Extremal Distribution for the Discrete 5-convex Stochastic Ordering and Applications

TIAN Yougong[1], JIAO Guimei[2]
(1. Department of Mathematics, Lanzhou Commercial College, Lanzhou 730020;
2. School of Mathematic and Statistic, Lanzhou University, Lanzhou 730000)

Abstract: In this paper, motivated by the idea from Courtois and Denuit et al.(2006), we derive extremal distribution for the discrete 5-convex stochastic ordering. As applications,We improve lower and upper bounds for the probability of extinction in a branching process and for the Lundberg's coefficient in the classical risk model with discrete claim amounts.

Key words: s-convex order; moment space; extremal distribution; branching process; Lundberg's coefficient

0 Introduction

For theoretical and practical purposes, the theory of stochastic orderings generates considerable interest in actuarial science (see books by Goovaerts et al. (1990), Kaas et al. (1994), Shaked and Shanthikumar (1994), De Vylder (1996) and Hürlimann (1998)). Especially, it is established that stochastic order relations constitute an important tool in the analysis of various actuarial problems. For example, they can be used to compare complex models with more tractable ones which are risker, thus leading to more conservative decisions. Quite recently, various discrete stochastic orderings have been introduced to compare random variables that are discrete by nature as counts. Such as, Fishburn and Lavalle(1995), Lefèvre and Utev(1996), and so on. A remarkable class investigated by Denuit and Lefèvre(1997) is the class of the discrete s-convex orderings among arithmetic random variables valued in some set $N_n = \{0,1,2,\cdots,n\}, n \in \mathbb{N}$, here s is any nonnegative integer smaller or equal to n. It is worth mentioning that these orderings have been generalized by Denuit et al.(1999), using the concept of divided difference operator, to compare any pairs of discrete random variables. The discrete s-convex extremal distribution have been derived for s = 1; 2; 3; 4 in Courtois et al.(2006).

Our purpose is to obtain the minimum and the maximum in the 5-convex sense for random variables valued in N_n. The paper is organized as follows: section 2 gives some basic notions and its propositions about the discrete s-convex order. Section 3 recall the cut-criterion. In section 4, we use the ideal from Courtois and Denuit et al.(2006) to find the 5-convex extrema. Finally, section 5 deals with applications of this theory. We also improve lower and upper bounds for the probability of extinction in a branching process and for the Lundberg's coe_cient in the classical insurance risk model .

1 Preliminaries

Given some class U of real-valued functions, which is often a convex cone in a function space, the random variable X is said to be smaller than Y if

$$E[v(X)] \le E[v(Y)], \quad \text{for all } v \in U,$$

such that the expectations exist, where U denotes a class of real-valued functions v, satisfying some desirable properties. As announced, random variables are assumed to take on values in the state space $N_n = \{0,1,2,\cdots,n\}$, for some $n \in \mathbb{N}$. Now let s be any fixed positive integer in

Brief author introduction:田有功（1977-），男，讲师，主要研究方向：概率统计应用，随机序，风险理论. E-mail: tianyg03@lzu.cn

N_n. Discrete s-convex orderings have been defined in Denuit and Lefèvre (1997). We denote by Δ the usual forward difference operator which is defined, for any function $v: N_n \to \mathbb{R}$ by $\Delta v(i) = v(i+1) - v(i)$ for all $i \in N_n$, such that $i+1 \in N_n$. Let $\Delta^k, k \in N_n$ be the **k**th order forward difference operator defined recursively by $\Delta^0 v = v$ and when $k \geq 1, \Delta^k v(i) = \Delta^{k-1} v(i+1) - \Delta^{k-1} v(i)$, for all $i \in N_n$, such that $i + k \in N_n$.

