Stability of nonlinear stochastic Volterra difference equations with continuous time

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Abstract: In recent years, many authors investigated the systems of stochastic difference equations with discrete time or the systems of numerical solution for stochastic difference equations with continue time. Lyapunov functionals are used to study the hereditary systems about problems of stability and optimal control. Besides, Lyapunov functionals construction has been widely used to discuss the stability for stochastic differential equations with delay and for stochastic difference equations with discrete time. Based on the general method of Lyapunov functionals construction, the stability of nonlinear stochastic Volterra difference equations is studied here. Particularly, the system considered here have continuous time. Sufficient conditions are obtained not only to ensure the mean square stability to nonlinear stochastic Volterra difference equations with continuous time but also the asymptotical mean square quasi-stability for this system. Furthermore, it is easy to know the considered system is mean square integrable when the system here is asymptotical mean square quasi-stable.
Keywords: Nonlinear stochastic difference equations; Stability; Lyapunov functional construction; Continuous time.

0 Introduction

The general method of Lyapunov functional construction, that was proposed by Kolmanovskii and Shaikhst and successfully used already for functional differential equations, difference equations with discrete time, difference equations with continuous time.

Difference equations with continuous time are difference equations in which the unknown function is a function of continuous time. In practice, time is often involved as the independent variable in difference equations with continuous time. In view of this fact, we may refer to them as difference equations with continuous time. Difference equations with continuous time appear as natural descriptions of observed evolution phenomena in many branches of natural science, see [1,2] and references therein. Deterministic and stochastic difference equations with continuous time are very popular with researchers, see [3,4,5,6] and references therein.

Motivated by the results in Shaikhet [6], concerning the mean square stability and asymptotically mean square quasistable of solutions of linear stochastic difference equations with continuous time, and in Luo [7], concerning the stability in probability of solutions of nonlinear stochastic Volterra difference equations with continuous time, in the present paper, we will be interested in the mean square stability and asymptotically mean square quasistable of solutions of nonlinear stochastic Volterra difference equations with continuous time.

1 Preliminaries

Let $\{\Omega, F, P\}$ be a probability space and $\{F_t\}_{t\geq 0}$ be a nondecreasing family of sub-σ-algebras of F, i.e., $F_{t_1} \subseteq F_{t_2}$ for $t_1 < t_2$.

Consider a stochastic Volterra difference equation with unbounded delay

$$x(t+\tau) = \sum_{i=0}^{m(t)} a_i x(t-h_i) + \sum_{i=0}^{m(t)} b_i x(t-h_i)\xi(t+\tau)$$

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$$+g(t,x(t),\ldots,x(t-h_{m(t)})),t>t_0-\tau, \quad (2.1)$$

with the initial condition

$$x(\theta)=\psi(\theta),\theta\in\Theta:=[t_0-\tau-h_{m(t)},t_0]. \quad (2.2)$$

Here

$$m(t)=m+[\frac{t-t_0}{\tau}], \quad (2.3)$$

m is a given nonnegative integer, $[t]$ denotes the largest integer less than or equal to t when $t\geq 0$, and $[t]$ is the smallest integer greater than or equal to t when $t<0$. The time increment τ per iteration step is a positive constant, the delays $h_i,i=0,1,\ldots,$ are constants satisfying the rational relation $h_i=i\tau$, a_i, b_i, $i=0,1,\ldots$ are known constants. The functional g satisfies the condition

$$|g(t,x(t),\ldots,x(t-h_{m(t)}))|\leq\sum_{i=0}^{m(t)}\gamma_i|x(t-h_i)|^{\nu_i},t>t_0-\tau,\gamma_i\geq 0,\nu_i>1,i=0,1,\ldots$$

(2.4)

$\psi(\theta),\theta\in\Theta$, is a F_{t_0}-measure function, the perturbation $\xi(t)\in R$ is a F_t measurable stationary stochastic process with conditions

$$E\xi(t+\tau)=0,E\xi^2(t+\tau)=1,t>t_0-\tau. \quad (2.5)$$

Furthermore, let

$$a=\sum_{i=0}^{m(t)}|a_i|,b=\sum_{i=0}^{m(t)}|b_i|,\gamma=\sum_{i=0}^{m(t)}\gamma_i,A=(a+\gamma)^2+b^2, \quad (2.6)$$

we assume $A<\infty$ and a solution of problem (2.1)-(2.2) is a F_t-measurable process $x(t)=x(t;t_0,\psi)$, which is equal to the initial function $\psi(t)$ from (2.2) for $t\leq t_0$ and with probability 1 is defined by Eq.(2.1) for $t>t_0$.

