Necessary and Sufficient Conditions for Positive and Negative Solutions of BSDEs with Continuous Coefficients

XU Yu hong
(Department of Mathematics, China University of Mining and Technology Xuzhou, 221008, P.R.China)
E-mail: xuyuhongmath@163.com

Abstract
As we know that, in stochastic finance, each pricing mechanism corresponds to a well-defined BSDE. The behaviors of g exert an influence to this mechanism. In some circumstances, to regulate or to design a pricing mechanism is in fact to find a suitable generating function g. By a technical result on the local limit of solutions to backward stochastic differential equations (BSDEs for short), this note gives necessary and sufficient conditions on g for positive and negative solutions of BSDEs with continuous coefficients, which implies that there is no arbitrage to the pricing mechanism characterized by these BSDEs.
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1 Introduction

Backward stochastic differential equations which arise in stochastic finance were first formulated by Pardoux-Peng [1] and Duffie-Epstein [2]. The first part of the solution to a BSDE represents the dynamical price of a financial position. If for a nonnegative position, the dynamical price is always nonnegative, then there is no arbitrage in this market. Thus an interesting question comes naturally, to what pricing mechanism, there will be no arbitrage for a financial position. In fact, each well-defined BSDE with a fixed generating function g forms a dynamic pricing mechanism. The behaviors of this mechanism are perfectly characterized by the behaviors of g. So in some circumstances, to regulate or to design a pricing mechanism is in fact to find a suitable generating function g. This note proves first a technical result on the local limit of solutions to BSDEs with continuous coefficients. This result is theoretically important because it can be used to test the generating function g by associated solutions. By this technical result, necessary and sufficient conditions on g are given for positive and negative solutions of BSDEs with continuous coefficients, which implies that there is no arbitrage in a financial market. Specially, for BSDEs with Lipschitz continuous coefficients, we prove that the unique solution is nonnegative if and only if for the generator g, we have $dP \times dt - a.s.$, $g(t, 0, 0) \geq 0$.

2 Preliminaries

Let $(B_t)_{t\in[0,T]}$ be a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ be the augmented Brownian filtration generated by $(B_t)_{t\in[0,T]}$. $T<0$ is a fixed time. For $x\in\mathbf{R}^d$, we define its norm $|x| = \sum_{i=1}^d |x_i|$. We denote by $\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ the space of all $\mathcal{F}_t$-progressively measurable $\mathbf{R}^d$-valued processes s.t. $\mathbf{E}\left[\int_0^T |\psi_t|^2 dt\right] < \infty$.

Consider the following one dimensional BSDE:

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\, ds - \int_t^T z_s dB_s,\ 0\le t\le T, \tag{2.1}$$

where the terminal variable $\xi \in L^2(\Omega,\mathcal{F},P)$ and the function g:$\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d \longmapsto \mathbf{R}$ is $\mathcal{F}_t$-progressively measurable. We call (ξ, g, T) the standard parameters of BSDE(2.1).

We assume that

(H1) $P-a.s., \forall t,\ g(t,y,z)$ is continuous in (y,z).

(H2) for any $(y,z)\in R\times R^d$, there exists a constant $K\geqslant 0$ such that $dP\times dt-a.s.$,

$$|g(t,y,z)| \le K(1+|y|+|z|).$$

Under (H1) and (H2), Lepeltier and San Martin [3] proved that BSDE(2.1) has a minimal solution $(\underline{Y}_t, \underline{Z}_t)_{t\in[0,T]} \in \mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R})\times\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$; In the same way, it is easy to know that it also has a maximal solution $\left(\bar{Y}_t, \bar{Z}_t\right)_{t\in[0,T]} \in \mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R})\times\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$.

Briand et al.[4, Proposition 2.2] gave a priori estimate for BSDEs with Lipschitz continuous generators. When using the Lipschitz condition in proof of the estimate, they in fact need the linear growth property from the Lipschitz condition. Thus by Briand et al.[4, Proposition 2.2], we immediately have the following estimate for BSDEs with linear growth generators.

Lemma 2.1. *Let (H1), (H2) hold for g and $\xi\in L^2(\Omega,\mathcal{F}_T,P)$, then the solution $(y_t,z_t)_{t\in[0,T]}$ of BSDE (2.1) satisfies*

$$\mathbf{E}\left[\sup_{t\le s\le T}\left(e^{\beta s}|y_s|^2\right) + \int_t^T e^{\beta s}|z_s|^2 ds|\mathcal{F}_t\right] \le C\mathbf{E}\left[e^{\beta T}|\xi|^2 + \left(\int_t^T \varphi_s e^{(\beta/2)s} ds\right)^2 |\mathcal{F}_t\right],$$

where $\beta = 2\left(K+K^2\right)$, C is a universal constant, K is the linear growth coefficient.