Definition 1.1 Let X and Y be two random variables valued in N_n, X is said to be smaller than Y in the s-convex sense (written as $X \leq_{s-cx}^{N_n} Y$) if $E[v(X)] \leq E[v(Y)]$, for all s-convex real function v on N_n, that is within the class

$$U_{s-cx}^{N_n} = \{v: N_n \to \mathbb{R} \mid \Delta^s v(i) \geq 0, \text{ } for \quad all \quad i \in N_{n-s}\}$$

Especially, since the power function $v(x) = x^k$ and $v(x) = -x^k$ both belong to $U_{s-cx}^{N_n}$, for $k = 1, 2, \cdots, s-1$. We get the necessary condition for $\leq_{s-cx}^{N_n}$:

$$X \leq_{s-cx}^{N_n} Y \Rightarrow E[X^k] = E[Y^k], \quad \text{for} \quad k = 1, 2, \cdots, s-1$$

Theorem 1.1 (Denuit and Lefèvre,1997) Let X and Y be two random variables valued in N_n,

$$X \leq_{s-cx}^{N_n} Y \Rightarrow \begin{cases} E[X^k] = E[Y^k], & k = 1, 2, \cdots, s-1, \\ E[X^k] \leq E[Y^k], & for\ all\ k \geq s. \end{cases}$$

That is to say, if $X \leq_{s-cx}^{N_n} Y$, then the s-1 first moments of X and Y necessarily match. Consequently, the ordering relation $\leq_{s-cx}^{N_n}$ can only be used to compare the random variable with the same first s-1 moments. So we introduce the moment space $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ which contains all random variables valued on N_n, such that the s-1 first moments are fixed to $E[X^k] = \mu_k, k = 1, 2, \cdots, s-1$, where s is a prescribed nonnegative integer.

2 Main results

Now we consider the class $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ of all the random variables X valued in N_n with prescribed s-1 first moments, $E[X^k] = \mu_k, k = 1, 2, \cdots, s-1$. Our purpose is to determine in $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ the minimum and the maximum with respect to the s-convex orderings, namely, there exist random variables $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ that belong to

\$ $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ and such that

$$X_{\min}^{(s)} \leq_{s-cx}^{N_n} X \leq_{s-cx}^{N_n} X_{\max}^{(s)} \quad \text{for all} \quad X \in D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1}).$$

In order to derive random variables $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ belong to $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ such that $X_{\min}^{(s)} \leq_{s-cx}^{N_n} X \leq_{s-cx}^{N_n} X_{\max}^{(s)}$, for all $X \in D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$, Courtois et al.(2006) constructed the following random variables that achieve the bounds

$$\underset{X \in D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})}{MIN} E[X^s] \text{ and } \underset{X \in D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})}{MAX} E[X^s]$$

the extrema $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ necessarily achieve the above bounds. To find the random variables that realize the above bounds, they gave the following results.

Theorem 2.1 (Courtois et al., 2006) (1) A random variable $X \in D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ achieves the maximum in (1) if and only if X is sup-admissible, that is, X is concentrated on the set

$$\{i \in N_n : i^s = c_0 + c_1 i + c_2 i^2 + \cdots + c_{s-1} i^{s-1}\}$$

Where the $c_i^{'}s$ are real constants such that

$$i^s \le c_0 + c_1 i + c_2 i^2 + \cdots + c_{s-1} i^{s-1}, \quad \text{for all} \quad i \in N_n$$

(2)A random variable $X \in D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ achieves the minimum in (1) if and only if X is sub-admissible, that is, X is concentrated on the set

$$\{i \in N_n : i^s = c_0 + c_1 i + c_2 i^2 + \cdots + c_{s-1} i^{s-1}\}$$

where the $c_i^{'}s$ are real constants such that

$$i^s \ge c_0 + c_1 i + c_2 i^2 + \cdots + c_{s-1} i^{s-1}, \text{ for all } \quad i \in N_n .$$

Theorem 2.2 (Courtois and Denuit et al., 2006) Let X be some random variable in $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$, then X is the s-convex minimum (respectively, maximum) if and only if $X = \underset{Z \in D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})}{MIN} E[Z^s]$ (respectively, $X = \underset{Z \in D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})}{MAX} E[Z^s]$).

Using the cut-criterion , Courtois et al.(2006) verified that the possible structure of the supports of the 5-convex discrete extrema takes the form $\{0, \xi_1, \xi_1 + 1, \xi_2, \xi_2 + 1\}$ or $\{\eta_1, \eta_1 + 1, \eta_2, \eta_2 + 1, n\}$, then this is done by computing the explicit probabilities associated with the support and by checking that the resulting probabilities are positive.

Theorem 2.3 Consider a moment space $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ with a given sequence of moments $\mu_1, \mu_2, \mu_3, \mu_4$.

