An Outer Commutator Multiplier and Capability of Finitely Generated Abelian Groups*

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Abstract

We present an explicit structure for the Baer invariant of a finitely generated abelian group with respect to the variety $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$, for all $c_2 \leq c_1 \leq 2c_2$. As a consequence we determine necessary and sufficient

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conditions for such groups to be $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable. We also show that if $c_1 \neq 1 \neq c_2$, then a finitely generated abelian group is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable if and only if it is capable. Finally we show that $\mathfrak{S}_2$-capability implies capability but there is a finitely generated abelian group which is capable but is not $\mathfrak{S}_2$-capable.

Key Words: Baer invariant; Finitely generated abelian group; Varietal capability; Outer commutator variety.
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1 Introduction and Preliminaries

An interesting problem connected to the notion of Baer invariants is the computation of Baer invariants for some natural classes of groups with respect to common varieties. The class of finitely generated abelian groups is an appropriate candidate because of their explicit structure theorem.

First of all Schur (1907), computed the Schur multiplier of a finite abelian group. The second author in a joint paper Mashayekhy and Moghaddam (1997), computed the $\mathfrak{N}_c$-multiplier of finitely generated abelian groups, where $\mathfrak{N}_c$ is the variety of all nilpotent groups of class at most c. The authors in 2006 (Mashayekhy and Parvizi, 2006), computed the polynilpotent multipliers of finitely generated abelian groups.

Another interesting problem is determining capable groups or more generally varietal capable groups. In 1938 Baer classified all capable groups among the direct sums of cyclic groups and in particular among the finitely generated abelian groups. Burns and Ellis (1998), extended the result for $\mathfrak{N}_c$-capability and recently the authors in a joint paper (Parvizi, et al.) with S. Kayvanfar classified all finitely generated abelian groups that are polynilpotent capable. Some work has been done in other classes of groups for example Magidin

(2005), worked on capability of the nilpotent product of cyclic groups.

We note that one reason for studying Baer invariants and varietal capability is their relevance to the isologism theory of P. Hall which is used to classify groups such as prime-power groups into a suitable equivalence classes coarser than isomorphism. The article of Leedham-Green and Mckay (1976), gives a fairly comprehensive account of these relationships.

In this paper we compute the multiplier of finitely generated abelian groups and determine all varietal capable finitely generated abelian groups with respect to the variety $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$, for all $c_2 \leq c_1 \leq 2c_2$.

In particular we show that:

a) if $c_1 \neq 1 \neq c_2$, then a finitely generated abelian group is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable if and only if it is capable;

b) every $\mathcal{S}_2$-capable group is a capable group and there is a finitely generated abelian group which is capable but is not $\mathcal{S}_2$-capable.

In the following there are some preliminaries which are needed.

Definition 1.1. Let G be any group with a free presentation $G \cong F/R$, where F is a free group. Then, after Baer (1945), the *Baer invariant* of G with respect to a variety of groups $\mathfrak{V}$, denoted by $\mathfrak{V}M(G)$, is defined to be

$$\mathfrak{V}M(G) = \frac{R \cap V(F)}{[RV^*F]} \ ,$$

where V is the set of words of the variety $\mathfrak{V}$, $V(F)$ is the verbal subgroup of F with respect to $\mathfrak{V}$ and

$$[RV^*F] = \Big\langle v(f_1, \ldots, f_{i-1}, f_i r, f_{i+1}, \ldots, f_n) v(f_1, \ldots, f_i, \ldots, f_n)^{-1} \ \Big| \ r \in R, v \in V, f_i \in F \text{ for all } 1 \leq i \leq n, n \in \mathbf{N} \Big\rangle.$$

As a special case, if $\mathfrak{V}$ is the variety of abelian groups, $\mathfrak{A}$, the Baer invariant of G is the well-known *Schur multiplier*

$$\frac{R \cap F'}{[R, F]}.$$

If $\mathfrak{N}_c$ is the variety of nilpotent groups of class at most $c \geq 1$, then the Baer invariant of G with respect to it, is called the *c-nilpotent multiplier* of G, is given by:

$$\mathfrak{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]},$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F and $[R,\ _1F] = [R, F], [R,\ _cF] = [[R,\ _{c-1}F], F]$, inductively.

Lemma 1.2. *(Hulse and Lennox 1976) If u and w are any two words and $v = [u, w]$ and K is a normal subgroup of a group G, then*

$$[Kv^*G] = [[Ku^*G], w(G)][u(G), [Kw^*G]].$$

Proof. See Hall and Senior (1964, Lemma 2.9). □

Now, using the above lemma, then the Baer invariant of a group G with respect to the outer commutator variety $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$, is as follows:

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \frac{R \cap [\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[R,\ _{c_1}F, \gamma_{c_2+1}(F)][R,\ _{c_2}F, \gamma_{c_1+1}(F)]}.$$

Definition 1.3. Let X be an independent subset of a free group, and select an arbitrary total order for X. We define the basic commutators on X, their weight *wt*, and the ordering among them as follows:

(1) The elements of X are basic commutators of weight one, ordered according to the total order previously chosen.

