UNIQUENESS OF THE FOLIATION OF CONSTANT MEAN CURVATURE SPHERES IN ASYMPTOTICALLY FLAT 3-MANIFOLDS

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Abstract

In this paper I study the constant mean curvature surface in asymptotically flat 3-manifolds with general asymptotics. Under some weak condition, I prove that outside some compact set in the asymptotically flat 3-manifold with positive mass, the foliation of stable spheres of constant mean curvature is unique.

1 Introduction

A three-manifold M with a Riemannian metric g and a two-tensor K is called an initial data set (M, g, K) if g and K satisfy the constraint equations

$$\begin{aligned} R_g - |K|_g^2 + (tr_g(K))^2 &= 16\pi\rho \\ div_g(K) - d(tr_g(K)) &= 8\pi J \end{aligned} \tag{1.1}$$

where R_g is the scalar curvature of the metric g, $tr_g(K)$ denotes $g^{ij}K_{ij}$, ρ is the observed energy density, and J is the observed momentum density.

Definition 1.1. Let $q \in (\frac{1}{2}, 1]$. We say (M, g, K) is asymptotically flat (AF) if it is a initial data set, and there is a compact subset $\widetilde{K} \subset M$ such that $M \setminus \widetilde{K}$ is diffeomorphic to $R^3 \setminus B_1(0)$ and there exists coordinate $\{x^i\}$ such that

$$g_{ij}(x) = \delta_{ij} + h_{ij}(x) \tag{1.2}$$

$$h_{ij}(x) = O_5(|x|^{-q}) \qquad K_{ij}(x) = O_1(|x|^{-1-q}) \tag{1.3}$$

Also, ρ and J satisfy

$$\rho(x) = O(|x|^{-2-2q}) \qquad J(x) = O(|x|^{-2-2q}) \tag{1.4}$$

Here, $f = O_k(|x|^{-q})$ means $\partial^l f = O(|x|^{-l-q})$ for $l = 0, \cdots, k$. $M \setminus \widetilde{K}$ is called an end of this asymptotically flat manifold.

We can define mass for the asymptotically flat manifolds as follows:

$$m = \lim_{r\to\infty} \frac{1}{16\pi} \int_{|x|=r} (h_{ij,j} - h_{jj,i}) v_g^i d\mu_g \tag{1.5}$$

where v_g and $d\mu_g$ are the normal vector and volume form with respect to the metric g. From [1],we know the mass is well defined when $q > 1/2$.

Definition 1.2. We say $(M, g, K)$is asymptotically flat satisfying the Regge-Teitelboim condition (AF-RT) if it is AF, and g, K satisfy these asymptotically even/odd conditions

$$h_{ij}^{odd}(x) = O_2(|x|^{-1-q}) \qquad K_{ij}^{even}(x) = O_1(|x|^{-2-q}) \tag{1.6}$$

Also, ρ and J satisfy

$$\rho^{odd}(x) = O(|x|^{-3-2q}) \qquad J^{odd}(x) = O(|x|^{-3-2q}) \tag{1.7}$$

where $f^{odd}(x) = f(x) - f(-x)$ and $f^{even}(x) = f(x) + f(-x)$.

For (AF-RT) manifolds, the center of mass C is defined by

$$C^\alpha = \frac{1}{16\pi m} \lim_{r\to\infty} (\int_{|x|=r} x^\alpha (h_{ij,i} - h_{ii,j}) v_g^j d\mu_g - \int_{|x|=r} (h_{i\alpha} v_g^i - h_{ii} v_g^\alpha) d\mu_g). \tag{1.8}$$

From [3], we know it is well defined.

The constant mean curvature surface is stable means the second variation operator has non-negative eigenvalues when restricted to the functions with 0 mean value, i.e.

$$\int_\Sigma (|A|^2 + Ric(v_g, v_g)) f^2 d\mu \le \int_\Sigma |\nabla f|^2 d\mu \tag{1.9}$$

for function f with $\int_\Sigma f d\mu = 0$, where A is the second fundamental form, and $Ric(v_g, v_g)$ is the Ricci curvature in the normal direction with respect to the metric g.

We discuss the existence and uniqueness of constant mean curvature spheres that separate the origin from the infinity in the AF-RT manifolds. The following two theorems are due to Lan-Hsuan Huang [2]:

Theorem 1.3. (Existence) If (M, g, K) is the AF-RT with $q \in (\frac{1}{2}, 1]$, there exists a foliation by spheres $\{\Sigma_R\}$ with constant mean curvature $H(\Sigma_R) = \frac{2}{R} + O(R^{-1-q})$ in the exterior region of M. Each leaf Σ_R is a $c_0 R^{1-q}$-graph over $S_R(C)$ and is strictly stable.

Set $r(x) = (\Sigma(x_i)^2)^{1/2}$. For the constant mean curvature sphere Σ which separates infinity from K, we define

$$\begin{aligned} r_0(\Sigma) &= \inf\{r(x) | x \in \Sigma\} \\ r_1(\Sigma) &= \sup\{r(x) | x \in \Sigma\} \end{aligned} \tag{1.10}$$

Theorem 1.4. (Uniqueness) Assume that (M, g, K) is AF-RT with $q \in (\frac{1}{2}, 1]$ and $m > 0$. There exists σ_1 and C_1 so that if Σ has the following properties:

- Σ is topologically a sphere
- Σ has constant mean curvature $H = H(\Sigma_R)$ for some $R \geq \sigma_1$
- Σ is stable
- $r_1 \leq C_1 r_0^{\frac{1}{a}}$ for some a satisfying $\frac{5-q}{2(2+q)} < a \leq 1$

then $\Sigma = \Sigma_R$.

Our main uniqueness result is

Theorem 1.5. Suppose (M, g, K) is AF-RT 3-manifold with positive mass, and g can be expressed on the end $M \setminus \widetilde{K}$ as follows:

$$g_{ij} = \delta_{ij} + h^1_{ij}(\theta)/r + Q \tag{1.11}$$

where $\theta = (\theta_1, \theta_2)$ is the coordinate on $S^2 \subset R^3$. If g satisfies the following properties:

- $h^1_{ij}(\theta) \in C^5(S^2)$
- $Q = O_5(|x|^{-2})$

Then for any $k > 2$, there exists some $\varepsilon > 0$ depending on k such that if

$$\|h_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{k,2}(S^2)} \leq \varepsilon, \tag{1.12}$$

there is a compact domain $\widetilde{K}$ such that if a foliation $\{\Sigma\}$ of stable constant mean curvature spheres which separates infinity from $\widetilde{K}$ have

$$\lim_{r_0 \to \infty} \frac{\log(r_1(\Sigma))}{r_0(\Sigma)^{1/4}} = 0 \tag{1.13}$$

then this foliation is the same one as in Theorem1.3.

Remark 1.6. If we replace $\|h_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{k,2}} \leq \varepsilon$ by $\|h_{ij}(\theta) - C\delta_{ij}(\theta)\|_{W^{k,2}} \leq \varepsilon$ for any constant $C > 0$, we can also get this theorem, but ε will depend on k and C.

Remark 1.7. RT condition is needed to apply the theorems of Huang and if we assume the scalar curvature satisfies $R = O(r^{-3-\varepsilon})$ for some $\varepsilon > 0$, then we do not need the constraint equation.

Remark 1.8. Here I can only deal with the case when $q = 1$. When $q \in (1/2, 1)$ it seems that $\|h_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{k,2}(S^2)} \leq \varepsilon$ is not a proper condition.

The above theorem is about the uniqueness of the foliation. For the uniqueness of a single CMC sphere we have:

Corollary 1.9. We assume the same condition on the metric as the above Theorem. Then for any constants $C > 0$ and $\beta > 0$, there exist some compact set $K(C, \beta) \subset M$, such that any stable sphere Σ that separates $K(C, \beta)$ from the infinity with

$$\frac{(\log(r_1(\Sigma)))^{1+\beta}}{r_0(\Sigma)^{1/4}} \leq C \tag{1.14}$$

belongs to the foliation in Theorem 1.3.

The paper is organized much like [9]: In Section 2 we do apriori estimate on the stable constant mean curvature sphere based on the Simon's identity. In Section 3, we introduce blow-down analysis in three different scales. In Section 4 we recall the asymptotic analysis from [10] and prove a technical lemma. In Section 5 we introduce the asymptotically harmonic coordinate. In Section 6 we introduce a sense of the center of mass and prove the theorem.

2 Curvature estimates

From now on let Σ be a constant mean curvature sphere in the asymptotically flat end (M, g)which separates the origin from the infinity. First we have the following estimate as Lemma 5.2 in [5].

Lemma 2.1. Let $X = x^i \frac{\partial}{\partial x^i}$ be the Euclidean coordinate vectorfield and $r = (\Sigma(x^i)^2)^{1/2}$ and with respect to the metric g, v is the outward normal vector field , $d\mu$ is the volume form of Σ. Then we have the estimate:

$$\int_\Sigma < X, v >^2 r^{-4} d\mu \leq H^2 |\Sigma| \tag{2.1}$$

Moreover for each $a \geq a_0 > 2$ and r_0 sufficiently large , we have:

$$\int_\Sigma r^{-a} d\mu \leq C(a_0) r_0^{2-a} H^2 |\Sigma| \tag{2.2}$$

Proof. Because the mean curvature H is constant, then for some smooth vector field Y on Σ , we have the divergence formula:

$$\int_\Sigma div_\Sigma Y d\mu = H \int_\Sigma < Y, v > d\mu. \tag{2.3}$$

We choose $Y = Xr^{-a}$, $a \geq 2$ and e_α is the orthonormal basis on Σ , $\alpha = 1, 2$. Suppose $e_\alpha = a_\alpha^i \frac{\partial}{\partial x^i}$, it is obvious that a_α^i is bounded because the manifold is asymptotically flat. Then we have:

$$\begin{aligned} div_\Sigma Y \quad &= div_\Sigma (Xr^{-a}) =< \nabla_{e_\alpha} (Xr^{-a}), e_\alpha > \\ &= r^{-a} div_\Sigma X - ar^{-a-2} a_\alpha^i a_\alpha^j x^i x^j + O(r^{-a-q}) \\ &= r^{-a} div_\Sigma X - \alpha r^{-a-2} |X^\tau|^2 + O(r^{-a-q}) \end{aligned} \tag{2.4}$$

where X^τ is the tangent projection of X.

$$|div_\Sigma X - 2| = O(r^{-q}) \tag{2.5}$$

Note that$|X^\tau|^2 = r^2 - <X, v>^2 + O(r^{2-q})$, then combine all of these we have:

$$|(2-a)\int_\Sigma r^{-a} d\mu + a\int_\Sigma <X, v>^2 r^{-a-2} d\mu - H\int_\Sigma <X, v> r^{-a} d\mu| \leq C\int_\Sigma r^{-a-q} d\mu \tag{2.6}$$

Choosing $a = 2$, from Hölder inequality , we have:

$$\int_\Sigma <X, v>^2 r^{-4} d\mu \leq \frac{1}{4}H^2|\Sigma| + C\int_\Sigma r^{-2-q} d\mu \tag{2.7}$$

then choose $a = 2 + q$,

$$\int_\Sigma r^{-2-q} d\mu \leq 4r_0^{-q}(\int_\Sigma <X, v>^2 r^{-4} d\mu + H^2|\Sigma| + C\int_\Sigma r^{-2-q} d\mu) \tag{2.8}$$

then combine this with (2.7),we have:

$$\int_\Sigma <X, v>^2 r^{-4} d\mu \leq H^2|\Sigma| \tag{2.9}$$

then again from (2.6), we have for $a \geq a_0 > 2$, we derive:

$$\int_\Sigma r^{-a} \leq C(a_0 - 2)^{-1} r_0^{2-a} H^2|\Sigma| \tag{2.10}$$

Then we can derive the integral estimate for $|\mathring{A}|$ from the stability of the surface as in [5] Proposition 5.3, i.e. we have

Lemma 2.2. Suppose Σ is a stable constant mean curvature sphere in the asymptotically flat manifold. We have for r_0 sufficiently large

$$\int_\Sigma |\mathring{A}|^2 d\mu \leq C r_0^{-q} \tag{2.11}$$

$$H^2|\Sigma| \leq C \tag{2.12}$$

$$\int_\Sigma H^2 d\mu = 16\pi + O(r_0^{-q}) \tag{2.13}$$

Proof. Since Σ is stable , we have

$$\int_\Sigma |\nabla f|^2 d\mu \geq \int_\Sigma (|A|^2 + Ric(v, v)) f^2 d\mu \tag{2.14}$$

for any function f , with$\int_\Sigma f d\mu = 0$, where A is the second fundamental form of Σ and Ric is the Ricci curvature of M

Choose ψ to be a conformal map of degree 1 from Σ to the standard S^2 in R^3. Each component ψ_i of ψ can be chosen such that $\int \psi_i d\mu = 0$, see [8] . We have for each ψ_i

$$\int_\Sigma |\nabla\psi_i|^2 d\mu = \frac{8\pi}{3} \tag{2.15}$$

since $\sum \psi_i^2 \equiv 1$ we conclude that

$$\int_\Sigma |A|^2 + Ric(v,v) d\mu \leq 8\pi \tag{2.16}$$

From Gauss equation

$$\frac{1}{2}|A|^2 + Ric(v,v) - \frac{1}{2}R + K = \frac{1}{2}H^2 \tag{2.17}$$

we have:

$$|A|^2 + Ric(v,v) = \frac{1}{2}|\mathring{A}|^2 + \frac{3}{4}H^2 + \frac{1}{2}R - K \tag{2.18}$$

where K is the Gauss curvature of Σ and $\mathring{A}$ is defined as $\mathring{A}_{ij} = A_{ij} - \frac{H}{2}g_{ij}$

Then we have:

$$\int_\Sigma \frac{1}{2}|\mathring{A}|^2 + \frac{3}{4}H^2|\Sigma| \leq 12\pi + r_0^{-q}H^2|\Sigma| \tag{2.19}$$

because $R = O(r^{-2-2q})$.