(1) If $\xi_1, \xi_2 \in N_n$ are such that $0 < \xi_1 < \xi_1 + 1 < \xi_2 < \xi_2 + 1 \le n$ and define:

$$\begin{aligned}
\alpha_1 &= \mu_4 - 2(\xi_1 + \xi_2 + 1)\mu_3 + [(\xi_1 + \xi_2 + 1)(\xi_1 + \xi_2 + 1) + \xi_1(\xi_2 + 1) + (\xi_1 + 1)\xi_2]\mu_2 - [\xi_1\xi_2(\xi_1 + 1) \\
&\quad + (\xi_1 + 1)(\xi_2 + 1)(\xi_1 + \xi_2) + \xi_1\xi_2(\xi_2 + 1)]\mu_1 + \xi_1(\xi_1 + 1)\xi_2(\xi_2 + 1) \\
\alpha_2 &= -\mu_4 + (\xi_1 + 2\xi_2 + 2)\mu_3 - [(\xi_1 + 1)\xi_2 + (\xi_1 + 1)(\xi_2 + 1) + \xi_2(\xi_2 + 1)]\mu_2 + (\xi_1 + 1)\xi_2(\xi_2 + 1)\mu_1 \\
\alpha_3 &= \mu_4 - (\xi_1 + 2\xi_2 + 1)\mu_3 + [\xi_1\xi_2 + \xi_1(\xi_2 + 1) + \xi_2(\xi_2 + 1)]\mu_2 - \xi_1\xi_2(\xi_2 + 1)\mu_1 \\
\alpha_4 &= -\mu_4 + (2\xi_1 + \xi_2 + 2)\mu_3 - [\xi_1(\xi_1 + 1) + \xi_1(\xi_2 + 1) + (\xi_1 + 1)(\xi_2 + 1)]\mu_2 + \xi_1(\xi_1 + 1)(\xi_2 + 1)\mu_1 \\
\alpha_5 &= \mu_4 - (2\xi_1 + \xi_2 + 1)\mu_3 + [\xi_1(\xi_1 + 1) + \xi_1\xi_2 + (\xi_1 + 1)\xi_2]\mu_2 - \xi_1(\xi_1 + 1)\xi_2\mu_1
\end{aligned}$$

that are positive , then the discrete 5-convex minimal distribution of $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ is given by

$$X_{\min}^{(5)} = \begin{cases}
0 & \text{with probability} \quad p_1 = \alpha_1 / \xi_1(\xi_1 + 1)\xi_2(\xi_2 + 1) \\
\xi_1 & \text{with probability} \quad p_2 = \alpha_2 / \xi_1(\xi_2 - \xi_1)(\xi_2 + 1 - \xi_1) \\
\xi_1 + 1 & \text{with probability} \quad p_3 = \alpha_3 / (\xi_1 + 1)(\xi_2 - \xi_1 - 1)(\xi_2 - \xi_1) \\
\xi_2 & \text{with probability} \quad p_4 = \alpha_4 / \xi_2(\xi_2 - \xi_1)(\xi_2 - \xi_1 - 1) \\
\xi_2 + 1 & \text{with probability} \quad p_5 = \alpha_5 / (\xi_2 + 1)(\xi_2 + 1 - \xi_1)(\xi_2 - \xi_1)
\end{cases}$$

(2) If $\eta_1,\eta_2 \in N_n$ are such that $0 \le \eta_1 < \eta_1+1 < \eta_2 < \eta_2+1 < n$ and define:

$$\beta_1 = \mu_4 - (\eta_1+2\eta_2+2+n)\mu_3 + [(n+\eta_2)(\eta_1+\eta_2+2)+(\eta_1+1)(\eta_2+1)+n\eta_2]\mu_2 - [(\eta_1+1)(\eta_2+1)(n+\eta_2)+n\eta_2(\eta_1+\eta_2+2)]\mu_1 + n(\eta_1+1)\eta_2(\eta_2+1)$$

$$\beta_2 = -\mu_4 + (\eta_1+2\eta_2+1+n)\mu_3 - [n(\eta_1+\eta_2+1)+(\eta_1+\eta_2)(\eta_2+1)+\eta_2(\eta_1+n)]\mu_2 + [(\eta_1\eta_2(n+\eta_2+1)+n(\eta_2+1)(\eta_1+\eta_2)]\mu_1 - n\eta_1\eta_2(\eta_2+1)$$