(2) Having defined the basic commutators of weight less than n, the basic commutators of weight n are the $c_k = [c_i, c_j]$, where:

(a) c_i and c_j are basic commutators and $wt(c_i) + wt(c_j) = n$, and

(b) $c_i > c_j$, and if $c_i = [c_s, c_t]$ then $c_j \geq c_t$.

(3) The basic commutators of weight n follow those of weight less than n. The basic commutators of weight n are ordered among themselves lexicographically; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight n, then $[b_1, a_1] \leq [b_2, a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

The next two theorems are vital in our investigation.

Theorem 1.4. *(Hall, 1959). Let $F = \langle x_1, x_2, \ldots, x_d \rangle$ be a free group, then*

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)} \quad , \qquad 1 \leq i \leq n$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on the letters $\{x_1, \ldots, x_d\}$.

Theorem 1.5. *(Witt Formula). The number of basic commutators of weight n on d generators is given by the following formula:*

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m},$$

where $\mu(m)$ is the Möbius function, which is defined to be

$$\mu(m) = \begin{cases} 1 & \textit{if} \;\; m = 1, \\ 0 & \textit{if} \;\; m = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \;\; \exists \alpha_i > 1, \\ (-1)^s & \textit{if} \;\; m = p_1 \ldots p_s, \end{cases}$$

where the p_i, $1 \leq i \leq k$, are the distinct primes dividing m.

Proof. See Hall (1959). □

The following definition will be used several times in this article.

Definition 1.6. Let $\mathfrak{V}$ be any variety of groups defined by a set of laws V, and G be any group. Extending the terminology of Hall and Senior (1964), G is called $\mathcal{V}$-capable if there exists a group E which satisfies $G \cong E/V^*(E)$, where $V^*(E)$ is the marginal subgroup of E with respect to $\mathfrak{V}$. (See also Moghaddam and Kayvanfar, 1997, for the definition of $\mathfrak{V}$-capability and Burns and Ellis, 1998, for $\mathfrak{N}_c$-capability.)

According to Definition 1.6 capable groups are $\mathfrak{A}$-capable groups, where $\mathfrak{A}$ is the variety of abelian groups.

The following definition and theorem are taken from Moghaddam and Kayvanfar (1997), and contains a necessary and sufficient condition for a group to be $\mathfrak{V}$-capable.

Definition 1.7. Let $\mathfrak{V}$ be any variety and G be any group. Define $V^{**}(G)$ as follows:

$$V^{**}(G) = \cap\{\psi(V^*(E)) \mid \psi : E \overset{onto}{\longrightarrow} G \ , \ ker\psi \subseteq V^*(E)\}.$$

Note that if $\mathfrak{V}$ is the variety of abelian groups, then the above notion which has been first studied in Beyl., et al. (1979), is denoted by $Z^*(G)$ and called epicenter in Burns and Ellis (1998). Also the above notion has been studied in Burns and Ellis (1998), for the variety $\mathfrak{N}_c$.

Theorem 1.8. *With the above notations and assumptions $G/V^{**}(G)$ is the largest quotient of G which is $\mathfrak{V}$-capable, and hence G is $\mathfrak{V}$-capable if and only if $V^{**}(G) = 1$.*

The following theorem and its conclusion state the relationship between $\mathfrak{V}$-capability and Baer invariants.

Theorem 1.9. *Let $\mathfrak{V}$ be any variety of groups, G be any group, and N be a normal subgroup of G contained in the marginal subgroup with respect to $\mathfrak{V}$. Then the natural homomorphism $\mathfrak{V}M(G) \longrightarrow \mathfrak{V}M(G/N)$ is injective if and only if $N \subseteq V^{**}(G)$, where $\mathfrak{V}M(G)$ is the Baer invariant of G with respect to $\mathfrak{V}$.*

Proof. See Moghaddam and Kayvanfar (1997). □

In the finite case the following theorem is easier to use than the proceeding ones.