So we have $H^2|\Sigma| \leq 16\pi$.

Using the Gauss equation in a different way, we have

$$\begin{aligned}
&\int_\Sigma |\mathring{A}|^2 d\mu = \int_\Sigma |A|^2 - \frac{H^2}{2} d\mu \\
&= \frac{1}{2}\int_\Sigma |A|^2 + Ric(v,v)d\mu + \frac{1}{2}\int_\Sigma R - 3Ric(v,v) - 2K d\mu \\
&\leq \int_\Sigma r^{-2-q} d\mu \\
&= O(r_0^{-q}).
\end{aligned} \tag{2.20}$$

Then from Gauss equation (2.17) again, we have:

$$\int_\Sigma H^2 d\mu = 4\int_\Sigma K d\mu + O(r_0^{-q}) = 16\pi + O(r_0^{-q}) \tag{2.21}$$

Lemma 2.3. Suppose that M is a constant mean curvature surface in an asymptotically flat end $(R^3 \setminus B_1(0), g)$. Then

$$\int_\Sigma H_e^2 d\mu_e = 16\pi + O(r_0^{-q}) \tag{2.22}$$

Proof. We follow the calculation of Huisken and Ilmanen [4],

$$g_{ij} = \delta_{ij} + h_{ij} \tag{2.23}$$

Suppose

$$g_{ij}|_\Sigma = f_{ij}, \delta_{ij}|_\Sigma = \varepsilon_{ij} \tag{2.24}$$

f^{ij} and ϵ^{ij} are the corresponding inverse matrices. $v, \omega, A, H, d\mu$ represents the normal vector , the dual form of v, the second fundamental form , the mean curvature and the volume form of Σ in the metric g. And $v_e, \omega_e, A_e, H_e, \mu_e$ represents the corresponding ones in Euclidean metric. Through easy calculation, we have

$$f^{ij} - \varepsilon^{ij} = -f^{ik}h_{kl}f^{lj} \pm C|h|^2 \tag{2.25}$$
$$g^{ij} - \delta^{ij} = -g^{ik}h_{kl}g^{lj} \pm C|h|^2 \tag{2.26}$$

$$\omega = \frac{\omega_e}{|\omega_e|} \qquad v^i = g^{ij}\omega_j \tag{2.27}$$

$$(\omega_e)_i = \omega_i \pm C|P| \quad v_e^i = v^i + C|h| \quad 1 - |\omega_e| = \frac{1}{2}h_{ij}v^iv^j \tag{2.28}$$

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\overline{\nabla}_i h_{jl} + \overline{\nabla}_j h_{il} - \overline{\nabla}_l h_{ij}) \pm C|h| \pm C|\overline{\nabla} h| \tag{2.29}$$

and $\Gamma_{ij}^k$is the Christoffel symbol for $\overline{\nabla} - \overline{\nabla}_e$,where we denote the gradient for the metric g and δ by $\overline{\nabla}$ and $\overline{\nabla}_e$.

We have the formula:

$$|\omega_e|_g A_{ij} = (A_e)_{ij} - (\omega_e)_k \Gamma_{ij}^k \tag{2.30}$$

So we have

$$\begin{aligned} H - H_e &= f^{ij}A_{ij} - \varepsilon^{ij}(A_e)_{ij} \\ &= (f^{ij} - \varepsilon^{ij})A_{ij} + \varepsilon^{ij}A_{ij}(1 - |\omega_e|_g) + \varepsilon^{ij}(|\omega_e|_g A_{ij} - (A_e)_{ij}) \end{aligned} \tag{2.31}$$

from (2.25)(2.26)(2.28), we have

$$\varepsilon^{ij}A_{ij}(1 - |\omega_e|_g) = \frac{1}{2}Hv^iv^jh_{ij} \pm C|h|^2|A| \tag{2.32}$$

and using (2.25)(2.26)(2.28)(2.29)(2.30)we have:

$$
\begin{aligned}
&\varepsilon^{ij}(|\omega_e|A_{ij}-(A_e)_{ij})\\
&=-\varepsilon^{ij}(\omega_e)_k\Gamma^k_{ij}\\
&=-\frac{1}{2}f^{ij}\omega_k g^{kl}(\overline{\nabla}_i h_{jl}+\overline{\nabla}_j h_{il}-\overline{\nabla}_l h_{ij})\pm C|h||\overline{\nabla}h|\\
&=-f^{ij}v^l\overline{\nabla}_i h_{jl}+\frac{1}{2}f^{ij}v^l\overline{\nabla}_l h_{ij}\pm C|h||\overline{\nabla}h| \qquad (2.33)
\end{aligned}
$$

At last , we have

$$
\begin{aligned}
&H-H_e=-f^{ik}h_{kl}f^{lj}A_{ij}+\frac{1}{2}Hv^iv^jh_{ij}-f^{ij}v^l\overline{\nabla}_i h_{jl}\\
&+\frac{1}{2}f^{ij}v^l\overline{\nabla}_l h_{ij}\pm C|h||\overline{\nabla}h|\pm C|h|^2|A| \qquad (2.34)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Sigma H_e^2 d\mu_e=(1+O(r_0^{-q}))\int_\Sigma H_e^2 d\mu\\
&\leq(1+O(r_0^{-q}))(\int_\Sigma H^2 d\mu+\int_\Sigma (H_e-H)^2+2|H(H_e-H)|d\mu)\\
&\leq(1+O(r_0^{-q}))(16\pi+O(r_0^{-q})+\int_\Sigma (H_e-H)^2\\
&+(\int_\Sigma H^2 d\mu)^{\frac{1}{2}}(\int_\Sigma (H_e-H)^2 d\mu)^{\frac{1}{2}}) \qquad (2.35)
\end{aligned}
$$

$$
\begin{aligned}
&\int (H_e-H)^2 d\mu\leq\int O(|x|^{-2q})|A|^2+H^2O(|x|^{-2q})+O(|x|^{-2-2q})d\mu\\
&\leq\int O(|x|^{-2q})H^2+O(|x|^{-2q})|\mathring{A}|^2+O(|x|^{-2-2q})d\mu\\
&=O(r_0^{-2q}) \qquad (2.36)
\end{aligned}
$$

so we have

$$
\int_\Sigma H_e^2 d\mu_e\leq 16\pi+O(r_0^{-q}) \qquad (2.37)
$$

On the other hand, by Euler formula,

$$
K_e=\frac{1}{4}H_e^2-\frac{1}{2}|\mathring{A}_e|^2. \qquad (2.38)
$$

So we have

$$
\int H_e^2 d\mu_e\geq 16\pi \qquad (2.39)
$$

which implies:

$$
\int_\Sigma H_e^2 d\mu_e=16\pi+O(r_0^{-q}) \qquad (2.40)
$$

Based on Michael and Simon, we have the following Sobolev inequality.

Lemma 2.4. Suppose that Σ is a constant mean curvature surface in an asymptotically flat end$(R^3\backslash B_1(0), g)$ with $r_0(\Sigma)$ sufficiently large, and that $\int_\Sigma H^2 \leq C$. Then

$$(\int_\Sigma f^2 d\mu)^{\frac{1}{2}} \leq C(\int_\Sigma |\nabla f| d\mu + \int_\Sigma H|f| d\mu). \tag{2.41}$$

Proof. Note that it is valid for the surface in Euclidean Space. So by the uniform equivalence of the metric g and δ , we have:

$$(\int |f|^2 d\mu)^{\frac{1}{2}} \leq C(\int |f|^2 d\mu_e)^{\frac{1}{2}} \leq C(\int |\nabla f| + H|f| + |H - H_e||f| d\mu) \tag{2.42}$$

To bound the last term on the right , we have:

$$\begin{aligned}\int |H - H_e||f| d\mu &\leq \int O(|x|^{-q})|A||f| + O(|x|^{-q})H|f| \\ &\quad + O(|x|^{-1-q})|f| d\mu \\ &\leq O(r_0^{-q}) \int H|f| + (\int |\mathring{A}|^2 d\mu)^{\frac{1}{2}} O(r_0^{-q}) \|f\|_{L^2} \\ &\quad + O(r_0^{-q}) \|f\|_{L^2}\end{aligned} \tag{2.43}$$

So we can choose r_0 sufficiently large and get the desired result.

Lemma 2.5. Suppose that Σ is a constant mean curvature surfaces in an asymptotically flat end $(R^3 \setminus B_1(0), g)$ with $r_0(\Sigma)$ sufficiently large, then:

$$C_1 H^{-1} \leq diam(\Sigma) \leq C_2 H^{-1} \tag{2.44}$$

In particular, if the surface Σ separates the infinity from the compact part, then:

$$C_1 H^{-1} \leq r_1(\Sigma) \leq C_2 H^{-1} \tag{2.45}$$

Proof. We already know that:

$$\int_\Sigma H_e^2 d\mu_e = 16\pi + O(r_0^{-q}) \tag{2.46}$$

Then from [7] Lemma 1.1, we know that

$$\sqrt{\frac{2|\Sigma|_e}{F(\Sigma)}} \leq diam(\Sigma) \leq C\sqrt{|\Sigma|_e F(\Sigma)} \tag{2.47}$$

where $F(\Sigma) = \frac{1}{2}\int_\Sigma H_e^2$ is the Willmore functional and $|\Sigma|_e$ is the volume of Σ with respect to the Euclidean metric. But the Euclidean metric is uniformly equivalent to g, so we get the result.

Now to get the pointwise estimate for $\mathring{A}$,we use the Simons identity and the Moser's iteration argument.

Lemma 2.6. (Simons identity [11]) Suppose N is a hypersurface in a Riemannian manifold (M,g) , then the second fundamental form satisfies the following identity:

$$\Delta A_{ij} = \nabla_i\nabla_j H + HA_{ik}A_{jk} - |A|^2 A_{ij} + HR_{3i3j} - A_{ij}R_{3k3k} + A_{jk}R_{klil} + A_{ik}R_{kljl} - 2A_{lk}R_{iljk} + \overline{\nabla}_j R_{3kik} + \overline{\nabla}_k R_{3ijk} \quad (2.48)$$

where R_{ijkl} and $\overline{\nabla}$ are the curvature and gradient operator of (M,g), then from this we easily deduce for constant mean curvature surface we have the next inequality for $\mathring{A}$:

$$-|\mathring{A}|\Delta|\mathring{A}| \leq |\mathring{A}|^4 + CH|\mathring{A}|^3 + CH^2|\mathring{A}|^2 + C|\mathring{A}|^2|x|^{-2-q} + CH|\mathring{A}||x|^{-2-q} + C|\mathring{A}||x|^{-3-q} \quad (2.49)$$

We also need an inequality for $\nabla\mathring{A}$ because we also want to estimate the higher derivative:

$$-|\nabla\mathring{A}|\Delta|\nabla\mathring{A}| \leq C|\nabla\mathring{A}|^2(|\mathring{A}|^2 + H|\mathring{A}| + H^2 + O(|x|^{-2-q})) \quad (2.50)$$
$$+|\nabla\mathring{A}|((|\mathring{A}|^2 + H|\mathring{A}| + H^2)O(|x|^{-2-q}) + (|\mathring{A}| + H)O(|x|^{-3-q}) + O(|x|^{-4-q}))$$

Lemma 2.7.

$$\|\mathring{A}^2\|_{L^2} + \|\nabla|\mathring{A}|\|_{L^2} + \|\nabla\mathring{A}\|_{L^2} + \|H|\mathring{A}|\|_{L^2} \leq Cr_0^{-1-q} \quad (2.51)$$

Proof. See [2] Lemma 4.5

Then we can get the pointwise estimates for $\mathring{A}$ and $\nabla\mathring{A}$.

