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Calculation of $K_2(\mathbb{F}_2[C_4 \times C_4])$ *

CHEN Hong[1†], GAO Yu-Bin[2], TANG Guo-Ping[1]

(1 *School of Mathematical Sciences, Graduate University, Chinese Academy of Sciences, Beijing* 100049, *China*;
2 *College of Mathematics and Information Science, Shaanxi Normal University, Xi'an* 710062, *China*)

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Abstract First, we reduce the calculation of $K_2(\mathbb{F}_2[C_4 \times C_4])$ to that of the relative K_2-group $K_2(\mathbb{F}_2C_4[t]/(t^4), (t))$ of the truncated polynomial ring $\mathbb{F}_2C_4[t]/(t^4)$. Then we give a minimal generating set of $K_2(\mathbb{F}_2[C_4 \times C_4])$ by subtle calculations of Dennis-Stein symbols. Finally we show that $K_2(\mathbb{F}_2[C_4 \times C_4]) = C_4^3 \oplus C_2^9$.

Key words K_2-group, Dennis-Stein symbols, group ring

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Let C_n denote a cyclic group of order n. An abelian group G is called an elementary abelian p-group if $x^p = 1$ for all $x \in G$. For any finite abelian group G, we write

$$p^i - rk(G) = \dim_{\mathbb{F}_p} G^{p^{i-1}}/G^{p^i},$$

i.e., the number of cyclic summands of order at least p^i in the decomposition of the Sylow p-subgroup of G. If G is a finite abelian group of 4-rank $\leqslant 1$, then by Theorem 5 in Ref. [1], $K_2(\mathbb{F}_2G)$ is an elementary abelian 2-group and its rank is given there. For odd prime p and G of p^2-rank $\leqslant 1$, it is proved in Corollary 3.4 of Ref. [2] that $K_2(\mathbb{F}_pG)$ is an elementary abelian p-group, and its rank is computed there. Now the question arises:

· determine the explicite structure of $K_2(\mathbb{F}_pG)$ when the p^2-rank of $G \geqslant 2$.

Perhaps the simplest example for G to be considered is $C_4 \times C_4$, and in this short paper we show that $K_2(\mathbb{F}_2[C_4 \times C_4]) = {C_4}^3 \oplus {C_2}^9$, so this K_2-group is not p-elementary.

When $I \subset \mathrm{rad}(R)$, the relative K_2-group $K_2(R,I)$ has a representation by generators and relations.

Theorem A ((1.4) in Ref. [3]) Let R be a ring with unit and $I \subseteq \mathrm{rad}(R)$. The relative K_2-group $K_2(R,I)$ is generated by Dennis-Stein symbols $\langle a,b \rangle$ with a or b in I, satisfying the following relations:

(DS1) $\langle a,b \rangle = -\langle b,a \rangle$, if $a \in I$;

(DS2) $\langle a,b \rangle + \langle a,c \rangle = \langle a, b+c-abc \rangle$, if $a \in I$ or $b,c \in I$;

(DS3) $\langle a,bc \rangle = \langle ab,c \rangle + \langle ac,b \rangle$, if $a \in I$.

The following two important relations are proved in Ref. [3].

(DS4) $p^r \langle a,b \rangle = \langle a^{p^r} b^{p^r-1}, b \rangle$, if the characteristic of R is $p > 0$;

(DS5) $\langle a,b^m \rangle = m \langle ab^{m-1}, b \rangle$, for any positive integer m.

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† E-mail: chenhong06@mails.gucas.ac.cn

Usually it is easy to find an exponent of a Dennis-Stein symbol, but difficult to determine its order, and this is almost the key obstruction in the process to determine the structure of such relative K_2 -group which can be generated by Dennis-Stein symbols. We do not know any literatures in which some elements in $K_2(\mathbb{F}_pG)$ of order bigger then p are given. In this paper we not only give some elements in $K_2(\mathbb{F}_2[C_4\times C_4])$ of order 4, but determine its structure completely. This is the main reason to write the paper. Let K be a finite unramified extension of p-adic field $\mathbb{Q}_p$ and A the ring of integers in K with $[A:\mathbb{Z}_p]=f$. Let $q=p^f$. Then $A/(p)\cong\mathbb{F}_q$, so for any finite abelian group G we have $K_2(\mathbb{F}_qG)\cong K_2((A/p)G)$. Using tools from p -adic analysis, group extension and group homology, Oliver R[4] has obtained the following powerful result, which enable us to determine the structure of $K_2(\mathbb{F}_qG)$ by simply knowing the exponent of some Dennis-Stein symbols.