Theorem 1.10. *Let $\mathfrak{V}$ be any variety and G be any group with $V(G) = 1$. If $\mathfrak{V}M(G)$ is finite, and N is a normal subgroup of G such that $\mathfrak{V}M(G/N)$ is also finite, then the natural homomorphism $\mathfrak{V}M(G) \longrightarrow \mathfrak{V}M(G/N)$ is injective if and only if $|\mathfrak{V}M(G/N)| = |\mathfrak{V}M(G)|$.*

Proof. It is easy to see that with the assumption of the theorem we have $\mathfrak{V}M(G) \cong V(F)/[RV^*F]$ and $\mathfrak{V}M(G/N) \cong V(F)/[SV^*F]$ in which $G \cong F/R$ is a free presentation for G and $N \cong S/R$. Therefore the kernel of the natural homomorphism $\mathfrak{V}M(G) \longrightarrow \mathfrak{V}M(G/N)$ is the group $[SV^*F]/[RV^*F]$. Considering the finiteness of $\mathfrak{V}M(G)$ and $\mathfrak{V}M(G/N)$, the result easily follows. □

As a useful consequence of Theorem 1.9 we have:

Corollary 1.11. *An abelian group G is $\mathfrak{V}$-capable if and only if the natural homomorphism $\mathfrak{V}M(G) \longrightarrow \mathfrak{V}M(G/\langle x\rangle)$ has a non-trivial kernel for all non-identity elements x in $V^*(G)$.*

The following fact is used in the last section [Stroud, (1965), Theorem 1.2(b)]

Theorem 1.12. *Let u and v be two words in independent variables and $w = [u, v]$. Then, in any group G,*
(i) $w(G) = [u(G), v(G)]$
(ii) if $A = C_G(u(G))$, $B = C_G(v(G))$, $L/A = v^*(G/A)$, and $M/B = u^*(G/B)$, then $w^*(G) = L \cup M$.

To use these results we need an explicit structure for the Baer invariants of finitely generated abelian groups with respect to the variety $\mathfrak{V}$ as defined. This will be done in Theorem 2.6.

2 Computing $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-Multipliers

Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group with $n_{i+1} \mid n_i$ for all $1 \leq i \leq t-1$, where for any group X, $X^{(n)}$ denotes the group $X \oplus X \oplus \cdots \oplus X$ (n copies). Let $F = F\langle x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+t}\rangle$ be the free group on the set $\{x_1, \ldots, x_{k+t}\}$. It is easy to see that

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1,$$

is a free presentation for G in which $R = \prod_{i=1}^{t} R_i\gamma_2(F)$, where $R_i = \langle x_{k+i}^{n_i}\rangle$, so the Baer invariant of G with respect to $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$ is

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \frac{R \cap [\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[R, \ _{c_1}F, \gamma_{c_2+1}(F)][R, \ _{c_2}F, \gamma_{c_1+1}(F)]}.$$

Since $R \supseteq \gamma_2(F)$ we have

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \frac{[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[R, \ _{c_1}F, \gamma_{c_2+1}(F)][R, \ _{c_2}F, \gamma_{c_1+1}(F)]}.$$

In order to find the structure of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)$, we need the following notation and lemmas. Using Definition 1.3, we define the following set when $c_1 \geq c_2$.

$$A = \{[\beta, \alpha] \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha, \\ wt(\beta) = c_1 + 1, \ wt(\alpha) = c_2 + 1 \ \}.$$

Lemma 2.1. *If $c_1 \leq 2c_2$, then every element of A is a basic commutator on X.*

Proof. Every element of A has the form $[\beta, \alpha]$, where β and α are basic commutators on X, $\beta > \alpha$ and $wt(\beta) = c_1 + 1, wt(\alpha) = c_2 + 1$. Now, let $\beta = [\beta_1, \beta_2]$, then in order to show that $[\beta, \alpha]$ is a basic commutator on X, it is enough to show that $\beta_2 \leq \alpha$. Since $\beta = [\beta_1, \beta_2]$ is a basic commutator on X,

$\beta_1 > \beta_2$ and hence $wt(\beta_2) \leq \frac{1}{2}wt(\beta)$. Now, if $c_1 \leq 2c_2$, then $\frac{1}{2}(c_1+1) < c_2+1$. Thus, since $wt(\beta) = c_1 + 1$, we have

$$wt(\beta_2) \leq \frac{1}{2}wt(\beta) = \frac{1}{2}(c_1+1) < c_2+1 = wt(\alpha).$$

Therefore $\beta_2 < \alpha$ and hence the result holds. □

Now put $H = [R, {}_{c_1}F, \gamma_{c_2+1}(F)][R, {}_{c_2}F, \gamma_{c_1+1}(F)] \cap \gamma_{c_1+c_2+3}(F)$ we have the following.

Lemma 2.2. $[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)] \equiv \langle A \rangle \pmod{H}$.