The following lemma is from Hewitt and Stromberg [5, Lemma 18.4].

Lemma 2.2. *Assume that the function f is Lebesgue integrable on $[a,b]$. $\forall\alpha\in R$, for almost all $t\in[a,b]$, we have*

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}|f(s)-\alpha| = |f(t)-\alpha|.$$

Similarly to Lepeltier and San Martin [3, Lemma 1], we have

Lemma 2.3. *Let (H1) and (H2) hold for g, then the following sequence*

$$g_n(t,y,z) = \inf\nolimits_{(a,b)\in Q^{1+d}}\left\{g(t,a,b)+n|y-a|+n|z-b|\right\},\ n\ge K, \tag{2.2}$$

is well defined for $n\ge K, dP\times dt-a.s.$, and we have $dP\times dt-a.s.$,

(i) Linear growth. $\forall\ (y,z)\in R\times R^d$, $|g_n(t,y,z)|\le \varphi_t + K(|y|+|z|)$.
(ii) Monotonicity in n. $\forall\ (y,z)\in R\times R^d$, $g_n(t,y,z)\nearrow$.
(iii) Lipschitz condition. $\forall y^i, z^i$, $i=1,2.$, $|g_n\left(t,y^1,z^1\right)-g_n\left(t,y^2,z^2\right)|\le n\left(|y^1-y^2|+|z^1-z^2|\right)$.
(iv) Strong convergence. if $(y_n,z_n)\to(y,z)$ as $n\to\infty$, then $g_n\left(t,y_n,z_n\right)\to g(t,y,z)$.

3 A technical result

Briand et al.[4] and Jiang [6] proved a local limit theorem for lipschitz continuous generator of a BSDE which has a unique solution. This section generalizes the limit theorem to the case where the coefficient is only continuous.

Proposition 3.1. *Suppose that the function $g:\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d\longmapsto\mathbf{R}$ satisfies (H1) and (H2). Let $\left(\underline{Y}_s^t\right)_{s\in[0,t]}$ and $\left(\bar{Y}_s^t\right)_{s\in[0,t]}$ be the minimal and maximal solutions of the following BSDE:*

$$y_s^t = y + z\cdot(B_t-B_s) + \int_s^t g\left(r,y_r^t,z_r^t\right)dr - \int_s^t z_r^t dB_r,\ 0\le s\le t. \tag{3.1}$$

where the terminal time $t\in[0,T]$. Then we have, for each $(y,z)\in\mathbf{R}\times\mathbf{R}^d$, $dt-a.e.$,

$$L^2-\lim_{s\to t^-}\frac{1}{t-s}\left[\bar{Y}_s^t - y\right] = g\left(t,y,z\right), \tag{3.2}$$

and

$$L^2-\lim_{s\to t^-}\frac{1}{t-s}\left[\underline{Y}_s^t - y\right] = g\left(t,y,z\right), \tag{3.3}$$

Proof. We only prove (3.2). For each given triplet $(t,y,z)\in[0,T]\times\mathbf{R}\times\mathbf{R}^d$, let $\left(\overline{y}_s^t,\ \overline{z}_s^t\right)_{s\in[0,t]}$ denote the maximal solution of BSDE(3.1). Note that BSDE(3.1) is not a fixed BSDE since the terminal datum $\xi = y + z\cdot(B_t-B_s)$ varies as s varies. However we can convert it into a fixed BSDE.

Set $\widetilde{y}_s^t = \overline{y}_s^t - y$, $\widetilde{z}_s^t = \overline{z}_s^t - z$, and $\widetilde{g}\left(s,\widetilde{y}_s^t,\widetilde{z}_s^t\right) = g\left(s,\widetilde{y}_s^t + y,\widetilde{z}_s^t + z\right)$, then obviously we get the following fixed BSDE:

$$\widetilde{y}_s^t = 0 + \int_s^t \widetilde{g}\left(r,\widetilde{y}_r^t,\widetilde{z}_r^t\right)dr - \int_s^t \widetilde{z}_s^t dB_r,\ 0\le s\le t. \tag{3.4}$$

One can easily check that $\widetilde{g}(t,\cdot,\cdot)$ is continuous $dP\times dt-a.s.$ and satisfies the following linear growth:

$$\left|\widetilde{g}\left(s,\widetilde{y}_s^t,\widetilde{z}_s^t\right)\right|\le\widetilde{\varphi}_s + K(\widetilde{y}_s^t+\widetilde{z}_s^t).$$

with $\widetilde{\varphi}_s=\varphi_s+K(|y|+|z|)$.