Theorem B (Proposition 6.3 in Ref. [4]) For any abelian p-group G and any unramified p-ring A with $[A:\mathbb{Z}_p]=f$, if $\exp(G)=p^e$ and $r_i=p^i-rk(G)$, $1\leqslant i\leqslant e$, then

$$\mathrm{ord}_p\mid K_2(A/p[G])\mid = f[(r_1-1)\mid G\mid-(r_1-r_2)\mid G^p\mid-\cdots-(r_{e-1}-r_e)\mid G^{p^{e-1}}\mid-(r_e-1)].$$

1 The result

Let $C_4\times C_4=\langle\sigma\rangle\times\langle\tau\rangle$ be the direct product of two cyclic groups of order 4 and $\mathbb{F}=\mathbb{F}_2$ the finite field of order 2. Obviously $\mathbb{F}[C_4\times C_4]\cong(\mathbb{F}C_4)[C_4]$. Let $R=\mathbb{F}C_4$ and x, t be indeterminants over R. Then $R[C_4]\cong R[x]/(x^4-1)$. Sending $x-1$ to t gives an isomorphism of rings $R[x]/(x^4-1)\cong R[t]/(t)$. In the sequel we also use t to denote its residue class in $R[t]/(t^4)$, therefore $t^4=0$. The following sequence is split exact,

$$0\to(t)\to R[t]/(t^4)\to R\to 0.$$

Since K_2 is functor from the category of rings to category of abelian groups, using the long exact sequence associated to $(R[t]/(t^4),(t))$ in K-theory we get the following split exact sequence

$$0\to K_2(R[t]/(t^4),(t))\to K_2(R[t]/(t^4))\to K_2(R)\to 0.$$

So we have $K_2(\mathbb{F}[C_4\times C_4])\cong K_2(R)\oplus K_2(R[t]/(t^4),(t))$. By Theorem 1 in Ref. [1] $K_2(R)=0$, this reduces the calculation of $K_2(\mathbb{F}[C_4\times C_4])$ to that of $K_2(R[t]/(t^4),(t))$.

Theorem 1.1 Let $C_4=\langle\sigma\rangle$ be a cyclic group of order 4 and $\mathbb{F}$ the finite field of order 2. Then $K_2(\mathbb{F}C_4[t]/(t^4),(t))$ has exponent 4 and can be generated by 12 Dennis-Stein symbols.

Proof By Theorem 3.1 in Ref. [2], $K_2(\mathbb{F}C_4[t]/(t^4),(t))$ has exponent 4. Since $t^4=0$ in $\mathbb{F}C_4[t]/(t^4)$ and $(t)\subseteq\mathrm{rad}(\mathbb{F}C_4[t]/(t^4))$, $K_2(\mathbb{F}C_4[t]/(t^4),(t))$ can be generated by Dennis-Stein symbols $\langle a,b\rangle$ with a or b in (t) by Theorem A. By Proposition 1.7 in Ref. [3], $K_2(\mathbb{F}C_4[t]/(t^4),(t))$ can be generated by:

$$\langle at^3,t\rangle,\ \langle at^2,t\rangle,\ \langle at,t\rangle,\ \langle at^3,b\rangle,\ \langle at^2,b\rangle,\ \langle at,b\rangle,$$

where a, $b\in\mathbb{F}C_4$. Since $\mid\mathbb{F}C_4\mid=16$, there are 816 such symbols which are too many to be a minimal generating set for $K_2(\mathbb{F}C_4[t]/(t^4),(t))$. We shall show that 12 of them are enough. Assume that a_1, $a_2\in\mathbb{F}C_4$. By (DS2),

$$\langle a_1t^3,t\rangle+\langle a_2t^3,t\rangle=\langle(a_1+a_2)t^3-a_1a_2t^7,t\rangle=\langle(a_1+a_2)t^3,t\rangle,$$

i.e. symbol $\langle at^3,t\rangle$ is additive in a. Although the Dennis-Stein symbol $\langle a,b\rangle$ is not additive in a and b in general, we can show that if we consider the six types of symbols step by step, the additive property holds in each type in the sense that after module the group generated by the symbols of the previous types.