Proof. Let $[\beta, \alpha]$ be a generator of the group $[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]$, so we have $\beta \in \gamma_{c_1+1}(F)$ and $\alpha \in \gamma_{c_2+1}(F)$. Now by Theorem 1.4 we can write $\beta = \beta_1\beta_2 \dots \beta_r\eta$ and $\alpha = \alpha_1\alpha_2 \dots \alpha_s\mu$ in which the β_j are basic commutators on X of weight c_1+1, the α_i are basic commutators on X of weight c_2+1, $\eta \in \gamma_{c_1+2}(F)$ and $\mu \in \gamma_{c_2+2}(F)$. Now $[\beta, \alpha]$will be a product of factors of the forms $[\beta_j, \alpha_i]^{f_{ij}}$, $[\beta_j, \mu]^{g_j}$, $[\eta, \alpha_i]^{h_i}$ and $[\eta, \mu]^k$, in which $f_{ij}, g_j, h_i, k \in \gamma_{c_2+1}(F)$. Now by the Three Subgroup Lemma it is easy to see that $[\beta_j, \alpha_i, f_{ij}], [\beta_j, \mu]^{g_j}, [\eta, \alpha_i]^{h_i}$ and $[\eta, \mu]^k \in H$. Hence the result holds. □

Now the group $[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]/H$ is the group generated by the set $\bar{A} = \{aH \mid a \in A\}$. The following shows that it is in fact the free abelian group with the basis $\bar{A}$.

Lemma 2.3. *With the above notation and assumptions* $[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]/H$ *is the free abelian group with the basis* $\bar{A}$.

Proof. The group is abelian and generated by $\bar{A}$, which is the image of the elements of A modulo H. The elements of A are basic commutators of weight c_1+c_2+2 (Lemma 2.1), and hence linearly independent over $\gamma_{c_1+c_2+3}(F)$; the latter contains H, so the elements of A are also linearly independent modulo H. □

Now by the isomorphism

$$[\mathfrak{N}_{c_1},\mathfrak{N}_{c_2}]M(G)\cong\frac{[\gamma_{c_1+1}(F),\gamma_{c_2+1}(F)]/H}{[R,\ _{c_1}F,\gamma_{c_2+1}(F)][R,\ _{c_2}F,\gamma_{c_1+1}(F)]/H},$$

in order to determine the explicit structure of $[\mathfrak{N}_{c_1},\mathfrak{N}_{c_2}]M(G)$ we only need to determine the structure of $[R,\ _{c_1}F,\gamma_{c_2+1}(F)][R,\ _{c_2}F,\gamma_{c_1+1}(F)]/H$. We actually show that the mentioned group is free abelian with basis $\cup_{i=1}^{t}\bar{B}_i$ where the B_i consist of n_ith powers of suitable elements of $\bar{A}$. To do this we need the following lemma.

Lemma 2.4. *With the previous notation we have*

$$[R,\ _{c_1}F,\gamma_{c_2+1}(F)]\equiv\prod[R_i,\ _{c_1}F,\gamma_{c_2+1}(F)]\pmod H$$

and

$$[R,\ _{c_2}F,\gamma_{c_1+1}(F)]\equiv\prod[R_i,\ _{c_2}F,\gamma_{c_1+1}(F)]\pmod H$$

Proof. By routine commutator calculus we have

$$\begin{aligned}[R,_{c-1}F,\gamma_{c_2+1}(F)] &= \prod_{i=1}^{t}[R_i^F\gamma_2(F),_{c-1}F,\gamma_{c_2+1}(F)]\\ &= \prod_{i=1}^{t}[R_i^F,_{c-1}F,\gamma_{c_2+1}(F)][\gamma_2(F),_{c-1}F,\gamma_{c_2+1}(F)]\\ &\equiv \prod_{i=1}^{t}[R_i,_{c-1}F,\gamma_{c_2+1}(F)]\pmod H.\end{aligned}$$

□

The following lemma will do most of the work.

Lemma 2.5. $[R,\ _{c_1}F,\gamma_{c_2+1}(F)][R,\ _{c_2}F,\gamma_{c_1+1}(F)]/H$ *is the free abelian group with the basis* $\cup_{j=1}^{t}\bar{B}_j$, *where*

$B_j=\{[\beta,\alpha]^{n_j}\mid[\beta,\alpha]\in A$ *and* x_{k+j} *does occur in* $[\beta,\alpha]\}$ *and* $^-$ *denotes the natural homomorphism* $[\gamma_{c_1+1}(F),\gamma_{c_2+1}(F)]\longrightarrow[\gamma_{c_1+1}(F),\gamma_{c_2+1}(F)]/H$.