$$\beta_3 = \mu_4 - (2\eta_1+\eta_2+2+n)\mu_3 + [n(\eta_1+\eta_2+2)+(\eta_1+1)(\eta_1+\eta_2+1)+\eta_1(\eta_2+1+n)]\mu_2 + [(\eta_1+1)(\eta_2+1)(n+\eta_1)+n\eta_1(\eta_1+\eta_2+2)]\mu_1 - n\eta_1(\eta_1+1)(\eta_2+1)$$

$$\beta_4 = -\mu_4 + (2\eta_1+\eta_2+1+n)\mu_3 - [\eta_1(\eta_1+\eta_2+1)+n(\eta_1+\eta_2+1)+n\eta_1+(\eta_1+1)\eta_2]\mu_2 + [\eta_1(\eta_1+1)(n+\eta_2)+n\eta_2(2\eta_1+1)]\mu_1 - n\eta_1(\eta_1+1)\eta_2$$

$$\alpha_5 = \mu_4 - 2(\eta_1+\eta_2+1)\mu_3 + [(\eta_1+\eta_2)(\eta_1+\eta_2+2)+(\eta_1+1)(\eta_2+1)+\eta_1\eta_2]\mu_2 + [\eta_1(\eta_1+1)(2\eta_2+1)+\eta_2(\eta_2+1)(2\eta_1+1)]\mu_1 - \eta_1(\eta_1+1)\eta_2(\eta_2+1)$$

that are positive, then the discrete 5-convex maximum distribution of $D_s(N_n;\mu_1,\mu_2,\cdots,\mu_{s-1})$ is given by

$$X_{\max}^{(5)} = \begin{cases} \eta_1 & \text{with probability} \quad p_1 = \beta_1/(\eta_2-\eta_1)(\eta_2+1-\eta_1)(n-\eta_1) \\ \eta_1+1 & \text{with probability} \quad p_2 = \beta_2/(\eta_2-\eta_1)(\eta_2-\eta_1-1)(n-\eta_1-1) \\ \eta_2 & \text{with probability} \quad p_3 = \beta_3/(\eta_2-\eta_1)(\eta_2-\eta_1-1)(n-\eta_2) \\ \eta_2+1 & \text{with probability} \quad p_4 = \beta_4/(\eta_2-\eta_1)(\eta_2+1-\eta_1)(n-\eta_2-1) \\ n & \text{with probability} \quad p_5 = \beta_5/(n-\eta_1)(n-\eta_1-1)(n-\eta_2)(n-\eta_2-1) \end{cases}$$

Proof (1) In Courtois and Denuit et al.(2006), we obtain the possible support of 5-convex extrema $X_{\min}^{(5)}$, so we need check it . By Theorem 4.1 and Theorem 4.2, we just compute the polynomials

$\mathrm{P}(i) = c_0 + c_1 i + c_2 i^2 + c_3 i^3 + c_4 i^4$ of degree 4 (c_0,c_1,c_2,c_3 and $c_4 \in \mathbb{R}$) such that $X_{\min}^{(5)} \in D_5(N_n;\mu_1,\mu_2,\mu_3,\mu_4)$ is concentrated on the set

$$\{i \in N_n : i^5 = c_0 + c_1 i + c_2 i^2 + c_3 i^3 + c_4 i^4\} = \{0,\xi_1,\xi_1+1,\xi_2,\xi_2+1\}$$
$$(0 < \xi_1 < \xi_1+1 < \xi_2 < \xi_2+1 \le n)$$

and $i^5 \le c_0 + c_1 i + c_2 i^2 + c_3 i^3 + c_4 i^4$ for all $i \in N_n$.

The only polynomial of degree 4 that fulfills the conditions:

$$\begin{aligned} 0 &= c_0 \\ \xi_1^5 &= c_0 + c_1\xi_1 + c_2\xi_1^2 + c_3\xi_1^3 + c_4\xi_1^4 \\ (\xi_1+1)^5 &= c_0 + c_1(\xi_1+1) + c_2(\xi_1+1)^2 + c_3(\xi_1+1)^3 + c_4(\xi_1+1)^4 \\ \xi_2^5 &= c_0 + c_1\xi_2 + c_2\xi_2^2 + c_3\xi_2^3 + c_4\xi_2^4 \\ (\xi_2+1)^5 &= c_0 + c_1(\xi_2+1) + c_2(\xi_2+1)^2 + c_3(\xi_2+1)^3 + c_4(\xi_2+1)^4 \end{aligned}$$