For example, consider the sixth type elements $\langle at,b\rangle$. Let H be the subgroup generated by the symbols of the first 5 types. We will show that for any b_1, $b_2\in\mathbb{F}C_4$,

$$\langle at,b_1\rangle+\langle at,b_2\rangle\equiv\langle at,b_1+b_2\rangle \bmod H.$$

Note that for any $d\in\mathbb{F}C_4$, by (DS2) one has

$$\langle at,dt\rangle=\langle adt,t\rangle+\langle at^2,d\rangle\in H,$$
$$\langle at,dt^2\rangle=\langle adt,t^2\rangle+\langle at^3,d\rangle=2\langle adt^2,t\rangle+\langle at^3,d\rangle\in H,$$

and

$$\langle at,dt^3\rangle=\langle adt,t^3\rangle+\langle at^4,d\rangle=3\langle adt^3,t\rangle\in H.$$

Using (DS2) repeatedly, one has

$$\begin{aligned}
&\langle at,b_1\rangle+\langle at,b_2\rangle\\
&=\langle at,b_1+b_2-ab_1b_2t\rangle\\
&\equiv\langle at,b_1+b_2-ab_1b_2t\rangle+\langle at,a^2b_1b_2(b_1+b_2)t^2\rangle\\
&\equiv\langle at,b_1+b_2-ab_1b_2t+a^2b_1b_2(b_1+b_2)t^2-a^3b_1b_2(b_1+b_2)^2t^3\rangle\\
&\equiv\langle at,b_1+b_2-ab_1b_2t+a^2b_1b_2(b_1+b_2)t^2\rangle+\langle at,-a^3b_1b_2(b_1+b_2)^2t^3\rangle\\
&\equiv\langle at,b_1+b_2-ab_1b_2t+a^2b_1b_2(b_1+b_2)t^2\rangle\\
&\equiv\langle at,b_1+b_2\rangle+\langle at,-ab_1b_2t\rangle\\
&\equiv\langle at,b_1+b_2\rangle\mod H.
\end{aligned}$$

Similarly, one can check that

$$\langle a_1t,b\rangle+\langle a_2t,b\rangle\equiv\langle(a_1+a_2)t,b\rangle\mod H.$$

This proves that $K_2(\mathbb{F}C_4[t]/(t^4),(t))$ is generated by $\langle\sigma^{j_1}t^i,t\rangle$ and $\langle\sigma^{j_1}t^i,\sigma^{j_2}\rangle$, $0\leqslant j_1,j_2\leqslant 3$, $1\leqslant i\leqslant 3$, since $\mathbb{F}C_4$ is generated by $1,\sigma,\sigma^2,\sigma^3$ over $\mathbb{F}_2$. By (DS5),

$$\langle\sigma^{j_1}t^i,\sigma^{j_2}\rangle=j_2\langle\sigma^{j_1+j_2-1}t^i,\sigma\rangle,$$

hence, we only need to consider $\langle\sigma^jt^i,t\rangle$ and $\langle\sigma^jt^i,\sigma\rangle$, $0\leqslant j\leqslant 3$, $1\leqslant i\leqslant 3$. The number of these symbols is 24 and we will eliminate 12 of them.

At first, we prove that four symbols $\langle t,t\rangle$, $\langle t^2,t\rangle$, $\langle t^3,t\rangle$ and $\langle\sigma^2t^2,t\rangle$ are all zero. Since $\sigma^4=1$, by (DS4), (DS1) and (DS5) it follows

$$\langle t,t\rangle=\langle\sigma^4t,t\rangle=2\langle\sigma^2,t\rangle=-2\langle t,\sigma^2\rangle=-4\langle\sigma t,\sigma\rangle=0,$$

since the exponent of $K_2(\mathbb{F}C_4[t]/(t^4),(t))$ is 4. By (DS1), (DS5) and (DS4) it follows

$$\langle t^2,t\rangle=-\langle t,t^2\rangle=-2\langle t^2,t\rangle=-\langle t^5,t\rangle=0,$$

and

$$\langle\sigma^2t^2,t\rangle=-\langle t,(\sigma t)^2\rangle=-2\langle\sigma t^2,\sigma t\rangle=-\langle\sigma^3t^5,\sigma t\rangle=0.$$