Proof. Clearly $[R,\ {}_{c_1}F] \equiv \prod_{i=k+1}^{k+t}[R_i,\ {}_{c_1}F] \pmod{\gamma_{c_1+2}(F)}$.
Also

$$[R_i,\ {}_{c_1}F] \equiv \langle \beta^{n_i} \mid \beta \text{ is a basic commutator of weight } c_1+1 \text{ on } X \text{ s.t. } x_{k+i} \text{ does appear in it} \rangle \pmod{\gamma_{c_1+2}(F)}.$$

Therefore

$$[R_i,\ {}_{c_1}F, \gamma_{c_2+1}(F)] \equiv \langle [\beta,\alpha]^{n_i} \mid\ [\beta,\alpha] \in A \text{ and } x_{k+i} \text{ does appear in } \beta\ \rangle \pmod{H}.$$

Similarly,

$$[R_i,\ {}_{c_2}F, \gamma_{c_1+1}(F)] \equiv \langle [\beta,\alpha]^{n_i} \mid\ [\beta,\alpha] \in A \text{ and } x_{k+i} \text{ does appear in } \alpha \rangle \pmod{H}.$$

Hence $[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)][R,\ {}_{c_2}F, \gamma_{c_1+1}(F)] \equiv \langle \bigcup B_j \rangle \pmod{H}$.

□

Now we are in a position to give an explicit structure for $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)$. It only remains to compute $|B_j|$ for $j = 1, ..., t$. Bearing in mind Lemma 2.3 it is clear that each $\bar{B}_j$ is linearly independent modulo H therefore the size of $\bar{B}_j$ is as same as that of B_j, so it is enough to compute the size of B_j. The cases $c_1 = c_2$ and $c_1 > c_2$ are essentially different in computing $|B_j|$. If $c_1 > c_2$ for an arbitrary j we can write $B_j = B_{1j} \cup B_{2j} \cup B_{3j}$ in which

$B_{1j} = \{[\beta,\alpha]^{n_j} \mid [\beta,\alpha]^{n_j} \in B_j \text{ and } x_{k+j} \text{ only appears in } \beta\}$,

$B_{2j} = \{[\beta,\alpha]^{n_j} \mid [\beta,\alpha]^{n_j} \in B_j \text{ and } x_{k+j} \text{ only appears in } \alpha\}$,

$B_{3j} = \{[\beta,\alpha]^{n_j} \mid [\beta,\alpha]^{n_j} \in B_j \text{ and } x_{k+j} \text{ appears in both } \beta \text{ and } \alpha\}$.

It is easy to see that the union is disjoint and we have

$|B_{1j}| = (\chi_{c_1+1}(k+j) - \chi_{c_1+1}(k+j-1))\chi_{c_2+1}(k+j-1)$,

$|B_{2j}| = \chi_{c_1+1}(k+j-1)(\chi_{c_2+1}(k+j) - \chi_{c_2+1}(k+j-1))$,

and

$|B_{3j}| = (\chi_{c_1+1}(k+j) - \chi_{c_1+1}(k+j-1))(\chi_{c_2+1}(k+j) - \chi_{c_2+1}(k+j-1))$,

so $|B_j| = \chi_{c_1+1}(k+j)\chi_{c_2+1}(k+j) - \chi_{c_1+1}(k+j-1)\chi_{c_2+1}(k+j-1)$.

In the case $c_1 = c_2$ it is easy to see that A is in fact the set of all basic commutators of weight 2 on the set of all basic commutators of weight c_1, so we have $|B_j| = \chi_2(\chi_{c_1+1}(k+j)) - \chi_2(\chi_{c_1+1}(k+j-1))$.

Now the following theorem gives the desired structure.

Theorem 2.6. *Let $G \cong \mathbf{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group with $n_{i+1} \mid n_i$ for all $1 \leq i \leq t-1$, if $c_2 \leq c_1 \leq 2c_2$ then,*

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \mathbb{Z}^{(b_k)} \oplus \mathbb{Z}_{n_1}^{(b_{k+1}-b_k)} \oplus \mathbb{Z}_{n_2}^{(b_{k+2}-b_{k+1})} \oplus \ldots \oplus \mathbb{Z}_{n_t}^{(b_{k+t}-b_{k+t-1})}$$

where $b_i = \chi_{c_1+1}(i)\chi_{c_2+1}(i)$, if $c_1 > c_2$ and $b_i = \chi_2(\chi_{c_1+1}(i))$ if $c_1 = c_2$.

Proof. The proof easily follows from Lemma 2.5. □

Comparing this theorem with the main theorem of Mashayekhy and Parvizi (2006), it is easy to see that they agree on the variety $\mathfrak{N}_{c,1}$ in the formula for the Baer invariant.