Theorem 2.8. [9]Suppose that $(R^3 \setminus B_1(0), g)$ is an asymptotically flat end. Then there exist positive numbers σ_0, δ_0 such that for any constant mean curvature surface in the end, which separates the infinity from the compact part, we have:

$$|\mathring{A}|^2(x) \leq C|x|^{-2}\int_{B_{\delta_0|x|}(x)} |\mathring{A}|^2 d\mu + C|x|^{-2-2q} \leq C|x|^{-2}r_0^{-q} \quad (2.52)$$

$$|\nabla\mathring{A}|^2(x) \leq C|x|^{-2}\int_{B_{\delta_0|x|}(x)} |\nabla\mathring{A}|^2 d\mu + C|x|^{-4-2q} \leq C|x|^{-2}r_0^{-2-2q} \quad (2.53)$$

provided that $r_0 \geq \sigma_0$.

Proof. In the Sobolev inequality (2.41) we take $f = u^2$, then we get:

$$(\int_\Sigma u^4 d\mu)^{\frac{1}{2}} \leq C(2\int_\Sigma |u||\nabla u| d\mu + \int_\Sigma Hu^2 d\mu)$$
$$\leq C(\int_\Sigma u^2)^{\frac{1}{2}}(\int_\Sigma |\nabla u|^2 d\mu)^{\frac{1}{2}} + C(\int_{supp(u)} H^2 d\mu)^{\frac{1}{2}}(\int_\Sigma u^4 d\mu)^{\frac{1}{2}} \quad (2.54)$$

Lemma 2.9. For any $\varepsilon > 0$, we can find a uniform δ_0 sufficiently small such that if for any $x \in \Sigma$, we have that:

$$\int_{B_{\delta_0|x|}(x)} H^2 \leq \varepsilon \tag{2.55}$$

Proof. In fact we need only to prove that there exist C

$$|B_{\delta_0|x|}(x)| \leq C\delta_0^2|x|^2 \tag{2.56}$$

because then,

$$H^2|B_{\delta_0|x|}(x)| \leq C\delta_0^2|x|^2H^2 \leq C\delta_0^2 \tag{2.57}$$

From [7] the proof of lemma 1.1, we know that, for any $x \in \Sigma$, $B_\sigma(x)$ denotes the Euclidean ball of radius σ with center x in R^3, $\Sigma_\sigma = \Sigma \cap B_\sigma(x)$, then there exists C such that for $0 < \sigma \leq \rho < \infty$

$$\sigma^{-2}|\Sigma_\sigma| \leq C(\rho^{-2}|\Sigma_\rho| + F(\Sigma_\rho)) \tag{2.58}$$

where $F(\Sigma_\rho)$ is the Willmore functional. C doesn't depend on Σ, σ, ρ.

Let $\rho \to \infty$, $\rho^{-2}|\Sigma_\rho| \to 0$, so we have:

$$\sigma^{-2}|\Sigma_\sigma| \leq CF(\Sigma) \leq C \tag{2.59}$$

so we prove the lemma.

So if $supp(u) \subset B_{\delta_0|x|}(x)$, we have the following scaling invariant Sobolev inequality:

$$(\int_\Sigma u^4 d\mu)^{\frac{1}{2}} \leq C(\int_\Sigma u^2)^{\frac{1}{2}}(\int_\Sigma |\nabla u|^2 d\mu)^{\frac{1}{2}} \tag{2.60}$$

Lemma 2.10. [9] Suppose that a nonnegative function $v \in L^2$ solves

$$-\Delta v \leq fv + h \tag{2.61}$$

on $B_{2R}(x_0)$, where

$$\int_{B_{2R}(x_0)} f^2 d\mu \leq CR^{-2} \tag{2.62}$$

and $h \in L^2(B_{2R}(x_0))$. And suppose that

$$(\int_\Sigma u^4 d\mu)^{\frac{1}{2}} \leq C(\int_\Sigma u^2)^{\frac{1}{2}}(\int_\Sigma |\nabla u|^2 d\mu)^{\frac{1}{2}} \tag{2.63}$$

holds for all u with support inside $B_{2R}(x_0)$. Then

$$\sup_{B_R(x_0)} v \leq CR^{-1}\|v\|_{L^2(B_{2R}(x_0))} + CR\|h\|_{L^2(B_{2R(x_0)})} \tag{2.64}$$

See [9] Lemma 2.6 for the proof of this lemma.

Then we find that:

$$
\begin{aligned}
&-\Delta|\mathring{A}| \le (|\mathring{A}|^2 + H^2 + H|\mathring{A}| + C|x|^{-2-q})|\mathring{A}| + CH|x|^{-2-q} + C|x|^{-3-q} \\
&= f_1|\mathring{A}| + h_1 \qquad (2.65)
\end{aligned}
$$

$$
\begin{aligned}
&-\Delta|\nabla\mathring{A}| \le C|\nabla\mathring{A}|(|\mathring{A}|^2 + H|\mathring{A}| + H^2 + O(|x|^{-3})) \\
&+((|\mathring{A}|^2 + H|\mathring{A}| + H^2)O(|x|^{-3}) + (|\mathring{A}| + H)O(|x|^{-4}) + O(|x|^{-5})) \\
&= f_2|\nabla\mathring{A}| + h_2. \qquad (2.66)
\end{aligned}
$$

We need to prove that $\|f_1\|^2_{L^2(B_{2\delta_0|x|}(x))}, \|f_2\|^2_{L^2(B_{2\delta_0|x|}(x))} \le C|x|^{-2}$, see [9] Theorem 2.5 for the proof. and it is easy to show that $\|h_1\|^2_{L^2(B_{2\delta_0|x|}(x))} = O(|x|^{-4-2q})$ and $\|h_2\|^2_{L^2(B_{2\delta_0|x|}(x))} = O(|x|^{-6-2q})$.

Remark 2.11. We can also do the same kind of estimate for $\nabla^2\mathring{A}$, where we need the third derivative of curvature. It is needed by the $C^{2,\alpha}$ convergence of the surface in the next section. This is the reason why we require the metric g to be smooth up to 5th order.

3 Blow down analysis

Now like [9], we blow down the surface in three different scales. First we consider

$$\widetilde{N} = \frac{1}{2}HN = \{\frac{1}{2}Hx : x \in N\} \qquad (3.1)$$

Suppose that there is a sequence of constant mean curvature surfaces $\{N_i\}$ such that

$$\lim_{i\to\infty} r_0(N_i) = \infty \qquad (3.2)$$

we have known that

$$\lim_{i\to\infty}\int_{N_i} H_e^2 d\sigma = 16\pi \qquad (3.3)$$

Hence, by the curvature estimates established in the previous section combining the proof of Theorem 3.1 in [7], we have

Lemma 3.1. Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$ and that

$$\lim_{i\to\infty} r_0(N_i) = \infty. \qquad (3.4)$$

And suppose that N_i separates the infinity from the compact part. Then, there is a subsequence of $\{\widetilde{N}_i\}$ which converges in Gromov-Hausdorff distance to a round sphere $S_1^2(a)$ of radius 1 and centered at $a \in R^3$. Moreover,the convergence is in $C^{2,\alpha}$ sense away from the origin.

Then, we use a smaller scale r_0 to blow down the surface

$$\widehat{N} = r_0(N)^{-1}N = \{r_0^{-1}x : x \in N\}. \tag{3.5}$$

Lemma 3.2. Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$ and that

$$\lim_{i\to\infty} r_0(N_i) = \infty. \tag{3.6}$$

And suppose that

$$\lim_{i\to\infty} r_0(N_i)H(N_i) = 0. \tag{3.7}$$

Then there is a subsuquence of $\{\widehat{N}_i\}$ converges to a 2-plane at distance 1 from the origin. Moreover the convergence is in $C^{2,\alpha}$ in any compact set of R^3.

We must understand the behavior of the surfaces N_i in the scales between $r_0(N_i)$ and $H^{-1}(N_i)$. We consider the scale r_i such that

$$\lim_{i\to\infty} \frac{r_0(N_i)}{r_i} = 0 \qquad \lim_{i\to\infty} r_i H(N_i) = 0 \tag{3.8}$$

and blow down the surfaces

$$\overline{N}_i = r_i^{-1}N = \{r_i^{-1}x : x \in N\}. \tag{3.9}$$

Lemma 3.3. Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$ and that

$$\lim_{i\to\infty} r_0(N_i) = \infty \tag{3.10}$$

And suppose that r_i are such that

$$\lim_{i\to\infty} \frac{r_0(N_i)}{r_i} = 0 \qquad \lim_{i\to\infty} r_i H(N_i) = 0 \tag{3.11}$$

Then there is a subsequence of $\{\overline{N}_i\}$ converges to a 2-plane at the origin in Gromov-Hausdorff distance. Moreover the convergence is $C^{2,\alpha}$ in any compact subset away from the origin.

4 Asymptotically analysis

First we revise Proposition 2.1 in [10]. We prove a different version. Let us denote:

$$\|u\|_{1,i}^2 = \int_{[(i-1)L, iL]\times S^1} |u|^2 + |\nabla u|^2 dt d\theta \tag{4.1}$$

Lemma 4.1. Suppose $u \in W^{1,2}(\Sigma, R^k)$ satisfies

$$\Delta u + A \cdot \nabla u + B \cdot u = h \tag{4.2}$$

in Σ, where $\Sigma = [0, 3L] \times S^1$. And suppose that L is given and large. Then there exists a positive number δ_0 such that if

$$|h|_{L^2(\Sigma)} \le \delta_0 \max_{1 \le i \le 3} |u|_{1,i} \tag{4.3}$$

and

$$|A|_{L^\infty(\Sigma)} \le \delta_0 \qquad |B|_{L^\infty(\Sigma)} \le \delta_0 \tag{4.4}$$

then,

(a)$\|u\|_{1,3} \le e^{-\frac{1}{2}L}\|u\|_{1,2}$ implies $\|u\|_{1,2} < e^{-\frac{1}{2}L}\|u\|_{1,1}$

(b)$\|u\|_{1,1} \le e^{-\frac{1}{2}L}\|u\|_{1,2}$ implies $\|u\|_{1,2} < e^{-\frac{1}{2}L}\|u\|_{1,3}$

(c)If both $\int_{L \times S^1} u d\theta$ and $\int_{2L \times S^1} u d\theta \le \delta_0 \max_{1 \le i \le 3} \|u\|_{1,i}$, then either $\|u\|_{1,2} < e^{-\frac{1}{2}L}\|u\|_{1,1}$ or $\|u\|_{1,2} < e^{-\frac{1}{2}L}\|u\|_{1,3}$

Proof. Suppose that $u \in W^{1,2}(\Sigma)$ and u is harmonic, we can deduce that u satisfies (a)(b)(c')with

(c')If both $\int_{L \times S^1} u d\theta$ and $\int_{2L \times S^1} u d\theta = 0$, then either $\|u\|_{1,2} < e^{-\frac{1}{2}L}\|u\|_{1,1}$ or $\|u\|_{1,2} < e^{-\frac{1}{2}L}\|u\|_{1,3}$

A harmonic function u can be written as:

$$u = a_0 + b_0 t + \sum_{n=1}^{\infty}\{e^{nt}(a_n \cos n\theta + b_n \sin n\theta) + e^{-nt}(a_{-n}\cos n\theta + b_{-n}\sin n\theta)\} \tag{4.5}$$