Since $\sigma^4=1$, by (DS4), it follows

$$\langle t^3,t\rangle=\langle\sigma^4t^3,t\rangle=4\langle\sigma,t\rangle=0,$$

By (DS4) it follows

$$\langle\sigma^2t^3,t\rangle=2\langle\sigma t,t\rangle,$$
$$\langle\sigma t^2,\sigma\rangle=2\langle t,\sigma\rangle,$$
$$\langle\sigma^3t^2,\sigma\rangle=2\langle\sigma t,\sigma\rangle.$$

By (DS4), and (DS1) it follows

$$\langle\sigma^2t,t\rangle=2\langle\sigma,t\rangle=-2\langle t,\sigma\rangle.$$

By (DS1), and (DS4) it follows

$$\langle t^3,\sigma\rangle=-\langle\sigma,t^3\rangle=-3\langle\sigma t^2,t\rangle,$$
$$\langle t^2,\sigma\rangle=-\langle\sigma,t^2\rangle=-2\langle\sigma t,t\rangle.$$

By (DS1), (DS5) and (DS3) it follows

$$\langle \sigma^2 t^3, \sigma \rangle = -\langle \sigma, (\sigma^2 t)^3 \rangle = -3\langle \sigma t^2, \sigma^2 t \rangle = -3(\langle \sigma^3 t^2, t \rangle + \langle \sigma t^3, \sigma^2 \rangle)$$

and

$$\langle \sigma^2 t^2, \sigma \rangle = -\langle \sigma, \sigma^2 t^2 \rangle = -2\langle \sigma^2 t, \sigma t \rangle = -2(\langle \sigma^3 t, t \rangle + \langle \sigma^2 t^2, \sigma \rangle).$$

However,

$$\langle \sigma t^3, \sigma^2 \rangle = 2\langle \sigma^2 t^3, \sigma \rangle = 0$$

and

$$2\langle \sigma^2 t^2, \sigma \rangle = \langle \sigma t^4, \sigma \rangle = 0,$$

hence

$$\langle \sigma^2 t^3, \sigma \rangle = -3\langle \sigma^3 t^2, t \rangle$$

and

$$\langle \sigma^2 t^2, \sigma \rangle = -2\langle \sigma^3 t, t \rangle.$$

Now only the following 12 Dennis-Stein symbols remain

$$\langle \sigma t^3, t \rangle,\ \langle \sigma^3 t^3, t \rangle,\ \langle \sigma t^2, t \rangle,\ \langle \sigma^3 t^2, t \rangle,$$
$$\langle \sigma t, t \rangle,\ \langle \sigma^3 t, t \rangle,\ \langle \sigma t^3, \sigma \rangle,\ \langle \sigma^3 t^3, \sigma \rangle,$$
$$\langle t, \sigma \rangle,\ \langle \sigma t, \sigma \rangle,\ \langle \sigma^2 t, \sigma \rangle,\ \langle \sigma^3 t, \sigma \rangle,$$

and $K_2(\mathbb{F}C_4[t]/(t^4),(t))$ is generated by these symbols. □

Theorem 1.2 Let C_n denote a cyclic group of order n. Then

$$K_2(\mathbb{F}_2[C_4 \times C_4]) = C_4^3 \oplus C_2^9.$$

Proof We have shown that $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}C_4[t]/(t^4),(t))$ which is generated by 12 Dennis-Stein symbols by Theorem 1.1. Now we will inspect the orders of these symbols. Obviously $\langle \sigma t^3, t \rangle$, $\langle \sigma^3 t^3, t \rangle$, $\langle \sigma t^2, t \rangle$, $\langle \sigma^3 t^2, t \rangle$, $\langle \sigma t^3, \sigma \rangle$ and $\langle \sigma^3 t^3, \sigma \rangle$ all have order $\leqslant 2$, since for example

$$2\langle \sigma t^3, t \rangle = \langle \sigma^2 t^7, t \rangle = \langle 0, t \rangle = 0.$$

The other 6 symbols are not linearly independent, and we give some relations of them below.