By computing , we get

$$\mathrm{P}(i) = -\xi_1\xi_2(\xi_1+1)(\xi_2+1)i + [(2\xi_1+1)\xi_2(\xi_2+1)+\xi_1(\xi_1+1)(2\xi_2+1)]i^2 - [\xi_1(\xi_1+2\xi_2+2)+(\xi_1+1)(2\xi_2+1)+\xi_2(\xi_2+1)]i^3 + 2(\xi_1+\xi_2+1)i^4$$

The zeroes of the polynomial $i^5 - \mathrm{P}(i)$ are of course $0,\xi_1,\xi_1+1,\xi_2,\xi_2+1$ are always positive on N_n. So we have checked that $i^5 \geq \mathrm{P}(i)$ on N_n, the random variable with support $\{0,\xi_1,\xi_1+1,\xi_2,\xi_2+1\}$ $(0<\xi_1<\xi_1+1<\xi_2<\xi_2+1\leq n)$ has to be $X_{\min}^{(5)}$.

Finally, we have to fix conditions on the support points to assure the non-negativity of their associated probabilities. We can not get explicit conditions on the support points, but we can easily obtained the distribution on the support points by testing each admissible pair (ξ_1,ξ_2) of N_n satisfying

$$\alpha_1\geq 0,\ \ \alpha_2\geq 0,\ \ \alpha_3\geq 0,\ \ \alpha_4\geq 0,\ \ \alpha_5\geq 0.$$

(2) Similarly, we just compute the polynomials $\mathrm{P}(i)=c_0+c_1 i+c_2 i^2+c_3 i^3+c_4 i^4$ of degree 4 (c_0,c_1,c_2,c_3 and $c_4\in\mathbb{R}$) such that $X_{\max}^{(5)}\in D_5(N_n;\mu_1,\mu_2,\mu_3,\mu_4)$ is concentrated on the set

$$\{i\in N_n : i^5=c_0+c_1 i+c_2 i^2+c_3 i^3+c_4 i^4\}=\{\eta_1,\eta_1+1,\eta_2,\eta_2+1,n\}$$
$$(0\leq\eta_1<\eta_1+1<\eta_2<\eta_2+1<n)$$

and $i^5\geq c_0+c_1 i+c_2 i^2+c_3 i^3+c_4 i^4$ for all $i\in N_n$.

The only polynomial of degree 4 that fulfills the conditions:

$$\begin{aligned}
\eta_1^5 &= c_0+c_1\eta_1+c_2\eta_1^2+c_3\eta_1^3+c_4\eta_1^4\\
(\eta_1+1)^5 &= c_0+c_1(\eta_1+1)+c_2(\eta_1+1)^2+c_3(\eta_1+1)^3+c_4(\eta_1+1)^4\\
\eta_2^5 &= c_0+c_1\eta_2+c_2\eta_2^2+c_3\eta_2^3+c_4\eta_2^4\\
(\eta_2+1)^5 &= c_0+c_1(\eta_2+1)+c_2(\eta_2+1)^2+c_3(\eta_2+1)^3+c_4(\eta_2+1)^4\\
n^5 &= c_0+c_1 n+c_2 n^2+c_3 n^3+c_4 n^4
\end{aligned}$$

By computing , we get

$$\begin{gathered}
\mathrm{P}(i)=n\eta_1\eta_2(\eta_1+1)(\eta_2+1)\\
-[n(\eta_2+1)[\eta_1(\eta_1+1)+\eta_1\eta_2+\eta_2(\eta_1+1)]+\eta_1\eta_2(\eta_1+1)(n+\eta_2+1)]i\\
+[n(\eta_2+1)(2\eta_1+\eta_2+1)+\eta_1(\eta_1+1)(n+2\eta_2+1)+\eta_2(2\eta_1+1)(n+\eta_2+1)]i^2\\
-[\eta_1(n+2\eta_2+\eta_1+2)+(\eta_1+1)(n+2\eta_2+1)+\eta_2(n+\eta_2+1)+n(\eta_2+1)]i^3\\
+(2\eta_1+2\eta_2+2+n)i^4
\end{gathered}$$

The zeroes of the polynomial $i^5-\mathrm{P}(i)$ are of course $\eta_1,\eta_1+1,\eta_2,\eta_2+1,n$ are always positive on N_n. So we have checked that $i^5\leq\mathrm{P}(i)$ on N_n, the random variable with support $\{\eta_1,\eta_1+1,\eta_2,\eta_2+1,n\}$ $(0\leq\eta_1<\eta_1+1<\eta_2<\eta_2+1<n)$ has to be $X_{\max}^{(5)}$.

