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On the Order of Schur Multipliers of Finite Abelian p-Groups

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Abstract

Let G be a finite p-group of order p^n with $|M(G)| = p^{\frac{n(n-1)}{2}-t}$, where $M(G)$ is the Schur multiplier of G. Ya.G. Berkovich, X. Zhou, and G. Ellis have determined the structure of G when $t = 0, 1, 2, 3$. In this paper, we are going to find some structures for an abelian p-group G with conditions on the exponents of $G, M(G)$, and $S_2M(G)$, where $S_2M(G)$ is the metabelian multiplier of G.

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1 Introduction and Preliminaries

Let G be any group with a presentation $G \cong F/R$, where F is a free group. Then the Baer invariant of G with respect to the variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]},$$

where V is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of F and

$$[RV^*F] = \langle v(f_1, ..., f_{i-1}, f_i r, f_{i+1}, ..., f_n)v(f_1, ..., f_i, ..., f_n)^{-1} | \\ r \in R, f_i \in F, v \in V, 1 \le i \le n, n \in N\rangle.$$

In particular, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer invariant of the group G will be $(R \cap F')/[R, F]$ which is isomorphic to the well-known notion the Schur multiplier of G, denoted by $M(G)$ (see [5,6] for further details).
If $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, ..., c_t)$, $\mathcal{N}_{c_1,c_2,...,c_t}$, then the Baer invariant of a group G with respect to this variety is as follows:

$$\mathcal{N}_{c_1,c_2,...,c_t}M(G) = \frac{R \cap \gamma_{c_t+1} \circ ... \circ \gamma_{c_1+1}(F)}{[R, {}_{c_1}F, {}_{c_2}\gamma_{c_1+1}(F), ..., {}_{c_t}\gamma_{c_{t-1}+1} \circ ... \circ \gamma_{c_1+1}(F)]}, \qquad (1)$$

where $\gamma_{c_t+1} \circ ... \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(...(\gamma_{c_1+1}(F))...))$ are the term of iterated lower central series of F. See [4] for the equality

$$[R\mathcal{N}^*_{c_1,...,c_t}F] = [R, {}_{c_1}F, {}_{c_2}\gamma_{c_1+1}(F), ..., {}_{c_t}\gamma_{c_{t-1}+1} \circ ... \circ \gamma_{c_1+1}(F)].$$

In particular, if $c_i = 1$ for $1 \le i \le t$, then $\mathcal{N}_{c_1,c_2,...,c_t}$ is the variety of solvable groups of length at most $t \ge 1, \mathcal{S}_t$.
In 1956, J.A. Green [3] showed that the order of the Schur multiplier of a finite p-group of order p^n is bounded by $p^{\frac{n(n-1)}{2}}$, and hence equals to $p^{\frac{n(n-1)}{2}-t}$, for some nonnegative integer t. In 1991, Ya.G. Berkovich [1] has determined all finite p-groups G for which $t = 0, 1$. The groups for which $t = 0$ are exactly elementary ablian p-groups, and the groups for which $t = 1$ are cyclic groups of order p^2 or the nonabelian group of order p^3 with exponent $p > 2$. In 1994, X. Zhou [7] found all finite p-groups for $t = 2$. He showed that these groups are the direct product of two cyclic groups of order p^2 and p or the direct product of a cyclic group of order p and the nonabelian group of order p^3 and exponent $p > 2$ or the dihedral group of order 8. G. Ellis [2] determined all finite p-groups G with $t = 0, 1, 2, 3$ in a quite different method to that of [1] and [7] as follows:

Theorem 1.1 *([2]). Let G be a group of prime-power order p^n. Suppose that $M(G)$ has order $p^{\frac{n(n-1)}{2}-t}$. Then $t \ge 0$ and*
(i) $t = 0$ if and only if G is elementary abelian;
(ii) $t = 1$ if and only if $G \cong \mathbf{Z}_{p^2}$ or $G \cong E_1$;
(iii)$t = 2$ if and only if $G \cong \mathbf{Z}_p \times \mathbf{Z}_{p^2}, G \cong D$ or $G \cong \mathbf{Z}_p \times E_1$;
(iv)$t = 3$ if and only if $G \cong \mathbf{Z}_{p^3}, G \cong \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_{p^2}, G \cong D \times \mathbf{Z}_{p^2}, G \cong E_2, G \cong Q$ or $G \cong \mathbf{Z}_p \times \mathbf{Z}_p \times E_1$.
Here $\mathbf{Z}_{p^m}$ denotes the cyclic group of order p^m, D denotes the dihedral group of order 8, Q denotes the quaternion group of order 8, E_1 denotes the extra special group of order p^3 with odd exponent p, and E_2 denotes the extra special

group of order p^3 with odd exponent p^2.