$$\begin{aligned} 2\langle \sigma t, t \rangle &= \langle \sigma^2 t^3, t \rangle = 2\langle \sigma^3 t, t \rangle; \\ 2\langle t, \sigma \rangle &= \langle \sigma t^2, \sigma \rangle = 2\langle \sigma^2 t, \sigma \rangle; \\ 2\langle \sigma t, \sigma \rangle &= \langle \sigma^3 t^2, \sigma \rangle = 2\langle \sigma^3 t, \sigma \rangle. \end{aligned} \tag{1}$$

So we can replace $\langle \sigma t, t \rangle$, $\langle t, \sigma \rangle$, $\langle \sigma t, \sigma \rangle$, $\langle \sigma^3 t, t \rangle$, $\langle \sigma^2 t, \sigma \rangle$ and $\langle \sigma^3 t, \sigma \rangle$ just by $\langle \sigma t, t \rangle$, $\langle t, \sigma \rangle$, $\langle \sigma t, \sigma \rangle$, $\langle \sigma t, t \rangle + \langle \sigma^3 t, t \rangle$, $\langle t, \sigma \rangle + \langle \sigma^2 t, \sigma \rangle$ and $\langle \sigma t, \sigma \rangle + \langle \sigma^3 t, \sigma \rangle$ since they can generate each other. By eq. (1),

$$2(\langle \sigma t, t \rangle + \langle \sigma^3 t, t \rangle) = 0,$$
$$2(\langle t, \sigma \rangle + \langle \sigma^2 t, \sigma \rangle) = 0,$$
$$2(\langle \sigma t, \sigma \rangle + \langle \sigma^3 t, \sigma \rangle) = 0.$$

So elements $\langle \sigma t, \sigma \rangle + \langle \sigma^3 t, t \rangle$, $\langle t, \sigma \rangle + \langle \sigma^2 t, \sigma \rangle$ and $\langle \sigma t, \sigma \rangle + \langle \sigma^3 t, \sigma \rangle$ have order $\leqslant 2$. Since all $\langle \sigma t, t \rangle$, $\langle t, \sigma \rangle$, $\langle \sigma t, \sigma \rangle$ have order $\leqslant 4$, which implies that $K_2(\mathbb{F}C_4[t]/(t^4),(t))$ has order $\leqslant 4^3 \times 2^9 = 2^{15}$. By Theorem B, its order is precisely 2^{15}, so $K_2(\mathbb{F}C_4[t]/(t^4),(t)) = C_4^3 \oplus C_2^9$ with generators listed below,

$$\langle \sigma t, t \rangle,\ \langle t, \sigma \rangle,\ \langle \sigma t, \sigma \rangle,$$
$$\langle \sigma t^3, t \rangle,\ \langle \sigma^3 t^3, t \rangle,\ \langle \sigma t^2, t \rangle,$$
$$\langle \sigma^3 t^2, t \rangle,\ \langle \sigma t^3, \sigma \rangle,\ \langle \sigma^3 t^3, \sigma \rangle,$$
$$\langle \sigma t, t \rangle + \langle \sigma^3 t, t \rangle,\ \langle t, \sigma \rangle + \langle \sigma^2 t, \sigma \rangle,\ \langle \sigma t, \sigma \rangle + \langle \sigma^3 t, \sigma \rangle.$$

This finishes the proof. □

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$K_2(\mathbb{F}_2[C_4 \times C_4])$的计算

陈 虹[1†], 高玉彬[2], 唐国平[1]

(1 中国科学院研究生院数学科学学院, 北京 100049; 2 陕西师范大学数学与信息科学学院, 西安 710062)

摘 要 把 $K_2(\mathbb{F}_2[C_4 \times C_4])$ 的计算归结为计算截断多项式环 $\mathbb{F}_2C_4[t]/(t^4)$ 的相对 K_2 -群 $K_2(\mathbb{F}_2C_4[t]/(t^4),(t))$. 运用 Dennis-Stein 符号及它们之间的关系进行细致的分析计算,给出了 $K_2(\mathbb{F}_2[C_4 \times C_4])$ 的一个极小生成元集并最终确定了 $K_2(\mathbb{F}_2[C_4 \times C_4]) = C_4^3 \oplus C_2^9$.

关键词 K_2-群,Dennis-Stein 符号,群环