Then it follows that:

$$\begin{aligned}
&\|u\|_{1,i}^2 = 2\pi((a_0^2 + b_0^2)L + a_0 b_0 L^2(2i-1) + \frac{1}{3}b_0^2 L^3(3i^2 - 3i + 1)) \\
&+\frac{\pi}{2}\sum_{n=1}^{\infty}\{\frac{e^{2nL-1}}{n}(e^{2(i-1)nL}(a_n^2 + b_n^2) + e^{-2niL}(a_{-n}^2 + b_{-n}^2)) + 4L(a_n a_{-n} + b_n b_{-n})\} \\
&+\pi\sum_{n=1}^{\infty}\{\frac{e^{2nL-1}}{n}(e^{2(i-1)nL}(n^2 a_n^2 + n^2 b_n^2) + e^{-2niL}(n^2 a_{-n}^2 + n^2 b_{-n}^2)) \\
&+4L(n^2 a_n a_{-n} + n^2 b_n b_{-n})\}
\end{aligned} \tag{4.6}$$

$i = 1, 2, 3$

If L is fixed and sufficiently large, then we have

$$\|u\|_{1,2}^2 < \frac{1}{2}(e^L\|u\|_{1,3}^2 + e^{-L}\|u\|_{1,1}^2) \tag{4.7}$$

which implies (a). We get (b) in the same way. For (c'), we have $a_0 = b_0 = 0$ then we have

$$\|u\|_{1,2}^2 < \frac{1}{2}e^{-L}(\|u\|_{1,3}^2 + \|u\|_{1,1}^2) \tag{4.8}$$

which implies (c')

The second step is to pass limits. If the proposition were false, then one would have a sequence of $\delta_k \to 0$ and a sequence of solution u_k with $\|h_k\|_{L^2} \leq \delta_k$ $|A_k| \leq \delta_k$ and $|B_k| \leq \delta_k$ solves:

$$\Delta u_k + A_k \cdot \nabla u_k + B_k \cdot u_k = h_k \tag{4.9}$$

We may assume $\max_{1\leq i\leq 3} \|u_k\|_{1,i} = 1$ otherwise we can normalize them. Then we know that there is a subsequence that converges to some $u \in W^{1,2}(\Sigma)$ weakly. And u is a harmonic function. From the interior $W^{2,p}$ estimate we know the convergence is strongly $W^{1,2}$ in I_2, which implies that u is not trivially zero. Because, with the assumption of the proof by contradiction, the middle one is the largest.

And because $u_i \rightharpoonup u$ weakly in $W^{1,2}(\Sigma)$ sense. So $u_i \rightharpoonup u$ in $W^{1,2}(I_1)$ and $W^{1,2}(I_3)$ sense, then we have:

$$\liminf_{i\to\infty} \|u_i\|_{1,1} \geq \|u\|_{1,1}, \liminf_{i\to\infty} \|u_i\|_{1,3} \geq \|u\|_{1,3} \tag{4.10}$$

and

$$\lim_{i\to\infty} \|u_i\|_{1,2} = \|u\|_{1,2} \tag{4.11}$$

then u_i converges to some non-trivial harmonic function u which violates one of (a)(b) or (c), which proves the lemma.

From now on we assume $q = 1$.

Given a surface N in R^3, recall from, for example, (8.5) in [6], that

$$\Delta_e v + |\nabla_e v|^2 v = \nabla_e H_e \tag{4.12}$$

where v is the Gauss map from $N \to S^2$. For the constant mean curvature surfaces in the asymptotically flat end $(R^3 \setminus B_1(0), g)$, we have

Lemma 4.2.

$$|\nabla_e H_e|(x) \leq C|x|^{-2} r_0^{-1} \tag{4.13}$$

Proof. Because the metric g and the Euclidean metric are uniformly equivalent. So we just prove that

$$|\nabla H_e|(x) \leq C|x|^{-2} r_0^{-1} \tag{4.14}$$

From (2.34), we know that:

$$
\begin{aligned}
|\nabla H_e| &\le |\overline{\nabla} h_{ij}||A| + |h_{ij}||A|^2 + |h_{ij}||\nabla \mathring{A}_{ij}| + H|A||h_{ij}| + H|\overline{\nabla} h_{ij}| \\
&+|A||\overline{\nabla} h_{ij}| + |\overline{\nabla}^2 h| \\
&\le |x|^{-2} r_0^{-1} \qquad (4.15)
\end{aligned}
$$

Suppose Σ is a constant mean curvature surface in the asymptotically flat end. Set

$$A_{r_1,r_2} = \{x \in \Sigma : r_1 \le |x| \le r_2\} \tag{4.16}$$

and $A^0_{r_1,r_2}$ stands for the standard annulus in R^2. We are concerned with the behavior of v on $A_{Kr_0(\Sigma),sH^{-1}(\Sigma)}$ of Σ where K will be fixed large and s will be fixed small. The lemma below gives us a good coordinate on the surface.

Lemma 4.3. Suppose Σ is a constant mean curvature surface in a given asymptotically flat end $(R^3 \setminus B_1(0), g)$. Then, for any $\varepsilon > 0$ and L fixed and large, there are M,s and K such that, if $r_0 \ge M$ and $Kr_0(\Sigma) < r < sH^{-1}(\Sigma)$, then $(r^{-1}A_{r,e^L r}, r^{-2}g_e)$ may be represented as $(A^0_{1,e^L}, \overline{g})$ and

$$\|\overline{g} - |dx|^2\|_{C^1(A^0_{1,e^L})} \le \varepsilon. \tag{4.17}$$

In other words, in the cylindrical coordinates $(S^1 \times [\log r, L + \log r, \overline{g}_c])$

$$\|\overline{g}_c - (dt^2 + d\theta^2)\|_{C^1(S^1\times[\log r, L+\log r])} \le \varepsilon \tag{4.18}$$

Proof. Suppose this is not true. Then we can assume that such K (or such s) cannot be found. Then by Lemma 3.2. for some $\varepsilon_0 > 0$, there is a sequence Σ_n with $r_0(\Sigma_n) \to \infty$, and $\widetilde{l}_n \to \infty$, such that:

$$((Kr_0 e^{\widetilde{l}_n L})^{-1} A_{Kr_0 e^{\widetilde{l}_n L}, Kr_0 e^{(\widetilde{l}_n+1)L}}, (Kr_0 e^{\widetilde{l}_n L})^{-2} g_e) \tag{4.19}$$

is not ε_0 close to $(A^0_{1,e^L}, \overline{g})$.

By Lemma 3.1. We know that

$$\frac{Kr_0 e^{\widetilde{l}_n L}}{sH^{-1}(\Sigma_n)} \to 0 \tag{4.20}$$

must hold because we have choose s sufficiently small.

So if we assume $r_n = Kr_0 e^{\widetilde{l}_n L}$, we have:

$$\lim_{n\to\infty} \frac{r_n}{Kr_0} = \infty, \lim_{n\to\infty} \frac{r_n}{sH^{-1}} = 0 \tag{4.21}$$

We blow down the surface using r_n, and have a contradiction with Lemma 3.3. This proves the lemma.

Now consider the cylindrical coordinates (t,θ) on $(S^1 \times [\log Kr_0, \log sH^{-1}])$, then the tension field

$$|\tau(v)| = r^2|\nabla_e H_e| \leq C r_0^{-1} \tag{4.22}$$

for $t \in [\log K r_0, \log sH^{-1}]$. Thus,

$$\int_{S^1\times[t,t+L]} |\tau(v)|^2 dt d\theta \leq C r_0^{-2} \tag{4.23}$$

Let I_i stand for $S^1 \times [\log K r_0 + (i-1)L, \log K r_0 + iL]$, and N_i stand for $I_{i-1} \cup I_i \cup I_{i+1}$. On Σ_n we assume $\log(sH^{-1}) - \log(K r_0) = l_n L$. And like [10], first we prove that,

Lemma 4.4. For each $i \in [3, l_n - 2]$, there exists a geodesic γ such that

$$\int_{I_i} |\widetilde{\nabla}(v-\gamma)|^2 dt d\theta \leq C(e^{-iL} + e^{-(l_n - i)L})s^2 + C r_0^{-1} \tag{4.24}$$

where $\widetilde{\nabla}$ is the gradient on $S^1 \times [\log(K r_0), \log(sH^{-1})]$

Proof. By Theorem2.8, we have

$$[v]_{C^\alpha(I_i)} \leq \|\widetilde{\nabla} v\|_{L^\infty} \leq C(r_0^{-\frac{1}{2}} + s) \tag{4.25}$$

then if r_0 sufficiently large and s sufficiently small, we have $[v]_{C^\alpha(N_i)}$ is very small.

To apply the Lemma 4.1 to prove this lemma we choose to points P and Q on S^2(the image of Gauss map) satisfying

$$\begin{aligned} |P - \frac{1}{2\pi}\int_{(i-1)L\times S^1} v d\theta| &\leq C \max_{(i-1)L\times S^1} |v-P|^2 \\ |Q - \frac{1}{2\pi}\int_{iL\times S^1} v d\theta| &\leq C \max_{iL\times S^1} |v-Q|^2 \end{aligned} \tag{4.26}$$

Note that S^2 is compact and smooth, so by (4.25) we can always find such P and Q and P,Q are very close. So there is a unique geodesic γ_i connecting P and Q whose velocity is sufficiently small.

So if we write down the equation satisfied by $v - \gamma_i$ on $S^1 \times [\log(K r_0), \log(sH^{-1})]$

$$\widetilde{\Delta} u + A \cdot \widetilde{\nabla} u + B \cdot u = \tau \tag{4.27}$$

where $u = v - \gamma_i$, we have:

$$\begin{aligned} |A| &\leq C(|\widetilde{\nabla} v| + |\widetilde{\nabla}\gamma_i|) \leq \delta_0 \\ |B| &\leq C\min\{|\widetilde{\nabla} v|^2, |\widetilde{\nabla}\gamma_i|^2\} \leq \delta_0 \end{aligned} \tag{4.28}$$

If Lemma 4.1 (C') cannot be used, the only reason is that

$$\|v - \gamma_i\|_{1,i} \leq C\|\tau\|_{L^2(N_i)} \tag{4.29}$$

which implies

$$\int_{I_i} |\widetilde{\nabla}(v-\gamma_i)|^2 dtd\theta \le Cr_0^{-2} \tag{4.30}$$

which implied (4.24).

If Lemma 4.1 (C') can be used, then applying it for $u = v - \gamma_i$ over N_i, we have either

$$\|u\|_{1,i} < e^{-\frac{1}{2}L}\|u\|_{1,i-1} \tag{4.31}$$

or

$$\|u\|_{1,i} < e^{-\frac{1}{2}L}\|u\|_{1,i+1}. \tag{4.32}$$

Suppose the first one happens (without loss of generality). Then we may push this relation to the left because (4.28) hold regardless of t's position. If the theorem can be used on N_{j+1} but not on N_j for some $j \ge 2$, then we have

$$\|u\|_{1,i} < e^{-\frac{1}{2}(i-j)L}\|u\|_{1,j} \le Ce^{-\frac{1}{2}(i-j)L}r_0^{-1} \le Cr_0^{-1}. \tag{4.33}$$

If the theorem can be used until I_2, then we have

$$\begin{aligned} &e^{\frac{L}{2}}\|u\|_{1,2} \le \|u\|_{1,1} = (\int_{I_1} u^2 dtd\theta)^{\frac{1}{2}} + (\int_{I_1} |\widetilde{\nabla} u|^2 dtd\theta)^{\frac{1}{2}} \\ &\le (\int_{I_2} u^2 dtd\theta)^{\frac{1}{2}} + (\int_{I_1} (u(t,\theta) - u(t+L,\theta))^2 dtd\theta)^{\frac{1}{2}} + (\int_{I_1} |\widetilde{\nabla} u|^2 dtd\theta)^{\frac{1}{2}} \end{aligned} \tag{4.34}$$

So we have

$$\begin{aligned} &(e^{\frac{L}{2}}-1)\|u\|_{1,2} \le (\int_{I_1}(\int_0^L |\frac{\partial u}{\partial t}(t+s,\theta)|ds)^2 dtd\theta)^{\frac{1}{2}} + (\int_{I_1} |\widetilde{\nabla} u|^2 dtd\theta)^{\frac{1}{2}} \\ &\le \int_0^L (\int_{I_1} |\frac{\partial u}{\partial t}(t+s,\theta)|^2 dtd\theta)^{\frac{1}{2}} ds + (\int_{I_1} |\widetilde{\nabla} u|^2 dtd\theta)^{\frac{1}{2}} \\ &\le C(\int_{I_1\cup I_2} |\widetilde{\nabla} u|^2 dtd\theta)^{\frac{1}{2}} \\ &\le C(\int_{I_1\cup I_2} |\widetilde{\nabla} v|^2 dtd\theta)^{\frac{1}{2}} + C(\int_{I_1\cup I_2} |\widetilde{\nabla} \gamma_i|^2 dtd\theta)^{\frac{1}{2}} \\ &\le C_(r_0^{-\frac{1}{2}} + s) \end{aligned} \tag{4.35}$$

So we have the estimate

$$\|u\|_{1,i} \le Ce^{-\frac{i-2}{2}L}\|u\|_{1,2} \le Ce^{-\frac{i}{2}L}(r_0^{-\frac{1}{2}} + s) \tag{4.36}$$

If $\|u\|_{1,i} < e^{-\frac{1}{2}L}\|u\|_{1,i+1}$ happens, we will have similarly

$$\|u\|_{1,i} \le Ce^{-\frac{l_n-i}{2}L}(r_0^{-\frac{1}{2}} + s) \tag{4.37}$$

Finally we get

$$\|u\|_{1,i} \le C(e^{-\frac{i}{2}L} + e^{-\frac{l_n-i}{2}L})s + Cr_0^{-\frac{1}{2}} \tag{4.38}$$

which implies (4.24).