Now, in this paper, we are going to find some structures for the p-group G when G is abelian and $|\Phi(G)| = p^a$ with conditions on the exponents of $G, M(G)$, and $S_2M(G)$. The following useful theorem of I. Schur is frequently used in our method.

Theorem 1.2 *(I. Schur [5]). Let $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus ... \oplus \mathbf{Z}_{n_k}$, where $n_{i+1}|n_i$ for all $i \in 1, 2, ..., k-1$ and $k \geq 2$, and let $\mathbf{Z}_n^{(m)}$ denote the direct product of m copies of $\mathbf{Z}_n$. Then*

$$M(G) \cong \mathbf{Z}_{n_2} \oplus \mathbf{Z}_{n_3}^{(2)} \oplus ... \oplus \mathbf{Z}_{n_k}^{(k-1)}$$

Remark 1.3. Let G be an abelian group with a free presentation F/R. Since $F' \leq R, \mathcal{N}_{c_1}M(G) = \gamma_{c_1+1}(F)/[R, {}_{c_1}F]$. Now, we can consider $\gamma_{c_1+1}(F)/[R, {}_{c_1}F]$ as a free presentation for $\mathcal{N}_{c_1}M(G)$ and hence

$$\mathcal{N}_{c_2}M(\mathcal{N}_{c_1}M(G)) = \frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F))}{[R, {}_{c_1}F, {}_{c_2}\gamma_{c_1+1}F]}.$$

Therefore by (1) we have

$$\mathcal{N}_{c_1,c_2}M(G) = \mathcal{N}_{c_2}M(\mathcal{N}_{c_1}M(G)).$$

By continuing the above process we can show that

$$\mathcal{N}_{c_1,c_2...,c_t}M(G) = \mathcal{N}_{c_t}M(...\mathcal{N}_{c_2}M(\mathcal{N}_{c_1}M(G))...).$$

In particular, if $c_1 = c_2 = 1$, then we have $S_2M(G) = M(M(G))$.

2 Main Results

Through out the paper we assume that G is an abelian p-group of order p^n with $|M(G)| = p^{\frac{n(n-1)}{2}-t}$.

Lemma 2.1 . *Let $\Phi(G)$, the Frattini subgroup of G, be of order p^a. Then $n = (a(a+1) + 2t + 2m)/2a$, for some $m \in \mathbf{N}_0$.*

Proof. Let $G = \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus ... \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}$, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{n-a}$. By Theorem 1.2 , $M(G) \cong \mathbf{Z}_{p^{\alpha_2}} \oplus \mathbf{Z}_{p^{\alpha_3}}^{(2)} \oplus ... \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}^{(n-a-1)}$ and so $|M(G)| =$

$p^{\alpha_2+2\alpha_3+...+(n-a-1)\alpha_{n-a}}$. But $M(G)$ has order $p^{\frac{n(n-1)}{2}-t}$. Therefore

$$\begin{aligned}\frac{n(n-1)}{2}-t &= \alpha_2+2\alpha_3+...+(n-a-1)\alpha_{n-a}\\ &\geq 1+2+...+(n-a-1)\\ &= \frac{(n-a)(n-a-1)}{2}\\ &= \frac{n^2-(2a+1)n+a(a+1)}{2}.\end{aligned}$$

Hence $2an \geq 2t+a(a+1)$, and the result holds.

Lemma 2.2 . *With the assumption and notation of the previous lemma we have the following inequalities for the exponent of* G,

$$p^{a-m+1} \leq exp(G) \leq p^{a+1}.$$

Proof. Clearly $G/\Phi(G)$ is an elementary abelian p-group of order p^{n-a} and so $exp(G) \leq p^{a+1}$.