Then by Lemma 2.1 and Schwartz's inequality, we have, for BSDE(3.4), $\mathbf{E}[\int_s^t|\widetilde{y}_r^t|^2+\int_s^t|\widetilde{z}_r^t|^2dr]\le C\mathbf{E}\left[\left(\int_s^t\widetilde{\varphi}_r e^{(\beta/2)r}dr\right)^2\right]\le Ce^{\beta T}(t-s)\mathbf{E}\left[\int_s^t|\widetilde{\varphi}_r|^2dr\right]$. Thus

$$\lim_{s\to t^-}\ \frac{1}{t-s}\int_s^t\mathbf{E}[|\widetilde{y}_r^t|^2+|\widetilde{z}_r^t|^2]dr\le\lim_{s\to t^-}Ce^{\beta T}\mathbf{E}\left[\int_s^t|\widetilde{\varphi}_r|^2dr\right]=0. \tag{3.5}$$

Taking conditional expectation of (3.4) with respect to $\mathcal{F}_s$, we get

$$\overline{y}_s^t - y = \mathbf{E}\left[\int_s^t g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) dr | \mathcal{F}_s\right]. \tag{3.6}$$

Then thanks to (3.6), we only need to prove that $dt - a.e.$,

$$L^2 - \lim_{s\to t^-} [\mathbf{E}\left[\frac{1}{t-s}\int_s^t g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) dr | \mathcal{F}_s\right] - g(t,y,z)] = 0.$$

By Jensen's inequality of conditional expectation, Schwartz's inequality, Fubini's theorem and Lemma 2.2, we have $dt - a.e.$,

$$\begin{aligned}
& \lim_{s\to t^-} \mathbf{E}\left|\mathbf{E}\left[\frac{1}{t-s}\int_s^t g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) dr | \mathcal{F}_s\right] - g(t,y,z)\right|^2 \\
= & \lim_{s\to t^-} \frac{1}{(t-s)^2}\mathbf{E}\left|\mathbf{E}[\int_s^t [g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) - g(t,y,z)]dr|\mathcal{F}_s]\right|^2 \\
\leq & \lim_{s\to t^-} \frac{1}{(t-s)^2}\mathbf{E}\left|\int_s^t [g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) - g(t,y,z)]dr\right|^2 \\
\leq & \lim_{s\to t^-} \frac{1}{(t-s)}\mathbf{E}\int_s^t \left|g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) - g(t,y,z)\right|^2 dr \\
\leq & 2\lim_{s\to t^-} \frac{1}{t-s}\int_s^t \mathbf{E}\left|g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) - g(r,y,z)\right|^2 dr \\
& +2\lim_{s\to t^-} \frac{1}{t-s}\int_s^t \mathbf{E}\left|g\left(r,y,z\right) - g(t,y,z)\right|^2 dr \\
= & 2\lim_{s\to t^-} \frac{1}{t-s}\int_s^t \mathbf{E}\left|g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) - g(r,y,z)\right|^2 dr \\
\leq & 8\left(\mathbf{I}_1^{(n)} + \mathbf{I}_2^{(n)} + \mathbf{I}_3^{(n)}\right), \ \forall n \geq K,
\end{aligned}$$

where we denote

$$\begin{aligned}
\mathbf{I}_1^{(n)} &= \lim_{s\to t^-} \frac{1}{t-s}\int_s^t \mathbf{E}\left|g_n\left(r, \overline{y}_r^t, \overline{z}_r^t\right) - g_n(r,y,z)\right|^2 dr, \\
\mathbf{I}_2^{(n)} &= \lim_{s\to t^-} \frac{1}{t-s}\int_s^t \mathbf{E}\left|g\left(r, \overline{y}_r^t, \overline{z}_r^t\right) - g_n\left(r, \overline{y}_r^t, \overline{z}_r^t\right)\right|^2 dr, \\
\mathbf{I}_3^{(n)} &= \lim_{s\to t^-} \frac{1}{t-s}\int_s^t \mathbf{E}\left|g_n(r,y,z) - g(r,y,z)\right|^2 dr,
\end{aligned}$$

where g_n is defined as in Lemma 2.3.