3 $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-Capability

The concept of capable groups occured in work of P. Hall for classifying p-groups. Determining such groups is an interesting problem to study. Some researches has done on it which for example see Burns and Ellis (1998), Ellis (1996), Moghaddam and Kayvanfar (1997), and Magidin (2005). In this section we explicitly determine the structure of all $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable groups in the class of finitely generated abelian groups. When $c_2 \leq c_1 \leq 2c_c$ to do this we wish to use Theorems 1.9, 1.10, and Corollary 1.11. To use them the structure of the subgroups of a finitely generated abelian group is needed as well as the structure of the Baer invariant of G. Theorem 2.6 gives the latter and the following will determine the structure of the desired subgroups.

Lemma 3.1. *Let G be a finitely generated abelian group and $H \leq G$. Then $r_0(G) = r_0(G/H) + r_0(H)$, where $r_0(X)$ is the torsion free rank of a finitely generated abelian group X.*

Proof. See Fuchs (1970). □

In the case of p-groups the following theorem has an important role in our investigation.

Theorem 3.2. *Let $G \cong \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p-group, where $\alpha_{i+1} \leq \alpha_i$ for all $1 \leq i \leq k-1$, and let H be a subgroup of G. Then $H \cong \mathbb{Z}_{p^{\beta_1}} \oplus \mathbb{Z}_{p^{\beta_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\beta_k}}$ where $\beta_{i+1} \leq \beta_i$ for all $1 \leq i \leq k-1$ and $0 \leq \beta_i \leq \alpha_i$ for $1 \leq i \leq k$.*

Proof. See Fuchs (1970). □

Theorem 3.3. *Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for $1 \leq i \leq t-1$, and let H be a finite subgroup of G. Then $H \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}$, where $m_{i+1} \mid m_i$ for all $1 \leq i \leq t-1$ and $m_i \mid n_i$ for all $1 \leq i \leq t$.*

Proof. Trivially $H \leq t(G)$, the maximal torsion subgroup of G, so without loss of generality we may assume that G is finite. It is well known that $G \cong S_{p_1} \oplus \cdots \oplus S_{p_t}$, where S_{p_i} is the p_i-Sylow subgroup of G. One may easily show that if $H \cong S'_{p_1} \oplus \cdots \oplus S'_{p_t}$ is the same decomposition for H, then $S'_{p_i} \leq S_{p_i}$ for all $1 \leq i \leq t$. Therefore it is enough to consider finite abelian p-groups. Now Theorem 3.2 completes the proof. □

Proceeding now to $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capability, note that $[\mathfrak{N}_1, \mathfrak{N}_1] = \mathfrak{S}_2$ is the variety of metabelian groups (that is groups of solvability length at most 2) and, according to Theorem 2.6, $\mathfrak{S}_2 M(G) = 0$ whenever G has at most two generators. But if $c_2 < c_1 \leq 2c_2$ or $c_1 = c_2 > 1$, then the Baer invariant is trivial only if G is cyclic. This suggests dealing with the two cases separately, and so we assume first that $c_2 < c_1 \leq 2c_2$ or $c_1 = c_2 > 1$.

The method we use here implies separating the cases which G is finite or infinite.

Case one: G is finite abelian group.

Theorem 3.4. *Let $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finite abelian group, where $n_{i+1} \mid n_i$ for $1 \leq i \leq t-1$, then G is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable if and only if $t \geq 2$ and $n_1 = n_2$.*

Proof. We will establish the necessity by contrapositive. If $t = 1$ then G and all its quotients are cyclic abelian groups so by Theorem 2.6 $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N) = 0$ for any normal subgroup N of G, hence by Corollary 1.11 G is not $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable. On the other hand if $n_1 \neq n_2$, then let $x = (\bar{n}_2, \bar{0}, \ldots, \bar{0})$, since $G/\langle x \rangle \cong \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_2} \cdots \oplus \mathbb{Z}_{n_t}$, Theorem 2.6 shows the Baer invariants for G and $G/\langle x \rangle$ have the same size. This shows G is not $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable in this case by Corollary 1.11.

For sufficiency, assume $t \geq 2$ and $n_1 = n_2$. By Corollary 1.11 it is enough to show that if $N < G$ and $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \longrightarrow [\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N)$ is injective, then N is trivial.

In finite abelian groups each quotient is isomorphic to a subgroup and vice versa. Now let $N < G$, then G/N is isomorphic to a subgroup of G, H say; so by Theorem 3.3 $H \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}$, where $m_{i+1} \mid m_i$ for all $1 \leq i \leq t-1$ and $m_i \mid n_i$ for all $1 \leq i \leq t$. Computing $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)$ and $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(H)$ using Theorem 2.6 shows that $|[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)| = |[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(H)|$ if and only if $m_i = n_i$ for all $2 \leq i \leq t$, but $n_1 = n_2$ by hypothesis which implies $n_1 = m_1$ which is equivalent to $H = G$ and hence $N = 0$. Therefore G is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable. □

Now we consider the infinite case.