For the other inequality let $G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus ... \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}$, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{n-a} \geq 1$. Then similar to the proof of previous lemma we have

$$\begin{aligned}\frac{n(n-1)}{2}-t &= \alpha_2+2\alpha_3+...+(n-a-1)\alpha_{n-a}\\ &\geq \alpha_2+(2+3+...+n-a-1)\\ &\geq \frac{2\alpha_2-2+n^2-(2a+1)n+a(a+1)}{2}\end{aligned}$$

Therefore $n \geq \frac{a(a+1)+2\alpha_2-2+2t}{2a}$ and hence by Lemma 2.1 we have $\alpha_2 \leq m+1$. Now, suppose by contrary $exp(G)=p^{a-k+1}$, where $k > m$, then $\alpha_3 \geq 2$. Thus by Theorem 1.2 we have

$$\begin{aligned}\frac{n(n-1)}{2}-t &= \alpha_2+2\alpha_3+...+(n-a-1)\alpha_{n-a}\\ &= (\alpha_2+\alpha_3+...+\alpha_{n-a})+\alpha_3+2\alpha_4+...+(n-a-2)\alpha_{n-a}\\ &\geq (n-a+k-1)+(2+2+3+...+n-a-2)\\ &= n-a+k+\frac{(n-a-2)(n-a-1)}{2}.\end{aligned}$$

Hence $n \geq \frac{2k+a(a+1)+2+2t}{2a} > \frac{2m+a(a+1)+2+2t}{2a}$ which is a contradiction by Lemma 2.1.

Theorem 2.3 . *With the above notation and assumptions, let G be of exponent p^{a-m+1}. Then* $G \cong \mathbf{Z}_{p^{a-m+1}} \oplus \mathbf{Z}_{p^{m+1}} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-2-copies}$.

Proof. let $G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus ... \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}$, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{n-a} \geq 1$. By the proof of previous lemma we have $\alpha_2 \leq m+1$. If $\alpha_2 \leq m$, then $\alpha_3 \geq 2$ and we have

$$\begin{aligned} \frac{n(n-1)}{2} - t &= \alpha_2 + 2\alpha_3 + ... + (n-a-1)\alpha_{n-a} \\ &= (\alpha_2 + \alpha_3 + ... + \alpha_{n-a}) + \alpha_3 + 2\alpha_4 + ... + (n-a-2)\alpha_{n-a} \\ &\geq (n-a+m-1) + (2+2+3+...+n-a-2) \\ &= n-a+m+\frac{(n-a-2)(n-a-1)}{2}. \end{aligned}$$

Therefore $n \geq \frac{2m+a(a+1)+2+2t}{2a}$ which is a contradiction by Lemma 2.1. Hence the result holds.

Theorem 2.4 . *Further to the previous notation and assumptions, let $m = k+s$ $(k, s \in \mathbf{N}_0)$, $exp(G) = p^{a-k+1}$, and $exp(M(G)) + exp(S_2M(G)) = p^{k+r}$. Then*

$$G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^x} \oplus \mathbf{Z}_{p^{k+r-x}} \oplus \mathbf{Z}_{p^{h_1}} \oplus ... \oplus \mathbf{Z}_{p^{h_f}} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-(f+3)-copies},$$

where
$x = k-s+2r-3+3(h_1-1)+...+(f+2)(h_f-1)$, $h_1 \geq h_2 \geq ... \geq h_f \geq 2$, *and* $f \leq -r+2$.

Proof. Let $G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus ... \oplus \mathbf{Z}_{p^{\alpha_{n-a}}}$, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{n-a}$. By Theorem 1.2 and Remark 1.3 it is easy to see that $exp(M(G)) = p^{\alpha_2}$, $exp(S_2M(G)) = p^{\alpha_3}$, and so by hypothesis $\alpha_2+\alpha_3 = k+r$, and $\alpha_1 = a-k+1$. If $\alpha_3 = 1$, then it is easy to see that $s = 0$ and so $exp(G) = a-m+1$. Hence by Theorem 2.3 $G \cong \mathbf{Z}_{p^{a-m+1}} \oplus \mathbf{Z}_{p^{m+1}} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-2-copies}$.

Now, we can assume that $\alpha_3 \geq 2$ and hence G may have the following structure:

$$G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^x} \oplus \mathbf{Z}_{p^{k+r-x}} \oplus \mathbf{Z}_{p^{h_1}} \oplus ... \oplus \mathbf{Z}_{p^{h_f}} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-(f+3)-copies},$$

where $f \geq 0$ and $h_1 \geq h_2 \geq ... \geq h_f \geq 2$. If $f \geq 1$ since G is a group of order p^n, we have

$(a-k+1)+(k+r)+(h_1+...+h_f)+(n-a-f-3) = n$ so $h_1+...+h_f = -r+f+2$ and hence $h_f = -r+f+2-h_1-...-h_{f-1}$.
But $h_1 \geq h_2 \geq ... \geq h_f \geq 2$, so that $-r+f+2 \geq 2f$ and so $0 \leq f \leq -r+2$.
Now, by Theorem 1.2 we have

$$\frac{n^2-n-2t}{2} = z + (1+2+...+n-a-1),$$

where

$$z = x + 2k + 2r - 2x + 3h_1 + 4h_2 + ... + (f+2)h_f - (1+2+...+f+2).$$

Thus

$$n = \frac{2z + a(a+1) + 2t}{2a}.$$

On the other hand by the hypothesis $n = \frac{2(k+s)+a(a+1)+2t}{2a}$, hence we have $z = k+s$, and the result follows.