Finally, we have to fix conditions on the support points to assure the non-negativity of their associated probabilities. We can not get explicit conditions on the support points, but we can easily obtained the distribution on the support points by testing each admissible pair (η_1,η_2) of N_n satisfying

$$\beta_1\geq 0,\ \ \beta_2\geq 0,\ \ \beta_3\geq 0,\ \ \beta_4\geq 0,\ \ \beta_5\geq 0.$$

3 Applications

In Courtois and Denuit et al.(2006), they gave theoretical background for the discrete s-convex stochastic ordering, now we simply recall it as follows: assumed a random variable X valued in N_n , being a positive integer, a classical problem consists in solving the equation $\varphi_N(z) = P_k(z)$

in the unknown z, where $\varphi_N(z) = E[z^N] = \sum_{k=0}^{n} z^k P[N = k], 0 \le z \le 1,$ is the probability generating function of N. When all that is only known about N is that it belongs to $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$, then (2) can not be solved explicitely. To show that the s-convex extrema described previously allow accurate approximations for the solution of (2). The idea is to construct two function $\varphi_{\min}^{(s)}(\cdot)$ and $\varphi_{\max}^{(s)}(\cdot)$ such that

$$\varphi_{\min}^{(s)}(z) \le \varphi_N(z) \le \varphi_{\max}^{(s)}(\cdot) \qquad \text{for all } 0 \le z \le 1.$$

3.1 Probability of ultimate extinction in a branching process

In a branching process, we assume that time $t = 0$, there exists an initial population M_0, $M_0 \ge 1$. During its life span, every individual gives birth to a random number of children and these children give birth to a random number of children, and so on. The reproduction rules are : (1) all individuals give birth according to the same probability law, independently of each other; (2) the number of children produced by an individual is independent of the number of individuals in their generation.

Let us assume that $M_0 = 1$. For $k \ge 1$, let M_k be the number of individuals in generation k and let N be random variable valued in N_n representing the number of children obtained by the individuals, $P[N = 1] < 1$. If we denote by α the probability of ultimate extinction of this process, $\alpha = P[M_k = 0$, for some $k]$, then α is the smallest non-negative root of the equation $z = \varphi_N(z)$; $\alpha = 1$ for $E[N] < 1$ and $\alpha < 1$ for $E[N] > 1$. Because the sequence $\{z^k, k \in N\}$ is completely monotonic for $0 \le z \le 1$,when s is even, $\varphi_{\min}^{(s)}(\cdot) = \varphi_{N_{\min}^{(s)}}(\cdot)$ and $\varphi_{\max}^{(s)}(\cdot) = \varphi_{N_{\max}^{(s)}}(\cdot)$, while s is odd, $\varphi_{\min}^{(s)}(\cdot) = \varphi_{N_{\max}^{(s)}}(\cdot)$ and $\varphi_{\max}^{(s)}(\cdot) = \varphi_{N_{\min}^{(s)}}(\cdot)$,where $N_{\min}^{(s)}$ and $N_{\max}^{(s)}$ are the stochastic extrema in $D_s(N_n; \mu_1, \mu_2, \cdots, \mu_{s-1})$ with respect to the discrete versions of the s-convex stochastic orderings. In order to illustrate the use of the s-convex extrema up to the order five, we also consider the following example from Guttorp (1991).

Example 1 Let us take n=10, $P[N = 0] = 0.4982, P[N = 1] = 0.2103,$ $P[N = 2] = 0.1270,$

$P[N = 3] = 0.0730,$ $P[N = 4] = 0.0418,$ $P[N = 5] = 0.0241,$ $P[N = 6] = 0.0132,$
$P[N = 7] = 0.0069,$ $P[N = 8] = 0.0035, P[N = 9] = 0.0015,$ $P[N = 10] = 0.0005.$

The exact extinction probability is α =0.879755. The 5-convex discrete extrema are as follows: $X_{\min}^{(5)}$ **and** $X_{\max}^{(5)}$ have respective supports {0,2,3,6,7} and {0,1,4,5,10}, associated probabilities {0.5551,0.3757,0.01120,0.0432,0.01410}and {0.4470,0.3742,0.1432,0.0316,0.0040}. The bounds obtained with the discrete s-convex extrema are displayed in the below Table 1.