Theorem 3.5. *Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be an infinite finitely generated abelian group, where $n_{i+1} \mid n_i$ for $1 \le i \le t-1$, then G is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable, if and only if $k \ge 2$.*

Proof. We first show that if $k = 1$, then there exists a nontrivial element x of G for which the natural homomorphism $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \longrightarrow [\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/\langle x\rangle)$ is injective, proving the necessity by contrapositive.

Suppose $k = 1$, then $G \cong \mathbb{Z} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$. Let $x = (n_1, \bar{0}, \ldots, \bar{0})$, so $G/\langle x\rangle \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$. Now by Theorem 2.6 we have $|[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)| = |[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/\langle x\rangle)|$, so the result follows.

For sufficiency, assume that $k \ge 2$. It is enough to show that there is no nontrivial subgroup N of G for which $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \longrightarrow [\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N)$ is injective. If N is an infinite subgroup then $r_0(G/N) < r_0(G)$, so by Theorem 2.6 the torsion free rank of the Baer invariant of G/N is strictly smaller than that of the invariant for G, so no injection is possible. On the other hand if N is contained in the torsion subgroup of G, so $G/N \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}$, where $m_{i+1} \mid m_i$ and $m_i \mid n_i$ for all $1 \le i \le t-1$, so by Theorem 2.5 we have $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \mathbb{Z}^{(b_k)} \oplus \mathbb{Z}_{n_1}^{(b_{k+1}-b_k)} \oplus \cdots \oplus \mathbb{Z}_{n_t}^{(b_{k+t}-b_{k+t-1})}$
and
$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N) \cong \mathbb{Z}^{(b_k)} \oplus \mathbb{Z}_{m_1}^{(b_{k+1}-b_k)} \oplus \cdots \oplus \mathbb{Z}_{m_t}^{(b_{k+t}-b_{k+t-1})}$. It is easy to show that

$$t([\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)) = \mathbb{Z}_{n_1}^{(b_{k+1}-b_k)} \oplus \cdots \oplus \mathbb{Z}_{n_t}^{(b_{k+t}-b_{k+t-1})}$$

and

$$t([\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N)) = \mathbb{Z}_{m_1}^{(b_{k+1}-b_k)} \oplus \cdots \oplus \mathbb{Z}_{m_t}^{(b_{k+t}-b_{k+t-1})}.$$

The image of the torsion subgroup of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)$ under the natural homomorphism must lie in the torsion subgroup of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N)$, so if the homomorphism is injective, then we must $t(G) = t(G/N)$; $t(G/N) = t(G)/N$, this proves that if the map is ingective then $N = 0$, completing the proof. □

Remark 3.6. Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group, with $n_{i+1} \mid n_i$ for all $1 \le i \le t-1$. Baer's result Baer (1938), implies that G is capable if and only if $k \ge 2$ or $k = 0$, $t \ge 2$ and $n_1 = n_2$. Burns and Ellis (1998), proved that G is $\mathfrak{N}_c$-capable if and only if it is capable. We now see that this also holds for $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capability with suitable conditions on c_1 and c_2.

In the case $c_1 = c_2 = 1$ we only state the characterization of the $\mathfrak{S}_2$-capable groups among finitely generated abelian groups. The proofs are similar to those of Theorems 3.4 and 3.5. The needed lemmas and their proofs can be restated with necessary changes similar to Theorems 3.4 and 3.5. Note that in this case the variety $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$ is actually the variety of metabelian groups $\mathfrak{S}_2$.

Theorem 3.7. *Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \le i \le t-1$. Then G is $\mathfrak{S}_2$-capable if and only if $k \ge 3$, or $k = 0$, $t \ge 3$, and $n_1 = n_2 = n_3$.*

4 The relation between capability and $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capability.

As before mentioned in the beginning of section 3, capability is one of the interesting concepts to study. Theorem 3.7 suggests to consider the relationship between capability and varietal capability. More precisely we may ask under what conditions capability implies varietal capability or vice versa? This section answers the above question in part and show the two concepts does not coincide in general. Having a review of what has been done, Burns and Ellis (1998), after introducing the concept of c-capability, showed that every $c+1$-capable group is c-capable group and hence is a capable group, but they construct a 2-group which is capable but is not 2-capable. This example shows that even in class of p-groups the capability does not imply c-capability. However as an interesting fact they proved for finitely generated abelian groups capability and c-capability are equivalent. Now we concentrate on $\mathfrak{S}_\ell$, the variety of solvable groups of length at most ℓ and prove the following theorem.