By Lemma 2.3 and limit (3.5), we deduce that

$$\begin{aligned}
\mathbf{I}_1^{(n)} &\leq 2n^2 \lim_{s\to t^-} \frac{1}{t-s}\int_s^t \mathbf{E}(\left|\overline{y}_r^t - y\right|^2 + \left|\overline{z}_r^t - z\right|^2)dr \\
&= 2n^2 \cdot 0 \\
&= 0.
\end{aligned}$$

Therefore, we now conclude that

$$\lim_{s\to t^-}\mathbf{E}\left|\mathbf{E}\left[\frac{1}{t-s}\int_s^t g\left(r,\overline{y}_r^t,\overline{z}_r^t\right)dr|\mathcal{F}_s\right]-g(t,y,z)\right|^2\leq 8\left(\mathbf{I}_2^{(n)}+\mathbf{I}_3^{(n)}\right),\ \forall n\geq K, \tag{3.7}$$

Take limit as $n\to\infty$ on both sides of the above inequality, we only need to prove that $\lim_{n\to\infty}\mathbf{I}_2^{(n)}=0$, $\lim_{n\to\infty}\mathbf{I}_3^{(n)}=0$ to complete the proof.

By Lemma 2.3 and the well known control convergence theorem, we obtain that $dt-a.e.$, $\mathbf{E}\left|g\left(r,\overline{y}_r^t,\overline{z}_r^t\right)-g_n\left(r,\overline{y}_r^t,\overline{z}_r^t\right)\right|^2$ converges to zero strongly as $n\to\infty$. Thus for any r, there is a Lebesgue integrable sequence $\{\varepsilon_n(r)\}_{n\geq K}^{\infty}$ such that $\varepsilon_n(r)\to 0$ as $n\to\infty$ and $dt-a.e.$, $\mathbf{E}\left|g\left(r,\overline{y}_r^t,\overline{z}_r^t\right)-g_n\left(r,\overline{y}_r^t,\overline{z}_r^t\right)\right|^2\leq\varepsilon_n(r)$. Then by Lemma 2.2, we have $dt-a.e.$, $\lim_{n\to\infty}\mathbf{I}_2^{(n)}\leq\lim_{n\to\infty}\lim_{s\to t^-}\frac{1}{t-s}\int_s^t\varepsilon_n(r)dr=\lim_{n\to\infty}\varepsilon_n(t)=0$. Similarly we can get that $dt-a.e.$, $\lim_{n\to\infty}\mathbf{I}_3^{(n)}=0$. This completes the proof.□

4 Necessary and sufficient conditions for positive and negative solutions of BSDEs

we now give necessary and sufficient conditions for positive and negative solutions of BSDEs with continuous coefficients.

Theorem 4.1. *Let (H1) and (H2) hold for g, $\forall t\in[0,T]$, for any $\xi\in L^2(\Omega,\mathcal{F}_t,P)$, Let $(\underline{Y}_s)_{s\in[0,t]}$ and $\left(\bar{Y}_s\right)_{s\in[0,t]}$ be the minimal solution and maximal solution of the following BSDE:*

$$y_s=\xi+\int_s^t g\left(r,y_r,z_r\right)du-\int_s^t z_r dB_r,\ 0\leq s\leq t, \tag{4.1}$$

respectively,

(i) $\forall t\in[0,T]$, for any $\xi\in L^2(\Omega,\mathcal{F}_t,P)$, such that $\xi\leq 0$, we have $\forall s\in[0,t], P-a.s.$,

$$\underline{Y}_s\leq 0,$$

if and only if $dP\times dt-a.s.$,

$$g(t,0,0)\leq 0.$$

(ii) $\forall t\in[0,T]$, for any $\xi\in L^2(\Omega,\mathcal{F}_t,P)$, such that $\xi\geq 0$, we have $\forall s\in[0,t]$, $P-a.s.$,

$$\bar{Y}_s\geq 0,$$

if and only if $dP\times dt-a.s.$,

$$g(t,0,0)\geq 0.$$

(iii) Assume moreover that there exists a constant $K\geq 0$ s.t. $\forall(y,z)\in R\times R^d$, $dP\times dt-a.s.$,

$$g\left(t,y,z\right)\geq g\left(t,0,0\right)-K\left(|y|+|z|\right). \tag{4.2}$$

Then $\forall t \in [0,T]$*, for any* $\xi \in L^2(\Omega, \mathcal{F}_t, P)$*, such that* $\xi \geq 0$*, we have* $\forall s \in [0,t]$*,* $P-a.s.$*,*