Corollary 2.5 . *With the notation and assumptions of previous theorem we have*

(i) $G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+1}} \oplus \mathbf{Z}_{p^{s+1}} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-3-copies}, \quad if \quad r = 2;$

(ii) $G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+2}} \oplus \mathbf{Z}_{p^{s-1}} \oplus \mathbf{Z}_{p^2} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-4-copies}, \quad if \quad r = 1;$

(iii) if $r = 0$, *then*

$$G \cong \begin{cases} \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+4}} \oplus \mathbf{Z}_{p^{s-4}} \oplus \mathbf{Z}_{p^2} \oplus \mathbf{Z}_{p^2} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-5-copies} \\ or \\ \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+3}} \oplus \mathbf{Z}_{p^{s-3}} \oplus \mathbf{Z}_{p^3} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-4-copies} \end{cases};$$

(iv) if $r = -1$, *then*

$$G \cong \begin{cases} \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+8}} \oplus \mathbf{Z}_{p^{s-9}} \oplus \mathbf{Z}_{p^4} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-4-copies} \\ \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+5}} \oplus \mathbf{Z}_{p^{s-6}} \oplus \mathbf{Z}_{p^3} \oplus \mathbf{Z}_{p^2} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-5-copies} \\ \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^{k-s+7}} \oplus \mathbf{Z}_{p^{s-8}} \oplus \mathbf{Z}_{p^2} \oplus \mathbf{Z}_{p^2} \oplus \mathbf{Z}_{p^2} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-6-copies} \end{cases}.$$

Proof. i) If $r = 2$, then $f = 0$. Therefore

$$G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^x} \oplus \mathbf{Z}_{p^{k+r-x}} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-3-copies},$$

where $x = k - s + 1$. Hence the result follows.
ii) If $r = 1$, then $f = 0, 1$. If $f = 0$, then $n = a - k + 1 + k + 1 + n - a - 3$ which is a contradiction. Then $f = 1$ and

$$G \cong \mathbf{Z}_{p^{a-k+1}} \oplus \mathbf{Z}_{p^x} \oplus \mathbf{Z}_{p^{k+r-x}} \oplus \mathbf{Z}_{p^{h_1}} \oplus \underbrace{\mathbf{Z}_p \oplus ... \oplus \mathbf{Z}_p}_{n-a-4-copies},$$

where $h_1 = -r + f + 2 = 2$ and $x = k - s + 2 + 6 - (1 + 2 + 3) = k - s + 2$. Hence the result follows.
iii) If $r = 0$, then $f = 0, 1, 2$. If $f = 0$, then $h_1 + h_2 + ... + h_f = 0$ but we have $h_1 + h_2 + ... + h_f = -r + f + 2 = 2$ which is a contradiction. If $f = 1$, then $x = k - s + 4$ and if $f = 2$, then $x = k - s + 3$.
iv)By a routine calculation similar to (ii) the result holds.

Note that we can continue the above corollary for other integers $r < -1$, but with a boring calculations.

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References

[1] Ya.G. Berkovich, On the order of the commutator subgroup and Schur multiplier of a finite p-group, *J. Algebra,* **144** (1991), 269-272.

[2] G.Ellis, On the Schur multiplier of p-groups, *Communications in Algebra,* **27** no. 9 (1999), 4173-4178.

[3] J.A. Green, On the number of outomorphisms of a finite p-group, *Proc. Roc,* A **237** (1958), 574-581.

[4] N.S. Hekster, Varieties of groups and isologism, *J. Austral. Math. Soc. (Series A),* **46** (1989), 22-60.

[5] G. Karpilovsky, *The Schur Multiplier,* London Math. Soc. Monographs, NewSeries No. 2, 1987.

[6] C.R. Leedham-Green and S. McKay, Baer-invarient, Isologism, Varietal Lows and Homology, *Acta Math.,* **137** (1976), 99-150.

[7] X. Zhou, On the order of Schur multipliers of finite p-groups, *Comm. Algebra,* **22** no. 1 (1994), 1-8.