Theorem 4.1. *Let $\mathfrak{S}_\ell$ be the variety of solvable groups of length at most ℓ. Then every $\mathfrak{S}_\ell$-capable group is $\mathfrak{S}_{\ell-1}$-capable.*

Proof. Using Theorem 1.12 we have

$$\frac{{\mathfrak{S}_\ell}^*(G)}{C_G(\mathfrak{S}_{\ell-1}(G))} = \frac{G}{C_G(\mathfrak{S}_{\ell-1}(G))}.$$

Now the result follows immediately. □

An immediate consequence of the above theorem is that every $\mathfrak{S}_\ell$-capable group is capable.

Comparing with c-capability, there is no difference in results. The next theorem shows the converse of Theorem 4.1 is not true in general, just the same as the result of Burns and Ellis. But the difference is that here the counter example is in the class of finitely generated abelian groups, exactly where the notions of capability and c-capability coincide.

Theorem 4.2. *Let n be a natural number, then the group $\mathbf{Z}_n \oplus \mathbf{Z}_n$ is capable but it is not $\mathfrak{S}_2$-capable*

Finally we consider $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capability. This can be considered as an special case of $[\mathfrak{V}, \mathfrak{W}]$-capability and one may suggest dealing with the relation between $\mathfrak{V}$-capability, $\mathfrak{W}$-capability and $[\mathfrak{V}, \mathfrak{W}]$-capability. Here, there can not be explained more about that situation except the following theorem which has a proof similar to that of Theorem 4.1.

Theorem 4.3. *Let $\mathfrak{V}$ be any variety then every $[\mathfrak{V}, \mathfrak{V}]$-capable group is $\mathfrak{V}$-capable group.*

As before stated the converse of the above theorem is not true in general, but in class of finitely generated abelian groups we have the following theorem.

Theorem 4.4. *Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq t-1$. Then the following are equivalent:*
(i) G is capable.
(ii) G is $\mathfrak{N}_c$-capable for some $c \geq 1$.
(iii) G is $\mathfrak{N}_c$-capable for all $c \geq 1$.
(iv) G is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable for all c_1, c_2 with $c_2 < c_1 \leq 2c_2$ or $c_1 = c_2 > 1$.
(v) $k \geq 2$, or $k = 0$, $t \geq 2$, and $n_1 = n_2$.

References

Baer, R. (1938). Groups with preassigned central and central quotient group. *Trans. Amer. Math. Soc.* 44:387-412.

Baer, R. (1945). Representations of groups as quotient groups I,II,III. *Trans. Amer. Math. Soc.* 58:295-419.

Beyl, F. R., Felgner, U., and Schmid, P. (1979). On Groups occuring as center factor groups. *J. of Algebra* 61:161-177.

Burns, J., Ellis, G. (1997). On the nilpotent multipliers of a group. *Math. Z.* 226:405-428.

Ellis, G. (1996) Capability, homology, and central series of a pair of groups. *J. of Algebra* 179:31-46.

Fuchs, L. (1970). Infinite abelian groups. *Academic Press, New York and London.*

Hall, M. (1959). The theory of groups. *MacMillan Company, New York.*

Hall, Jr. M., Senior, J. K. (1964). The groups of order 2^n $(n \leq 6)$.

MacMillan, New York.

Hulse, J. A., Lennox, J. C. (1976). Marginal series in groups. *Proc. the Royal Soc. of Edinburgh* 76A:139-154.

Leedham-Green, C. R., McKay, S. (1976). Baer-invariant, isologism, varietal Laws and homology. *Acta Math.* 137:99-150.

Magidin, A. Capability of nilpotent products of cyclic groups. *J. Group Theory* 8:4:431-452.

Mashayekhy, B., Moghaddam, M. R. R. (1997). Higher Schur-multiplicator of a finite abelian group. *Algebra Colloquium* 4:3:317-322.

Mashayekhy, B., Parvizi, M. (2006). Polynilpotent multipliers of finitely generated abelian groups. *J. Math. Game Theory Algebra* 16:93-102.

Moghaddam, M. R. R., Kayvanfar, S. (1997). A new notion derived from varieties of groups. *Algebra Colloquium* 4:1:1-11.

Parvizi, M., Mashayekhy, B., and Kayvanfar, S. Polynilpotent capability of finitely generated abelian groups. Submitted.

Schur, I. (1907). Untersuchungen über die darstellung der endlichen gruppen durch gebrochene lineare substitutionen. *J. Reine Angew. Math.* 132:85-137.

Stroud, P. W. (1965). On a property of verbal and marginal subgroups. *Proc. Cambridge Phil. Soc.* 61:41-48.

Teague, T. K. (1974). On the Engel margin. *Pacific Journal of Mathematics* 50:1:205-214.