$$\underline{Y}_s \geq 0,$$

if and only if $dP \times dt - a.s.$*,*

$$g(t,0,0) \geq 0.$$

Proof. (i) Lepeltier and San Martin [3] showed that the unique solution (Y^n, Z^n) of BSDE (ξ, g_n, T) converges to $(\underline{Y}_t, \underline{Z}_t)_{t\in[0,s]}$ in $\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}) \times \mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ where g_n is defined as in Lemma 2.3. Observe that $g_n(t,0,0) \leq g(t,0,0) \leq 0$ and by Lemma 2.3(iii) we have

$$g_n(t,y,z) \leq g_n(t,0,0) + n|y| + n|z| \tag{4.3}$$

Compare solutions of the following two BSDEs:

$$y^n_s = \xi + \int_s^t (g_n(r,0,0) + n|y^n_r| + n|z^n_r|)\, dr - \int_s^t z^n_r dB_r,\ 0 \leq s \leq t, \tag{4.4}$$

$$Y^n_s = \xi + \int_s^t g_n(r, Y^n_r, Z^n_r)\, dr - \int_s^t Z^n_r dB_r,\ 0 \leq s \leq t, \tag{4.5}$$

by the comparison theorem for Lipschitz continuous BSDE in EPQ[7, Theorem 2.2], we have $y^n_s \leq 0$ and $Y^n_s \leq y^n_s$, therefor $\underline{Y}_s = \lim_{n\to\infty} Y^n_s \leq 0$.

The converse is a sequence of taking $y = z = 0$ in (3.3).

(ii) Observe that the following sequence

$$\bar{g}_n(t,y,z) = \sup_{(a,b)\in Q^{1+d}} \{g(t,a,b) - n|y-a| - n|z-b|\},\ n \geq K, \tag{4.6}$$

is decreasing and converges to g $dP \times dt - a.s.$ and satisfies the Lipschitz condition, similarly to (i), we can prove (ii).

(iii) Obviously if $\underline{Y}_s \geq 0$, $\bar{Y}_s \geq \underline{Y}_s \geq 0$, thus by (ii) we get that $dP \times dt - a.s.$, $g(t,0,0) \geq 0$. Conversely, consider the following two BSDEs:

$$\underline{Y}_s = \xi + \int_s^t g(r, \underline{Y}_r, \underline{Z}_r)\, dr - \int_s^t \underline{Z}_r dB_r,\ 0 \leq s \leq t, \tag{4.7}$$

$$y_s = \xi + \int_s^t (-K|y_r| - K|z_r| + g(r,0,0))\, dr - \int_s^t z_r dB_r,\ 0 \leq s \leq t, \tag{4.8}$$

by the comparison theorem in Liu and Ren [8] and (4.2), we obtain that $\forall s \in [0,t]$, $\underline{Y}_s \geq y_s$, $P-a.s.$ and by the well known comparison theorem (see EPQ[7, theorem 2.2]) for BSDEs with Lipscitz continuous generators, if $\xi \geq 0$, $P-a.s.$ and $g(t,0,0) \geq 0$, $dP \times dt - a.s.$, then $\forall s$, $y_s \geq 0$, $P-a.s.$, thus $\forall s$, $\underline{Y}_s \geq 0$, $P-a.s.$.□

It is just a sequence of Theorem 4.1 that,

Corollary 4.1. *Assume that there exists a constant* $K \geq 0$ *s.t.* $\forall y^1, y^2, z^1, z^2$*,* $dP \times dt - a.s.$*,* $|g(t,y^1,z^1) - g(t,y^2,z^2)| \leq K(|y^1 - y^2| + |z^1 - z^2|)$ *and* $|g(t,0,0)| \leq K$*. Then*

(i) $\forall t \in [0,T]$*, for any* $\xi \in L^2(\Omega, \mathcal{F}_t, P)$*, such that* $\xi \geq 0$*, we have* $\forall s \in [0,t]$*,* $P-a.s.$*, for the unique solution of BSDE(4.1),*

$$Y_s \geq 0,$$

if and only if $dP \times dt - a.s.,$

$$g(t,0,0) \geq 0.$$

(ii) $\forall t \in [0,T]$, *for any* $\xi \in L^2(\Omega, \mathcal{F}_t, P)$, *such that* $\xi \leq 0$, *we have* $\forall s \in [0,t]$, $P - a.s.$, *for the unique solution of BSDE(4.1),*

$$Y_s \leq 0,$$

if and only if $dP \times dt - a.s.,$

$$g(t,0,0) \leq 0.$$

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Author Brief Introduction: XU Yu-hong (1981-), male, major in backward stochastic differential equation and stochastic finance.