Then to get the energy decay, we use the Hopf differential

$$\Phi = |\partial_t v|^2 - |\partial_\theta v|^2 - 2\sqrt{-1}\partial_t v \cdot \partial_\theta v \tag{4.39}$$

We know that the L^1 norm of Φ is invariant under conformal change of the coordinates. (t,θ) is the coordinate of $A_{Kr_0e^{(i-2)L},Kr_0e^{(i+1)L}}$, we find another coordinate for it: set $r_i = Kr_0e^{iL}$, then $(r_i^{-1}A_{Kr_0e^{(i-2)L},Kr_0e^{(i+1)L}}, r_i^{-2}g_e)$ can be represented as $(A^0_{e^{-2L},e^L}, \overline{g})$, where $\|\overline{g} - |dx|^2\|_{C^1(A^0_{e^{-2L},e^L})} \le \varepsilon$. Assume this Euclidean coordinate is (x,y), so:

$$\int_{S^1\times[\log Kr_0+(i-1)L,\log Kr_0+iL]} |\Phi| dt d\theta = \int_{A^0_{e^{-L},1}} |\Phi| dx dy \tag{4.40}$$

To estimate the right hand side, we use the Cauchy integral formula on $\Omega = A^0_{e^{-2L},e^L}$, and set $\Omega' = A^0_{e^{-L},1}$, for any $z \in \Omega'$

$$\Phi(v)(z) = \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Omega}\frac{\Phi(w)}{w-z}dw + \frac{1}{2\pi\sqrt{-1}}\int_{\Omega}\frac{\partial\Phi(w)}{\partial\overline{w}}\frac{dw\wedge d\overline{w}}{w-z} \tag{4.41}$$

We know

$$|\partial_x v|, |\partial_y v| \le CKr_0e^{iL}|A| \le CKr_0e^{iL}(|x|^{-1}r_0^{-\frac{1}{2}} + r_1^{-1}) \le C(r_0^{-\frac{1}{2}} + se^{-(l_n-i)L}) \tag{4.42}$$

so we have:

$$\left|\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Omega}\frac{\Phi(w)}{w-z}dw\right| \le C(r_0^{-1} + s^2e^{-2(l_n-i)L}) \tag{4.43}$$

For the second term, notice that by easy calculation

$$\frac{\partial\Phi(w)}{\partial\overline{w}} = \partial v \cdot \overline{\tau}(v) \tag{4.44}$$

where $\overline{\tau}(v)$ is the tension field under this coordinate. And

$$|\overline{\tau}(v)| \le (Kr_0e^{il})^2|\nabla_e H_e| \le Cr_0^{-1} \tag{4.45}$$

so we have:

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Omega}\frac{\partial\Phi(w)}{\partial\overline{w}}\frac{dw\wedge d\overline{w}}{w-z}\leq Cr_0^{-1} \tag{4.46}$$

Then we get:

$$\int_{\Omega'}|\Phi|\leq C(r_0^{-1}+s^2e^{-2(l_n-i)L}) \tag{4.47}$$

By direct calculation

$$\begin{aligned}&\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\partial_t v|^2dtd\theta\\ &\leq\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\Phi|dtd\theta+\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\partial_\theta v|^2dtd\theta\end{aligned} \tag{4.48}$$

and we can get the estimate of $\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\partial_\theta v|^2dtd\theta$ directly by (4.24). So we get the estimate:

$$\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\widetilde{\nabla}v|^2dtd\theta\leq C(e^{-iL}+e^{-(l_n-i)L})s^2+Cr_0^{-1} \tag{4.49}$$

Proposition 4.5. Suppose that$\{\Sigma_n\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(R^3\setminus B_1(0),g)$ and that

$$\lim_{i\to\infty}r_0(\Sigma_n)=\infty \tag{4.50}$$

And suppose that

$$\lim_{n\to\infty}r_0(\Sigma_n)H(\Sigma_n)=0 \tag{4.51}$$

Then there exist a large number K, a small number s and n_0 such that,when $n\geq n_0$,

$$\max_{I_i}|\widetilde{\nabla}v|\leq C(e^{-\frac{i}{2}L}+e^{-\frac{(l_n-i)}{2}L})s+Cr_0^{-\frac{1}{2}} \tag{4.52}$$

where

$$I_i=S^1\times[\log(Kr_0(\Sigma_n))+(i-1)L,\log(Kr_0(\Sigma_n))+iL] \tag{4.53}$$

and

$$i\in[0,l_n]\qquad\log(Kr_0(\Sigma_n))+l_nL=\log(sH^{-1}(\Sigma_n)) \tag{4.54}$$

Proof. We just use the interior estimate of the elliptic equation

$$\widetilde{\Delta} v + |\widetilde{\nabla} v|^2 v = \tau \tag{4.55}$$

We know $\|\widetilde{\nabla} v\|_\infty \leq C(r_0^{-\frac{1}{2}} + s)$, and $\|\tau\|_\infty \leq C r_0^{-1}$. Assume that :

$$I_i \subset\subset \widetilde{I}_i \subset\subset N_i \tag{4.56}$$

then for some $p > 2$

$$\begin{aligned} \sup_{I_i} |\widetilde{\nabla} v| &\leq C\|\widetilde{\nabla} v\|_{W^{1,p}(I_i)} \leq C(\|v\|_{L^p(\widetilde{I}_i)} + r_0^{-1}) \leq C(\|v\|_{L^2(N_i)} + r_0^{-1}) \\ &\leq C(e^{-\frac{i}{2}L} + e^{-\frac{(l_n - i)}{2}L})s + C r_0^{-\frac{1}{2}} \end{aligned} \tag{4.57}$$

This analysis improves our understanding of the blowdowns that we discussed in the previous section. Namely,

Corollary 4.6. Assume the same condition as the above proposition and in addition $\lim_{r_0 \to \infty} \frac{log(r_1)}{r_0^{1/4}} = 0$. Then the limit plane in Lemma3.2 and Lemma3.3 are all orthogonal to the same vector a. In fact, we may choose s small and i large enough so that,

$$|v(x) + a| \leq \varepsilon \tag{4.58}$$

for all $x \in \Sigma_n$ and $|x| \leq sH^{-1}(\Sigma_n)$

Proof. We want to prove that

$$Osc_{B_{sH^{-1}} \cap \Sigma_n} v \tag{4.59}$$

is sufficiently small if $r_0(\Sigma_n)$ large and s small. We already know that

$$Osc_{B_{Kr_0} \cap \Sigma_n} v \tag{4.60}$$

is very small from Lemma 3.2, so we need only to prove that

$$Osc_{(B_{sH^{-1}} \setminus B_{Kr_0}) \cap \Sigma_n} v \tag{4.61}$$

is small.

From the proposition above we find that

$$\begin{aligned} Osc_{(B_{sH^{-1}} \setminus B_{Kr_0}) \cap \Sigma_n} v &\leq \sum_{i=1}^{l_n} Osc_{I_i} v \leq C \sum_{i=1}^{l_n} \sup_{I_i} |\widetilde{\nabla} v| \\ &\leq C \sum_{i=1}^{l_n} ((e^{-\frac{i}{2}L} + e^{-\frac{(l_n - i)}{2}L})s + r_0^{-\frac{1}{2}}) \leq Cs + l_n r_0^{-\frac{1}{2}} \end{aligned} \tag{4.62}$$

From $C^{-1}r_1 \le H^{-1} \le Cr_1$ and the condition $\lim_{r_0\to\infty} \frac{\log(r_1)}{r_0^{1/4}} = 0$, we have

$$l_n r_0^{-\frac{1}{2}} = L^{-1}(\log(sH^{-1}) - \log(Kr_0))r_0^{-\frac{1}{2}} \le C\frac{\log r_1}{r_0^{\frac{1}{2}}} \to 0 \tag{4.63}$$

as $r_0 \to \infty$, so we prove the lemma.

Corollary 4.7. Assume the same condition as Proposition 4.5. Let $v_n = v(p_n)$ for some $p_n \in I_{\frac{l_n}{2}}$. Then

$$\sup_{I_i} |v - v_n| \le C(e^{-\frac{1}{2}iL} + e^{-\frac{1}{4}l_n L})s + l_n r_0^{-\frac{1}{2}} \tag{4.64}$$

for $i \in [0, \frac{1}{2}l_n]$

$$\sup_{I_i} |v - v_n| \le C(e^{-\frac{1}{4}l_n L} + e^{-\frac{1}{2}(l_n - i)L})s + l_n r_0^{-\frac{1}{2}} \tag{4.65}$$

for $i \in [\frac{1}{2}l_n, l_n]$

5 Harmonic Coordinates

We assume that the metric g can be expanded in the coordinate $\{x_i\}$ as

$$g_{ij} = \delta_{ij} + h_{ij} = \delta_{ij} + h^1_{ij}(\theta)/r + Q$$

where θ is the coordinate on the unit sphere S^2, and $h^1_{ij}(\theta)$ is a function extended constantly along the radius direction. And Q satisfies

$$\sup r^{2+k}|\partial^k Q| \le C \tag{5.1}$$

for $k = 0, 1, \cdots, 5$

First, note that:

$$\begin{aligned}\Delta_g x_k &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}(\sqrt{g}g^{ij}\frac{\partial}{\partial x_j}x_k) \\ &= \frac{\partial}{\partial x_i}g^{ik} + \frac{1}{2}g^{ik}g^{mn}g_{mn,i} \\ &= -g^{mn}\Gamma^k_{mn} = O(|x|^{-2})\end{aligned} \tag{5.2}$$

Now our aim is to find asymptotically harmonic coordinate, i.e. some coordinate y^i such that $\Delta_g y^k = O(|x|^{-3})$

$$\begin{aligned}\Delta_g x^k &= -g^{jl}g^{ik}\frac{1}{2}(\frac{\partial}{\partial x_j}h_{li} + \frac{\partial}{\partial x_l}h_{ji} - \frac{\partial}{\partial x_i}h_{jl}) \\ &= -g^{jl}g^{ik}\frac{1}{2}(r^{-2}((h^1_{li,j}(\theta) - h^1_{li}(\theta)\frac{x_j}{r}) \\ &+(h^1_{ji,l}(\theta) - h^1_{ji}(\theta)\frac{x_l}{r}) - (h^1_{jl,i}(\theta) - h^1_{jl}(\theta)\frac{x_i}{r}))) + \partial Q \\ &= -g^{jl}g^{ik}\frac{1}{2}r^{-2}f^1_{lij}(\theta) + O(|x|^{-3})\end{aligned} \tag{5.3}$$

We also know that $g^{ij} = \delta^{ij} - h^1_{ij}(\theta)/r + O(r^{-2})$
Then :

$$\Delta_g x^k = -\frac{1}{2} r^{-2} f^1_{jkj}(\theta) + O(r^{-3}) \tag{5.4}$$

Suppose $0 = \xi_0 > \xi_1 \geq \xi_2 \geq \cdots$ are the eigenvalues of $\Delta|_{S^2}$, and $A_n(\theta)$ are the corresponding orthonormal eigenvectors.