Definition 1. The trivial solution of Eq.(2.1)-(2.2) is called p-stable, $p>0$, if for any $\varepsilon>0$ and $t_0\geq 0$ there exists a $\delta=\delta(\varepsilon,t_0)$ such that $E|x(t;t_0,\psi)|^p<\varepsilon$ for all $t\geq t_0$ if $\|\psi\|^p=\sup_{\theta\in\Theta}E|\psi(\theta)|^p<\delta$.

Definition 2. The trivial solution of Eq.(2.1)-(2.2) is called asymptotically p-stable, $p>0$, if it is p-stable and for all initial functions ψ

$$\lim_{t\to\infty}E|x(t;t_0,\psi)|^p=0. \quad (2.7)$$

Definition 3. The trivial solution of Eq.(2.1)-(2.2) is called asymptotically p-quasistable, $p>0$, if it is p-stable and for each $t\in[t_0,t_0+h_0)$ and all initial functions ψ

$$\lim_{j\to\infty}E|x(t+jh_0;t_0,\psi)|^p=0. \quad (2.8)$$

Definition 4. The solution of Eq.(2.1) with initial condition (2.2) is called p-integrable, $p>0$, if for all initial functions ψ

$$\int_{t_0}^{\infty}E|x(t;t_0,\psi)|^p\,dt<\infty. \quad (2.9)$$

If in Definitions 2.1-2.4 p =2, then the solution is called correspondingly mean square stable, asymptotically mean square stable, asymptotically mean square quasistable, mean square integrable.

2 The general theorem

Theorem 1. Let there exist a nonnegative functional $V(t)=V(t,x(t),...,x(t-h_{m(t)}))$ and positive numbers c_1, c_2, such that

$$EV(t)\le c_1 \sup_{s\le t} E|x(s)|^2, t\in[t_0,t_0+\tau), \quad (3.1)$$

and

$$E\Delta V(t)\le -c_2 E|x(t)|^2, t\ge t_0, \quad (3.2)$$

if

$$x(s)\in U_\varepsilon=\{x:|x|\le\varepsilon\le 1\}, s\le t,$$

where

$$\Delta V(t)=V(t+\tau)-V(t). \quad (3.3)$$

Then the trivial solution of Eq.(2.1)-(2.2) is asymptotically mean square quasistable.

Proof. From condition (3.2), we can easily get

$$\sum_{j=0}^{i} E\Delta V(t+j\tau)\le \sum_{j=0}^{i}(-c_2)E|x(t+j\tau)|^2, t\ge t_0,$$

which together with (3.3) yields

$$c_2\sum_{j=0}^{i} E|x(t+j\tau)|^2\le EV(t)-EV(t+(i+1)\tau)\le EV(t), t\ge t_0.$$

Let $i\to\infty$, then

$$c_2\sum_{j=0}^{\infty} E|x(t+j\tau)|^2\le EV(t), t\ge t_0. \quad (3.4)$$

From (3.2) we also obtain

$$EV(t+\tau)\le EV(t), t\ge t_0, \quad (3.5)$$

and

$$c_2 E|x(t)|^2\le EV(t), t\ge t_0. \quad (3.6)$$

(3.5) implies that

$$EV(t)\le EV(t-\tau)\le EV(t-2\tau)\le ...\le EV(s), t\ge t_0, \quad (3.7)$$

where $s=t-[\frac{t-t_0}{\tau}]\tau\in[t_0,t_0+\tau)$. Besides, from (3.1) it follows

$$\sup_{s\in[t_0,t_0+\tau)} EV(s)\le c_1 \sup_{t\le t_0+\tau} E|x(t)|^2 . \quad (3.8)$$

Using (2.1)-(2.5), for $t\le t_0+\tau$ we obtain

$$E|x(t)|^2 = E[\sum_{i=0}^{m(t)} a_i x(t-\tau-h_i)]^2 + E[\sum_{i=0}^{m(t)} b_i x(t-\tau-h_i)]^2$$
$$+E[g(t-\tau, x(t-\tau), \ldots, x(t-\tau-h_{m(t)}))]^2$$
$$+2E[\sum_{i=0}^{m(t)} a_i x(t-\tau-h_i) g(t-\tau, x(t-\tau), \ldots, x(t-\tau-h_{m(t)}))].$$

Then by (2.6) and Cauchy inequality we have

$$[\sum_{i=0}^{m(t)} a_i x(t-\tau-h_i)]^2 \le a \sum_{i=0}^{m(t)} |a_i| x^2(t-\tau-h_i),$$
$$[\sum_{i=0}^{m(t)} b_i x(t-\tau-h_i)]^2 \le b \sum_{i=0}^{m(t)} |b_i| x^2(t-\tau-h_i),$$
$$[g(t-\tau, x(t-\tau), \ldots, x(t-\tau-h_{m(t)}))]^2 \le \gamma \sum_{i=0}^{m(t)} \gamma_i x^2(t-\tau-h_i).$$