Table 1: Bounds on the probability of ultimate extinction α

s	$\alpha_{\min}^{(s)}$	$\alpha_{\max}^{(s)}$
3	0.8414716	0.8868653
4	0.8791374	0.8807095
5	0.8796994	0.8797986

It shows that the bounds obtained with s=5 are remarkably accurate(the bounds for s=3 and s=4 can be found in Courtois et al.(2006)).

3.2 Ruin probability---Binomial risk model

In the classical discrete binomial risk model (see 1989,1993), the discrete claim amounts $X_1, X_2, \cdots$ recorded by an insurance company are assumed to be independent and identically distributed with common distribution function F having finite $s-1$ moments, such that $F(0)=0$. The number of claims in the time interval $[0,t]$ is assumed to be independent of the individual claim amounts and to form a binomial process $N(t), t \in \mathbb{N}$ with parameter $q, 0<q<1$ (in any time period there occur 1 or 0 claims with probabilities q and $1-q$, respectively, and occurrences of claims in different time intervals are independent events). We assume that the premium received in each period is equal to 1 and is larger than the net premium, which means that $qE[X_1]<1$.

Let $\psi(\kappa)$ be the ultimate ruin probability with an initial capital κ; that is to say, the probability that the process $Z(t)=\kappa+t-\sum_{i=1}^{N(t)} X_i, t \in \mathbb{N}$, describing the wealth of the insurance company, ever falls below zero. If the moment generating function of X exists, Lundberg's inequality provides an exponential upper bound on $\psi(\kappa)$, that is , $\psi(\kappa) \le e^{-z\kappa}$, where z is Lundberg's adjustment coefficient satisfying the integral equation $\phi_N(z)=E[e^{zN}]=e^z$ with N denoting the aggregate claim amount in the t-th time interval. As we are dealing with a compound binomial model, it comes out easily that z is the solution of the equation $1-q+qE[e^{zX}]=e^z$ where $E[e^{zX}]$ is the moment generating function of the discrete claim amounts $X_1, X_2, \cdots$ .Since the sequence $\{e^{kz}, k \in N\}$ is absolutely monotonic, $\phi_{\min}^{(s)}(\cdot) \le \phi_N^{(s)}(\cdot) \le \phi_{\max}^{(s)}(\cdot)$ with $\phi_{\min}^{(s)}(\cdot)=\phi_{N_{\min}^{(s)}}(\cdot)$ and $\phi_{\max}^{(s)}(\cdot)=\phi_{N_{\max}^{(s)}}(\cdot)$

Example Let us take n = 11, $P[N=0]=0.4982, P[N=1]=0.2103,$ $P[N=2]=0.1270,$

$P[N=3]=0.0730,\ P[N=4]=0.0418,\ P[N=5]=0.0241,\ P[N=6]=0.0132,$
$P[N=7]=0.0069,\ P[N=8]=0.0035, P[N=9]=0.0015,\ P[N=10]=0.0005.$

Thus, the first moments of the discrete claim amounts are fixed at $\mu_1=2.145,\ \mu_2=7.1454,\ \mu_3=33.4896, \mu_4=195.4362.$ Let $q=0.4$, Lundberg's adjustment coefficient is equal to $z=0.1163$ and the ruin probabilities $\psi(\kappa)$ for some initial surplus level κ are depicted in Table 2. The 5-convex discrete extrema are given as follows: $X_{\min}^{(5)}$ and $X_{\max}^{(5)}$ have respective supports {0, 1, 2, 6, 7} and {1, 2, 5, 6, 11} and associated probabilities {0.1266, 0.0400, 0.7372, 0.0429, 0.0533} and {0.4470, 0.3742, 0.1432, 0.0316, 0.0040}. The extremal 5-convex adjustment coefficients are respectively equal to $z_{\min}^{(5)}=0.11628$ and $z_{\max}^{(5)}=0.11634$. The exponential upper bounds obtained using these extrema are displayed in Table 2(the first five columns datas see Courtois et al.(2006)).