Set:

$$y^k = x^k + \sum_{n=0}^{\infty} f^k_n(r) A_n(\theta) \tag{5.5}$$

We have:

$$\Delta_g y^k = \Delta_g x^k + \sum_{n=0}^{\infty} \Delta_{R^3}(f^k_n(r)A_n(\theta)) + \sum_{n=0}^{\infty} (\Delta_g - \Delta_{R^3})(f^k_n(r)A_n(\theta)) \tag{5.6}$$

Solve the equation:

$$\Delta_g x^k + \sum_{n=0}^{\infty} \Delta_{R^3}(f^k_n(r)A_n(\theta)) = O(|x|^{-3}) \tag{5.7}$$

Assume

$$\frac{1}{2} f^1_{jkj}(\theta) = \sum_{n=0}^{\infty} \lambda^k_n A_n(\theta) \tag{5.8}$$

so we have:

$$\sum_{n=0}^{\infty} \Delta_{R^3}(f^k_n(r)A_n(\theta)) = r^{-2} \sum_{n=0}^{\infty} \lambda^k_n A_n(\theta) \tag{5.9}$$

$$\frac{1}{r^2}(2r f^{k\prime}_n + r^2 f^{k\prime\prime}_n + f^k_n(r)\xi_n) = \lambda^k_n, n = 0, \cdots, \infty \tag{5.10}$$

$$n = 0, \quad f^k_0 = \lambda^k_0 \log(r) \tag{5.11}$$

$$n > 0, \quad f^k_n = \frac{\lambda^k_n}{\xi_n} \tag{5.12}$$

and this solution satisfies that:

$$\sum_{n=0}^{\infty} (\Delta_g - \Delta_{R^3})(f^k_n(r)A_n(\theta)) = O(|x|^{-3}) \tag{5.13}$$

so if

$$y^k = x^k + \frac{1}{2\sqrt{\pi}} \lambda^k_0 \log r + \sum_{n=1}^{\infty} \frac{\lambda^k_n}{\xi_n} A_n(\theta) \tag{5.14}$$

then we must have:

$$\Delta y^k = O(|x|^{-3}) \tag{5.15}$$

Note that

$$\Delta|_{S^2}\sum_{n=1}^{\infty}\frac{\lambda_n^k}{\xi_n}A_n(\theta)=\sum_{n=1}^{\infty}\lambda_n^kA_n(\theta)=\frac{1}{2}f_{jkj}^1(\theta)-\frac{1}{2}\overline{f_{jkj}^1(\theta)} \tag{5.16}$$

where $\overline{f_{jkj}^1(\theta)}$ is its mean value on the unit sphere.

Set

$$g_k^1(\theta)=\sum_{n=1}^{\infty}\frac{\lambda_n^k}{\xi_n}A_n(\theta)=\Delta^{-1}(\frac{1}{2}f_{jkj}^1(\theta)-\frac{1}{2}\overline{f_{jkj}^1(\theta)}) \tag{5.17}$$

$$\frac{\partial y^k}{\partial x^i}=\delta_{ik}+\frac{\lambda_0^k}{2\sqrt{\pi}}\frac{1}{r}\frac{x^i}{r}+g_k^1(\theta)_i\frac{1}{r} \tag{5.18}$$

$$\frac{\partial x^i}{\partial y^k}=\delta_{ik}+O(|x|^{-1}) \tag{5.19}$$

So we get:

$$\widetilde{g}_{ij}=g(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})=\delta_{ij}+O(|x|^{-1}) \tag{5.20}$$

Suppose

$$\widetilde{g}_{ij}=\delta_{ij}+\widetilde{h}_{ij} \tag{5.21}$$

Now I want to discuss the ellipticity of $\widetilde{h}_{ij}$

$$\widetilde{h}_{ij}=h_{ij}-\frac{1}{2r\sqrt{\pi}}(\lambda_0^i\frac{x^j}{r}+\lambda_0^j\frac{x^i}{r})-\frac{(g_{i,j}^1(\theta)+g_{j,i}^1(\theta))}{r} \tag{5.22}$$

Where $g_{i,j}^1(\theta)$ denotes the constant extention along the radius direction of function $\frac{\partial g_i^1(\theta)}{\partial x_j}|_{S^2}$

Example 5.1. : For the metric $g_{ij}=\delta_{ij}+\frac{\delta_{ij}}{r}$, we have:

$$\Delta_g x^k=-\frac{1}{2}\frac{x^k}{r^3}+O(|x|^{-3}) \tag{5.23}$$

We know that on S^2, we have $\Delta|_{S^2}x^k=-2x^k$. So if we let:

$$y^k=x^k-\frac{1}{4}\frac{x^k}{r} \tag{5.24}$$

We have $\Delta_g y^k=O(|x|^{-3})$, then:

$$\frac{\partial y^k}{\partial x^i}=\delta_{ki}-\frac{1}{4}(\frac{\delta_{ki}}{r}-\frac{x^kx^i}{r^3}) \tag{5.25}$$

$$\widetilde{h}_{ij}=\frac{3\delta_{ij}}{2r}-\frac{x^ix^j}{2r^3}+O(r^{-2}) \tag{5.26}$$

Lemma 5.2. Suppose in some coordinate $\{x^i\}$, $g_{ij}=\delta_{ij}+h^1_{ij}(\theta)/r+Q$, then for any $m>2$ there exists $\varepsilon>0$, if $\|h^1_{ij}(\theta)-\delta_{ij}(\theta)\|_{W^{m,2}(S^2)}\le\varepsilon$ then in the asymptotically harmonic coordinate $\{y^i\}$we get above , we have

$$\widetilde{g}_{ij}=\delta_{ij}+\widetilde{h}_{ij} \tag{5.27}$$

where $\widetilde{h}_{ij}=O(|y|^{-1})$, and $|y|\widetilde{h}_{ij}$ is uniformly elliptic.

Proof: We know easily from (5.18) that $\widetilde{h}_{ij}=O(|x|^{-1})$ and that $\lim_{|x|\to\infty}\frac{|y|}{|x|}=1$,then $\widetilde{h}_{ij}=O(|y|^{-1})$. So we need only to prove that $|y|\widetilde{h}_{ij}$ is uniformly elliptic.

First we know from $\|h^1_{ij}(\theta)-\delta_{ij}(\theta)\|_{W^{m,2}(S^2)}\le\varepsilon$ that

$$\|\frac{1}{2}f^1_{jkj}(\theta)-\frac{1}{2}\frac{x^k}{r}\|_{W^{m-1,2}(S^2)}\le C\varepsilon \tag{5.28}$$

Note that $\frac{1}{2}f^1_{jkj}(\theta)=\sum_{n=0}^{\infty}\lambda^k_nA_n(\theta)$ and x^k is an eigenvector of Δ_{S^2} , so we can assume that $A_1(\theta)=C_kx^k|_{S^2}$ without loss of generality.

$$\|\lambda^k_0A_0(\theta)+(\lambda^k_1C_k-\frac{1}{2})x^k+\sum_{n=2}^{\infty}\lambda^k_nA_n(\theta)\|_{W^{m-1,2}(S^2)}\le\varepsilon \tag{5.29}$$

so we get

$$|\lambda^k_0|\le\varepsilon,(\lambda^k_1C_k-\frac{1}{2})\le\varepsilon,\sum_{n=2}^{\infty}(|\xi_n|^{\frac{m-1}{2}}\lambda^k_n)^2\le\varepsilon \tag{5.30}$$

Note that from (5.14)

$$\frac{\partial y^k}{\partial x^i}=\delta_{ik}+\frac{\lambda^k_0}{2\sqrt{\pi}}\frac{1}{r}\frac{x^i}{r}-\frac{1}{2}(\frac{1}{2}\pm\varepsilon)(\frac{\delta_{ik}}{r}-\frac{x_ix_k}{r^3})+\sum_{n=2}^{\infty}\frac{\lambda^k_n}{\xi_n}\frac{\partial A_n(\theta)}{\partial x_i} \tag{5.31}$$

where the last term on the right can be estimated, for some $p>0$

$$\begin{aligned}
&|\sum_{n=2}^{\infty}\frac{\lambda^k_n}{\xi_n}\frac{\partial A_n(\theta)}{\partial x_i}|\le\sum_{n=2}^{\infty}\frac{|\lambda^k_n|}{|\xi_n|}\frac{|\nabla_{S^2}A_n(\theta)|}{r}\\
&\le\sum_{n=2}^{\infty}\frac{|\lambda^k_n|}{|\xi_n|}\frac{\|A_n(\theta)\|_{W^{2+p,2}}}{r}\\
&\le\sum_{n=2}^{\infty}\frac{|\lambda^k_n|}{|\xi_n|}\frac{|\xi_n|^{1+\frac{p}{2}}\|A_n(\theta)\|_{L^2}}{r}\\
&\le\frac{1}{r}\sum_{n=2}^{\infty}|\lambda^k_n||\xi_n|^{\frac{m-1}{2}}|\xi_n|^{\frac{p-m+1}{2}}\\
&\le\frac{1}{r}(\sum_{n=2}^{\infty}(|\lambda^k_n||\xi_n|^{\frac{m-1}{2}})^2)^{\frac{1}{2}}(\sum_{n=2}^{\infty}|\xi_n|^{p-m+1})^{\frac{1}{2}}
\end{aligned} \tag{5.32}$$

let $p=\frac{m-2}{2}$, then from $\xi_n=O(n)$ we have

$$\sum_{n=2}^{\infty}|\xi_n|^{p-m+1}\leq C \tag{5.33}$$

so

$$|\sum_{n=2}^{\infty}\frac{\lambda_n^k}{\xi_n}\frac{\partial A_n(\theta)}{\partial x_i}|\leq\frac{C\varepsilon}{r} \tag{5.34}$$

then we have:

$$\frac{\partial y^k}{\partial x^i}=\delta_{ik}-\frac{1}{4}(\frac{\delta_{ik}}{r}-\frac{x_ix_k}{r^3})+\frac{C\varepsilon}{r} \tag{5.35}$$

so we can deduce that:

$$\widetilde{h}_{ij}=h_{ij}+\frac{\delta_{ij}}{2r}-\frac{x^ix^j}{2r^3}+\frac{C\varepsilon}{r} \tag{5.36}$$

because $|h^1_{ij}(\theta)-\delta_{ij}(\theta)|_{W^{m,2}(S^2)}\leq\varepsilon$, we have rh_{ij} is uniformly elliptic. And the eigenvalues of $\frac{x^ix^j}{r^2}$ are between 0 and 1, so $|y|\widetilde{h}_{ij}$ is uniformly elliptic from $\lim_{r\to\infty}\frac{|y|}{r}=1$ for ε sufficiently small.

So all the analysis in Section 2,3,4 can be done in the asymptotically harmonic coordinate $\{y_i\}$.

Lemma 5.3. In the asymptotically harmonic coordinate $\{y^i\}$, we have that

$$-\frac{1}{2}\Delta_g\log|\widetilde{g}|=R(g)+O(|y|^{-4}) \tag{5.37}$$

Proof. From direct calculation we have

$$R(g)=\widetilde{g}^{jk}\widetilde{g}^{il}\widetilde{g}_{ml}(\frac{\partial\widetilde{\Gamma}^m_{jk}}{\partial y^i}-\frac{\partial\widetilde{\Gamma}^m_{ik}}{\partial y^j})+O(|y|^{-4}) \tag{5.38}$$

$$\begin{aligned}\widetilde{g}^{jk}\widetilde{g}^{il}\widetilde{g}_{ml}\frac{\partial\widetilde{\Gamma}^m_{jk}}{\partial y^i}&=\widetilde{g}^{il}\widetilde{g}_{ml}\frac{\partial(\widetilde{g}^{jk}\widetilde{\Gamma}^m_{jk})}{\partial y^i}+O(|y|^{-4})\\&=-\widetilde{g}^{il}\widetilde{g}_{ml}\frac{\partial\Delta_g y^m}{\partial y^i}+O(|y|^{-4})=O(|y|^{-4})\end{aligned} \tag{5.39}$$

$$\begin{aligned}-\widetilde{g}^{jk}\widetilde{g}^{il}\widetilde{g}_{ml}\frac{\partial\widetilde{\Gamma}^m_{ik}}{\partial y^j}&=-\frac{1}{2}\widetilde{g}^{jk}\widetilde{g}^{ip}\frac{\partial^2\widetilde{g}_{ip}}{\partial y^j\partial y^k}+O(|y|^{-4})\\&=-\frac{1}{2}\Delta_g\log|\widetilde{g}|+O(|y|^{-4})\end{aligned} \tag{5.40}$$

so we prove the lemma.