On the other hand, it is easy to derive that

$$2\sum_{i=0}^{m(t)} a_i x(t-\tau-h_i) g(t-\tau, x(t-\tau), \ldots, x(t-\tau-h_{m(t)}))$$
$$\le \sum_{i=0}^{m(t)} |a_i| \sum_{k=0}^{m(t)} \gamma_k \varepsilon^{v_k-1}[x^2(t-\tau-h_k) + x^2(t-\tau-h_i)]$$
$$\le \sum_{i=0}^{m(t)} (\gamma |a_i| + a\gamma_i) x^2(t-\tau-h_i).$$

Hence

$$E|x(t)|^2 \le E\sum_{i=0}^{m(t)} \{(a+\gamma)|a_i| + b|b_i| + (a+\gamma)\gamma_i\} x^2(t-\tau-h_i) \le A\|\psi\|^2, t \ge t_0. \quad (3.9)$$

By using this fact and (3.4)-(3.8) we have

$$c_2 \sum_{j=0}^{\infty} E|x(t+j\tau)|^2 \le EV(t) \le EV(s) \le \sup_{s\in[t_0, t_0+\tau)} EV(s) \le c_1 \sup_{t\le t_0+\tau} Ex^2(t) \le c_1 A\|\psi\|^2, t \ge t_0,$$

i.e.,

$$\sum_{j=0}^{\infty} E|x(t+j\tau)|^2 \le (\frac{c_1}{c_2} A)\|\psi\|^2, t \ge t_0, \quad (3.10)$$

and also we can get

$$c_2 Ex^2(t) \le EV(t) \le c_1 A\|\psi\|^2, t \ge t_0, \quad (3.11)$$

which means the trivial solution of Eq.(2.1)-(2.2) is mean square stable. Combing (2.6) and (3.10) we have that for each $t \ge t_0$, $\lim_{j\to\infty} E|x(t+j\tau)|^2 = 0$. Therefore, the trivial solution of Eq.(2.1)-(2.2) is asymptotically mean square quasistable. This completes the proof of Theorem 1.

Remark 2. If the conditions of Theorem 1 hold then the solution of Eq.(2.1) for each initial function (2.2) is mean square integrable. Really, integrating (3.2) from $t = t_0$ to $t = T$, by virtue of (3.3) we have

$$\int_T^{T+\tau} EV(t)dt - \int_{t_0}^{t_0+\tau} EV(t)dt \le -c_2 \int_{t_0}^{T} E|x(t)|^2 dt. \quad (3.12)$$

From here, (2.6), (3.8) and (3.9) it follows

$$c_2\int_{t_0}^{T} E\,|\,x(t)\,|^2\,dt \le \int_{t_0}^{t_0+\tau} EV(t)dt \le c_1 A\|\psi\|^2\,\tau < \infty, \tag{3.13}$$

and by $T \to \infty$ we obtain (2.8).

Corollary 3. Let there exist a functional $V(t)=V(t,x(t),x(t-h_1),...x(t-h_{m(t)}))$ and positive numbers c_1, c_2, p, such that conditions (3.1) and (3.6) and $E\Delta V(t)\le 0$ hold. Then the trivial solution of Eq.(2.1) is mean square stable.

3 Conclusion

In our paper, based on the general method of Lyapunov functional construction, we mainly study the stability of nonlinear stochastic Volterra difference equations. Especially, the considered system in this paper have continuous time.

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非线性随机连续 Volterra 差分方程的稳定性

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摘要：近几年来，很多科研工作者研究了离散时间型的随机差分方程系统以及连续时间型随机差分方程系统的数值解。李雅普诺夫函数法通常用来研究遗传系统的稳定性和最优控制。此外，李雅普诺夫函数构造法已经广泛地用于研究时滞随机微分方程的稳定性以及离散时间型的随机差分方程的稳定性。基于一般的李雅普诺夫函数构造法，在这里研究了非线性随机 Volterra 差分方程的稳定性，特别地，所研究的方程是带有连续时间的。得到了充分的条件不仅保证了非线性随机 Volterra 差分方程的均方稳定性，也确保了非线性随机 Volterra 差分方程的渐近均方拟稳定性。此外，当这个所研究的系统是渐近均方拟稳定的，那么很容易知道这个所考虑的系统也是均方可积的。

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