Table 2: Ruin probabilities and Lundberg's bounds when n = 11, q =0.4.

Initial surplus level κ	$\psi(\kappa)$	$e^{-\kappa z}$	$e^{-\kappa z_{\min}^{(3)}}$	$e^{-\kappa z_{\min}^{(4)}}$	$e^{-\kappa z_{\min}^{(5)}}$
0	0.7633	1	1	1	1
1	0.6842	0.8902	0.9003	0.8906	0.8902
2	0.6117	0.7925	0.8101	0.7933	0.7925
3	0.5461	0.7054	0.7291	0.7066	0.7055
4	0.4869	0.6280	0.6562	0.6294	0.6281
5	0.4338	0.5590	0.5906	0.5606	0.5591
6	0.3862	0.4977	0.5315	0.4993	0.4977
7	0.3438	0.4430	0.4784	0.4447	0.4431
8	0.3060	0.3944	0.4306	0.3961	0.3945
9	0.2724	0.3511	0.3875	0.3528	0.3512
10	0.2425	0.3125	0.3488	0.3142	0.3126
15	0.1355	0.1747	0.2060	0.1761	0.1748
20	0.0758	0.0977	0.1217	0.0987	0.0977
30	0.0237	0.0305	0.0425	0.0310	0.0305
40	0.0074	0.0095	0.0148	0.0097	0.0096
50	0.0023	0.0030	0.0052	0.0031	0.0030

References

[1] Courtois CI, Denuit M and Bellegem S V. Discrete s-convex extremal distributions: Theory and applications[J]. Applied Mathematics Letters ,2006, 19:1367-1377

[2] Lefèvre CI, Utev S. Comparing sums of exchangeable Bernoulli random variables[J]. Appl. Probab,1996,33: 285-310.

[3] Vylder F E D. Advanced Risk Theory, A Self-Contained Introduction, Editions de l'Universitè Libre de Bruxelles-Swiss Association of Actuaries, Bruxelles.1996.

[4] Shiu E S W. The probability of eventual ruin in the compound binomial model[J]. Astin Bull,1989,19: 179-190.

[5] Fishburn P C, Lavalle I H. Stochastic dominance on unidimensional grid[J].Math. Oper. Res. 1995,20: 513–525.

[6] Willmot G E. Ruin probabilities in the compound binomial model[J]. Insurance, Math.

Econom,1993,12:133-142.
[7]Hürlimann W. Extremal Moment Methods and Stochastic Orders[M]. Application in Actuarial Sciences. Monograph,1998.
[8]Denuit M, Lefévre CI. Some new classes of stochastic order relations among arithmetic random variables, with applications in actuarial sciences[J]. Insurance Math. Econom,1997,12:197-214.
[9]Denuit M,Lefévre CI,Mesfioui M. On s-convex stochastic extrema for arithmetic risks[J].Insurance Math. Econom,1999,25: 143-155.
[10] Denuit M, Vylder F E D, Lefévre CI. Extremal generators and extremal distributions for the continuous s-convex stochastic orderings[J]. Insurance Math. Econom,1999,24: 201-217.
[11]Shaked M, Shanthikumar J G.Stochastic Orders and their Applications[C].Academic Press, New York, 1994.
[12]Goovaerts M J, Kaas R, Heerwaarden A E V, Bauwelinckx T.Effective Actuarial Methods[C]. North-Holland, Amsterdam,1990.
[13] Guttorp P. Statistical Inference for Branching Processes[M] Wiley, New York, 1991.
[14]Kaas R, Heerwaarden A E V, Goovaerts M J. Ordering of Actuarial Risks[M]. CAIRE, Brussels,1994.

离散的 5 阶凸随机序的极值分布及其应用

田有功[1]，焦桂梅[2]
（1.兰州商学院数学系，兰州 730020;
2.兰州大学数学与统计学院，兰州 730000）

摘要：本论文受 Courtois 和 Denuit 等人（2006）思想的启发，我们获得了离散的 5 阶凸随机序的极值分布。作为应用，我们改进了分支过程中灭绝概率的上下界以及经典的离散风险模型中 Lundberg’s 系数的上下界

关键词： s 阶凸随机序；矩空间；极值分布；分支过程；Lundberg’s 系数
中图分类号：O211.6