Corollary 5.4. If in addition $R = O(|x|^{-3-\tau})$ for some $\tau > 0$, then in the asymptotically harmonic coordinate $\{y^i\}$, we have

$$\sum_{i=1}^{3} \widetilde{h}_{ii} = 8m(g)/|y| + o(|y|^{-1-\frac{\tau}{2}}) \tag{5.41}$$

Proof: First we know that

$$\lim_{|x|\to\infty} \frac{|y|}{|x|} = 1, \tag{5.42}$$

then from the lemma above that in the coordinate $\{y^i\}$, we have

$$\Delta_g \log|\widetilde{g}| = O(|y|^{-3-\tau}) \tag{5.43}$$

We know that

$$\log|\widetilde{g}| = O(|y|^{-1}) \tag{5.44}$$

From the theory of harmonic functions in R^n , we have there exist some constant C such that:

$$\log|\widetilde{g}| = \frac{C}{|y|} + o(|y|^{-1-\frac{\tau}{2}}) \tag{5.45}$$

From Bartnik's result , we know the mass is invariant under the change of coordinates because $R(g) \in L^1$.

$$m(g) = \lim_{R\to\infty} \frac{1}{16\pi} \int_{s_R} (\widetilde{h}_{ij,j} - \widetilde{h}_{jj,i}) v_g^i d\mu \tag{5.46}$$

Now we have

$$\begin{aligned} &\widetilde{g}_{ik,k} - \frac{1}{2}\widetilde{g}_{kk,i} = \widetilde{g}^{ij}\widetilde{g}^{kl}(\widetilde{g}_{jk,l} - \frac{1}{2}\widetilde{g}_{kl,j}) + O(|y|^{-3}) \\ &= -\Delta_g y^i + O(|y|^{-3}) = O(|y|^{-3}) \end{aligned} \tag{5.47}$$

So we have:

$$\begin{aligned} m(g) \quad &= \lim_{R\to\infty} \frac{1}{16\pi} \int_{s_R} (-\frac{1}{2}\widetilde{h}_{jj,i}) v_g^i d\mu \\ &= -\lim_{R\to\infty} \frac{1}{32\pi} \int_{s_R} \frac{\partial \log|\widetilde{g}|}{\partial y^i} v_g^i d\mu \\ &= \lim_{R\to\infty} \frac{1}{32\pi} \int_{s_R} \frac{Cy^i}{|y|^3} v_g^i d\mu \\ &= \frac{C}{8} \end{aligned} \tag{5.48}$$

So we get the result by easy calculation .

Remark 5.5. In fact we can replace the constraint equation by the condition

$$R = O(|x|^{-3-\tau}) \tag{5.49}$$

for some $\tau > 0$.

6 Proof of the Theorem

Now let's prove Theorem 1.5.

First recall that, for any surface Σ embedded in $\mathbb{R}^3$ and any given vector $b \in \mathbb{R}^3$, one has

$$\int_\Sigma H_e < v_e \cdot b >_e d\mu_e = 0 \tag{6.1}$$

where H_e and v_e denote the mean curvature and normal vector field with respect to the Euclidean metric.

On the other hand , if Σ is a constant mean curvature surface in the asymptotically flat end , then

$$\int_\Sigma H < v_e \cdot b >_e d\mu_e = 0 \tag{6.2}$$

So we have

$$\int_\Sigma (H - H_e) < v_e \cdot b >_e d\mu_e = 0 \tag{6.3}$$

From now on , our calculation is in the coordinate $\{x^i\}$,which is assumed to be the asymptotically harmonic coordinate. We have calculated $H - H_e$, so we have

$$\int_\Sigma (H - H_e) < v_e \cdot b >_e d\mu_e = \int_\Sigma (-f^{ik} h_{kl} f^{lj} A_{ij} + \frac{1}{2} H v^i v^j h_{ij} - f^{ij} v^l \overline{\nabla}_i h_{jl}$$
$$+ \frac{1}{2} f^{ij} v^l \overline{\nabla}_l h_{ij} \pm C|h||\overline{\nabla} h| \pm C|h|^2|A|) < v_e \cdot b >_e d\mu_e \tag{6.4}$$

We assume that there exists a sequence of constant mean curvature surfaces Σ_n with

$$\lim_{n\to\infty} r_0(\Sigma_n) = \infty \qquad \lim_{n\to\infty} H(\Sigma_n) r_0(\Sigma_n) = 0 \tag{6.5}$$

otherwise we have get the result from the uniqueness theorem of Lan-Hsuan Huang. So we can choose s sufficiently small and K sufficiently large with $sH^{-1} > Kr_0$ for r_0 sufficiently large.

We know that

$$|h| = O(|x|^{-1}), |\overline{\nabla} h| = O(|x|^{-2}), |A| \le CH + C|\mathring{A}| \tag{6.6}$$

from the estimate

$$|\mathring{A}| \le r_0^{-\frac{1}{2}} O(|x|^{-1}) \tag{6.7}$$

we have

$$| \int_\Sigma (\pm C|h||\overline{\nabla} h| \pm C|h|^2|A|) < v_e \cdot b >_e d\mu_e| \le C \int_\Sigma (H|x|^{-2} + |x|^{-3})$$
$$= O(r_0^{-1}) \tag{6.8}$$

by the estimates in Section 2.

Now we calculate other terms in (6.4)

$$\begin{aligned}
&\int_{\Sigma_n} -f^{ij} v^l (\overline{\nabla}_i h_{jl}) v^m b^m d\mu_e \\
= \quad & \frac{1}{2}\int_{\Sigma_n} (f^{ij} h_{jk} f^{kl} A_{li} - H v^j v^l h_{jl}) v^m b^m d\mu_e + \frac{1}{2}\int_{\Sigma_n} f^{ij} v^l h_{jl} A_{ik} f^{km} b^m d\mu_e \\
& -\frac{1}{2}\int_{\Sigma_n} f^{ij} v^l (\overline{\nabla}_i h_{jl}) v^m b^m d\mu_e \qquad (6.9)
\end{aligned}$$

because $d\mu_e = (1 + O(r^{-1}))d\mu$, $v_e = (1 + O(r^{-1}))v$ and $< v_e \cdot b >_e =< v \cdot b >_g$ $+O(r^{-1})$.

So we have

$$\begin{aligned}
\int_{\Sigma_n} (H - H_e) < v_e \cdot b >_e d\mu_e = \int_{\Sigma_n} & -\frac{1}{2} f^{ik} h_{kl} f^{lj} A_{ij} v^m b^m + f^{ij} v^l h_{jl} A_{ik} f^{km} b^m \\
& -\frac{1}{2} f^{ij} v^l \overline{\nabla}_i h_{jl} v^m b^m + \frac{1}{2} f^{ij} v^l \overline{\nabla}_l h_{ij} v^m b^m + O(r_0^{-1}) d\overline{\mu}
\end{aligned} \qquad (6.10)$$

Note that

$$A_{ij} = \mathring{A}_{ij} + \frac{f_{ij}}{2} H, \quad \sup|\mathring{A}| \le r_0^{-\frac{1}{2}} O(|x|^{-1}) \qquad (6.11)$$

So we have

$$\begin{aligned}
&\int_{\Sigma_n} (H - H_e) < v_e \cdot b >_e d\mu_e = \int_{\Sigma_n} -\frac{H}{4} f^{kl} h_{kl} v^m b^m + \frac{H}{4} f^{jm} h_{jl} v^l b^m \\
&+\frac{1}{2} f^{ij} (\overline{\nabla}_l h_{ij}) v^l v^m b^m - \frac{1}{2} f^{ij} (\overline{\nabla}_i h_{jl}) v^l v^m b^m \\
&\pm C \int_{\Sigma_n} |x|^{-2} r_0^{-\frac{1}{2}} + O(r_0^{-1}) \qquad (6.12)
\end{aligned}$$

In this case we calculate

$$\int_{\Sigma_n} |x|^{-2} r_0^{-\frac{1}{2}} d\mu_e \qquad (6.13)$$

We divide the integral into three parts:

$$\int_{\Sigma_n} |x|^{-2} r_0^{-\frac{1}{2}} = \int_{\Sigma_n \cap B^c_{sH^{-1}}(0)} + \int_{\Sigma_n \cap B_{Kr_0}(0)} + \int_{\Sigma_n \cap (B_{sH^{-1}} \setminus B_{Kr_0})} |x|^{-2} r_0^{-\frac{1}{2}} \qquad (6.14)$$

Then by the blowdown results in Section 3 we have

$$\int_{\Sigma_n \cap B^c_{sH^{-1}}(0)} |x|^{-2} r_0^{-\frac{1}{2}} d\mu_e = \int_{\widetilde{\Sigma}_n \cap B^c_s(0)} |\widetilde{x}|^{-2} r_0^{-\frac{1}{2}} d\widetilde{\mu} \le C r_0^{-\frac{1}{2}} \qquad (6.15)$$

$$\int_{\Sigma_n\cap B_{Kr_0}(0)} |x|^{-2} r_0^{-\frac{1}{2}} d\mu_e = \int_{\widehat{\Sigma}_n\cap B_k(0)} |\widehat{x}|^{-2} r_0^{-\frac{1}{2}} d\widehat{\mu} \leq C r_0^{-\frac{1}{2}} \tag{6.16}$$

$$\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})} |x|^{-2} r_0^{-\frac{1}{2}} d\mu_e = \sum_{i=0}^{n} \int_{\Sigma_n\cap(B_{Kr_0 e^{4iL}}\setminus B_{Kr_0 e^{4(i-1)L}})} |x|^{-2} r_0^{-\frac{1}{2}} d\mu_e$$
$$\leq C \sum_{i=0}^{n} \int_{B_{e^{4L}}\setminus B_1} |\overline{x}|^{-2} r_0^{-\frac{1}{2}} d\overline{\mu} \leq C r_0^{-\frac{1}{2}} l_n L \tag{6.17}$$

where $e^{l_n L} K r_0 = sH^{-1}$

so if

$$\lim_{r_0\to 0} \frac{|\log H|}{r_0^{\frac{1}{2}}} = 0 \tag{6.18}$$

in other words

$$\lim_{r_0\to 0} \frac{|\log r_1|}{r_0^{\frac{1}{2}}} = 0 \tag{6.19}$$

we have

$$\int_{\Sigma} |x|^{-2} r_0^{-\frac{1}{2}} d\overline{\mu} \to 0 \tag{6.20}$$

as $r_0 \to \infty$

From the property of the asymptotically harmonic coordinate

$$g^{ij} h_{ij} = \frac{8m(g)}{r} + o(r^{-1-\frac{\tau}{2}}) \tag{6.21}$$

$$g^{kl}(g_{ik,l} - \frac{1}{2} g_{kl,i}) = O(|x|^{-3}) \tag{6.22}$$

$$\begin{aligned}
&\int_{\Sigma_n} -\frac{H}{4} f^{kl} h_{kl} v^m b^m + \frac{H}{4} f^{jm} h_{jl} v^l b^m + \frac{1}{2} f^{ij} (\overline{\nabla}_l h_{ij} - \overline{\nabla}_i h_{jl}) v^l v^m b^m \\
&= \int_{\Sigma_n} -\frac{H}{4} g^{kl} h_{kl} v^m b^m + \frac{H}{4} g^{jm} h_{jl} v^l b^m \\
&+\frac{1}{2} g^{ij} (\overline{\nabla}_l h_{ij} - \overline{\nabla}_i h_{jl}) v^l v^m b^m + O(|r_0|^{-1}) \\
&= -2m(g) \int_{\Sigma_n} (\frac{H}{r} < v_e \cdot b_e >_e + \frac{< x \cdot v_e >_e < v_e \cdot b >_e}{r^3}) \\
&+ \int_{\Sigma_n} \frac{H}{4} h_{ml} v^l b^m + o(1).
\end{aligned} \tag{6.23}$$

