbol{x}^{-1}) + \cdots + u^n(\boldsymbol{y}^n, \boldsymbol{x}^{-n})\Big) \\ s.t. \ \boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i}) \ \ \forall i \in \boldsymbol{I} \end{array} \end{array} \right. \end{array}$$

Similar to the above discuss in section 3, we have the following conclusions, we list them without detail proofs.

Lemma 4.1 *For the given* $\mathbf{x}^* \in \mathbf{R}^m$, $P(\mathbf{x}^*)$ *has optimal solution if and only if* $\forall i \in \mathbf{I}$, $EP_{(\mathbf{x}^{-i*})}$ *has optimal solution.*

*Proof:*We just need to prove that $P(\boldsymbol{x}^*)$ has optimal solution is equivalent with the multi-objective programming $V(\boldsymbol{x}^*)$ has efficient solution.

If $P(\boldsymbol{x}^*)$ has optimal solution $\boldsymbol{y}^*$, it is obviously that $\boldsymbol{y}^*$ is an efficient solution to $V(\boldsymbol{x}^*)$.

Conversely, if $V(\boldsymbol{x}^*)$ has an efficient solution $\boldsymbol{y}^*$, it means that$\forall \boldsymbol{y} = (\boldsymbol{y}^1, \boldsymbol{y}^2, \cdots, \boldsymbol{y}^n) \in \boldsymbol{R}^m$ such that $\forall i \in \boldsymbol{I}$, $\boldsymbol{y}^i \in K^i(\boldsymbol{x}^{-i})$, the following inequalities hold:

$$u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i*}) \le u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*}), \quad \forall i \in \boldsymbol{I}$$

By adding them up, we have the following inequality:

$$\sum_{i=1}^{n} u^i(\boldsymbol{y}^i, \boldsymbol{x}^{-i*}) \le \sum_{i=1}^{n} u^i(\boldsymbol{y}^{i*}, \boldsymbol{x}^{-i*})$$

It just shows that $\boldsymbol{y}^*$ is an optimal solution to $P(\boldsymbol{x}^*)$. Thus we complete the conclusion.

Theorem 4.2 $\mathbf{y}^* \in \mathbf{R}^m$ *is a generalized Nash equilibrium point to GNEP if and only if there exists* $\mathbf{x}^* \in \mathbf{R}^m$ *such that* $(\mathbf{x}^*, \mathbf{y}^*)$ *solves* (BP_2) *and the optimal value is zero.*

5 conclusion

In this paper, we consider the transforming form of generalized Nash equilibrium problem. We construct a special bilevel programming model and establish some results about the relations between solutions of the two models with strictly proof. Due to the special structure of the bilevel programming problem in this paper, the properties and the calculation of the general Nash equilibrium point becomes possible and easy. Of course, much more research is needed in order to provide algorithmic tools to effectively solve GNEP. In this regard we feel it deserves further investigations in the special bilevel programming (BP_1) and (BP_2).

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