So we have:

$$\lim_{n\to\infty} (-2m(g) \int_{\Sigma_n} (\frac{H}{r} < v_e \cdot b >_e + \frac{< x \cdot v_e >_e < v_e \cdot b >_e}{r^3}) + \int_{\Sigma_n} \frac{H}{4} h_{ml} v^l b^m) = 0 \tag{6.24}$$

Note that:

$$h_{ml}v^l = (h_{ml} - \frac{tr(h)}{2}\delta_{ml})v^l + \frac{tr(h)}{2}v^m \tag{6.25}$$

where $tr(h) = g^{ij}h_{ij}$

Assume that the three eigenvalues of h_{ml} are

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \tag{6.26}$$

For $p \in \Sigma$ fixed , choose coordinate properly such that

$$h_{ml} - \frac{tr(h)}{2}\delta_{ml} \tag{6.27}$$

can be written as

$$\begin{pmatrix} \lambda_1 - \frac{tr(h)}{2} & 0 & 0 \\ 0 & \lambda_2 - \frac{tr(h)}{2} & 0 \\ 0 & 0 & \lambda_3 - \frac{tr(h)}{2} \end{pmatrix} \tag{6.28}$$

Assume $v = (\widetilde{v}^1, \widetilde{v}^2, \widetilde{v}^3)$,and $(\widetilde{v}^1)^2 + (\widetilde{v}^2)^2 + (\widetilde{v}^3)^2 = 1$. Then we have

$$\sum_{i=1}^{3}((\lambda_i - \frac{tr(h)}{2})\widetilde{v}^i)^2 = \frac{(tr(h))^2}{4} - \sum_{i=1}^{3}\lambda_i(tr(h) - \lambda_i)(\widetilde{v}^i)^2 \tag{6.29}$$

Because of the uniformly ellipticity we have there exists $C > 0$, such that

$$\frac{th(h)}{C} \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq (1 - \frac{1}{C})th(h) \tag{6.30}$$

so

$$\lambda_i(th(h) - \lambda_i) \geq \frac{1}{C}(1 - \frac{1}{C})(tr(h))^2 \tag{6.31}$$

hence

$$\sum_{i=1}^{3}((\lambda_i - \frac{tr(h)}{2})\widetilde{v}^i)^2 \leq (\frac{1}{4} - \frac{1}{C}(1 - \frac{1}{C}))(tr(h))^2 \tag{6.32}$$

$$\begin{aligned} \int_{\Sigma_n} \frac{H}{4}h_{ml}v^l b^m &= \int_{\Sigma_n} \frac{H}{4}(\frac{tr(h)}{2} < v_e \cdot b >_e + (h_{ml} - \frac{tr(h)}{2}\delta_{ml})v^l b^m) \\ &\leq \int_{\Sigma_n} \frac{Htr(h)}{4}(\frac{1}{2} < v_e \cdot b >_e + \sqrt{\frac{1}{4} - \frac{1}{C}(1 - \frac{1}{C})}) \\ &= \int_{\Sigma_n} \frac{Hm(g)}{r}(< v_e \cdot b >_e + 1 - \frac{2}{C}) \end{aligned} \tag{6.33}$$

so we have

$$\int_{\Sigma_n}(H-H_e) < v_e\cdot b >_e \leq -m\int_{\Sigma_n}\frac{H}{r} < v_e\cdot b >_e$$
$$+\frac{2}{r^3} < x\cdot v_e >_e < v_e\cdot b >_e d\mu_e + (1-\frac{2}{C})m(g)\int_{\Sigma_n}\frac{H}{r}d\mu_e + o(1) \tag{6.34}$$

as $n\to\infty$

From Lemma 3.1, we have $\frac{H}{2}\Sigma_n$ subconverges to some sphere $S_1^2(a)$ with $|a|=1$. Now we choose $b=-a$. Then from the calculation in [9], we have

$$-m(g)\int_{\Sigma_n}\frac{H}{r} < v_e\cdot b >_e \quad\to\quad -\frac{8}{3}\pi m(g) \tag{6.35}$$
$$-m(g)\int_{\Sigma_n}\frac{2}{r^3} < x\cdot v_e >_e < v_e\cdot b >_e \quad\to\quad -\frac{16}{3}\pi m(g) \tag{6.36}$$
$$(1-\frac{2}{C})m(g)\int_{\Sigma_n}\frac{H}{r} \quad\to\quad (1-\frac{2}{C})8\pi m(g) \tag{6.37}$$

as $n\to\infty$

Because there is a little difference from [9],we prove them again. We notice from Lemma 3.1, we have $\frac{H}{2}\Sigma_n$ subconverges to some sphere $S_1(a)$ with $|a|=1$, and the first and third integral converges to $-m(g)\int_{S_1(a)}\frac{2}{r} < v_e\cdot b >_e = -\frac{8}{3}\pi m(g)$ and $(1-\frac{2}{C})m(g)\int_{S_1(a)}\frac{2}{r} = (1-\frac{2}{C})8\pi m(g)$ respectively.

To deal with the (6.36), first we notice that

$$\int_{S^2(a)}\frac{2}{r^3} < x\cdot v_e >_e < v_e\cdot b >_e d\mu_e = \frac{4}{3}\pi \tag{6.38}$$

then we break up the integral (6.36) into three parts.

$$\int_{\Sigma_n}\frac{2}{r^3} < x\cdot v_e >_e < v_e\cdot b >_e d\mu_e$$
$$=\int_{\Sigma_n\cap B^c_{sH^{-1}}(0)}+\int_{\Sigma_n\cap B_{Kr_0}(0)}+\int_{\Sigma_n\cap B_{sH^{-1}}\setminus B_{Kr_0}}\frac{2}{r^3} < x\cdot v_e >_e < v_e\cdot b >_e d\mu_e \tag{6.39}$$

Then

$$\lim_{n\to\infty}\int_{\Sigma_n\cap B^c_{sH^{-1}}(0)}\frac{2}{r^3} < x\cdot v_e >_e < v_e\cdot b >_e d\mu_e$$
$$=\int_{S^2(a)\cap B^c_s}\frac{2}{r^3} < x\cdot v_e >_e < v_e\cdot b >_e d\mu_e \tag{6.40}$$

and

$$\lim_{n\to\infty}\int_{\Sigma_n\cap B_{Kr_0}(0)}\frac{2}{r^3}<x\cdot v_e>_e<v_e\cdot b>_e\,d\mu_e$$
$$=\int_{P\cap B_K(0)}\frac{2}{r^3}<x\cdot v_e>_e<v_e\cdot b>_e\,d\mu_e, \qquad (6.41)$$

where P is the limit plane in Lemma 3.2. From Corollary4.6, we know the normal vector of P is v_e. Then due to an easy calculation we know

$$\int_P\frac{2}{r^3}<x\cdot v_e>_e<v_e\cdot b>_e\,d\mu_e=4\pi \qquad (6.42)$$

From the divergence theorem we have

$$\int_{\Sigma_n}\frac{2}{r^3}<x\cdot v_e>_e\,d\mu_e=8\pi \qquad (6.43)$$

for any n and

$$\int_{S^2(a)}\frac{2}{r^3}<x\cdot v_e>_e\,d\mu_e=4\pi \qquad (6.44)$$

because the origin is on the sphere $S^2(a)$. Since

$$\lim_{n\to\infty}\int_{\Sigma_n\cap B^c_{sH^{-1}}(0)}\frac{2}{r^3}<x\cdot v_e>_e\,d\mu_e=\int_{S^2(a)\cap B^c_s(0)}\frac{2}{r^3}<x\cdot v_e>_e\,d\mu_e \qquad (6.45)$$

$$\lim_{n\to\infty}\int_{\Sigma_n\cap B_{Kr_0}(0)}\frac{2}{r^3}<x\cdot v_e>_e\,d\mu_e=\int_{P\cap B_K(0)}\frac{2}{r^3}<x\cdot v_e>_e\,d\mu_e \qquad (6.46)$$

and

$$\int_P\frac{2}{r^3}<x\cdot v_e>_e\,d\mu_e=4\pi \qquad (6.47)$$

then we have

$$\lim_{s\to 0,K\to\infty}\limsup_{n\to\infty}\left|\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})}\frac{2}{r^3}<x\cdot v_e>_e\,d\mu_e\right|=0 \qquad (6.48)$$

Now we want to prove that

$$\lim_{s\to 0,K\to\infty}\limsup_{n\to\infty}\left|\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})}\frac{2}{r^3}<x\cdot v_e>_e<v_e\cdot b>_e\,d\mu_e\right|=0 \qquad (6.49)$$

We use Lemma 4.7 to get (6.49) from (6.48), but there is a bit difference from [9].

$$\begin{aligned}
&\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})}\frac{2}{r^3}<x\cdot v_e>_e<v_e\cdot b>_e\,d\mu_e\\
&=<v_n\cdot b>_e\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})}\frac{2}{r^3}<v_e\cdot b>_e\,d\mu_e\\
&+\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})}\frac{2}{r^3}<x\cdot v_e>_e<(v_e-v_n)\cdot b>_e\,d\mu_e
\end{aligned}\tag{6.50}$$

The first term will converge to 0. For the second term, we deal with it in the cylinder coordinate in Section 4:

$$\begin{aligned}
&|\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})}\frac{2}{r^3}<x\cdot v_e>_e<(v_e-v_n)\cdot b>_e\,d\mu_e|\\
&=|\sum_{j=1}^{l_n}\int_{A_{Kr_0e^{(j-1)L},Kr_0e^{jL}}}\frac{2}{r^3}<x\cdot v_e>_e<(v_e-v_n)\cdot b>_e\,d\mu_e|\\
&\leq C\sum_{j=1}^{l_n}L\max_{I_j}|v_e-v_n|\\
&=C\sum_{j=1}^{l_n/2}L\max_{I_j}|v_e-v_n|+C\sum_{j=l_n/2+1}^{l_n}L\max_{I_j}|v_e-v_n|
\end{aligned}\tag{6.51}$$

From Lemma 4.7

$$\begin{aligned}
&CL\sum_{i=1}^{l_n/2}\sup_{I_i}|v-v_n|+CL\sum_{i=\frac{l_n}{2}+1}^{l_n}\sup_{I_i}|v-v_n|\\
&\leq C(l_ne^{-\frac{1}{4}l_nL}+C)s+l_n^2r_0^{-\frac{1}{2}}
\end{aligned}\tag{6.52}$$

But from the condition

$$\lim_{n\to\infty}\frac{\log(r_1(\Sigma_n))}{r_0(\Sigma_n)^{1/4}}=0\tag{6.53}$$

we know

$$\lim_{n\to\infty}l_n^2r_0^{-\frac{1}{2}}=\lim_{n\to\infty}(\frac{L^{-1}(\log sH^{-1}-\log Kr_0)}{r_0^{\frac{1}{4}}})^2=0\tag{6.54}$$

so (6.49) holds.

Then

$$0\leq-\frac{8}{3}\pi m(g)-\frac{16}{3}\pi m(g)+(1-\frac{2}{C})8\pi m(g)=-\frac{16}{C}\pi m(g)\tag{6.55}$$

but $m(g) > 0$, this is a contradiction. So for the stable constant mean curvature foliation there exists some constant $C > 0$ such that for any sphere Σ in the foliation,

$$\frac{r_0(\Sigma)}{r_1(\Sigma)} \geq C. \tag{6.56}$$

Then the uniqueness follows from Theorem 1.4.

Proof of the Corollary 1.9. Suppose there is not such $K(C, \beta)$, then we can find a sequence of constant mean curvature spheres Σ_n, with

$$\lim_{n\to\infty} r_0(\Sigma_n) = \infty \qquad \lim_{n\to\infty} \frac{\log(r_1)}{r_0^{\frac{1}{4}}} = 0 \tag{6.57}$$

and Σ_n do not belong to the foliation. But from the argument above we know this sequence satisfies

$$\frac{r_0(\Sigma_n)}{r_1(\Sigma_n)} \geq C. \tag{6.58}$$

So when n is sufficiently large, Σ_n must belong to the foliation, which ends the proof.

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