Some Inequalities for Nilpotent Multipliers of Finite Groups

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Abstract

In this paper we present some inequalities for the order, the exponent and the number of generators of the c-nilpotent multiplier (the Baer invariant with respect to the variety of nilpotent groups of class at most $c \geq 1$) of a finite group and specially of a finite p-group. Our results generalize some previous related results of M.R. Jones and M.R.R. Moghaddam. Also, we show that our results improve some of the previous inequalities.

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1 Introduction and Motivation

Let G be a group with a presentation F/R, where F is a free group. Then the Baer invariant of G with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]},$$

where $V(F)$ is the verbal subgroup of F and

$$[RV^*F] = \langle v(f_1, ..., f_{i-1}, f_i r, f_{i+1}, ..., f_n) v(f_1, ..., f_i, ..., f_n)^{-1} |$$

$$r \in R, f_i \in F, v \in V, 1 \leq i \leq n, n \in N \rangle.$$

One can see that the Bear invariant of a group G is always abelian and independent of the choice of the presentation of G. In particular, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer invariant of the group G will be

$$\frac{R \cap F'}{[R,F]},$$

which is isomorphic to the Schur multiplier of G , denoted by $M(G)$. Also, if $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, then the Baer invariant of the group G will be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]}.$$

We also call it the c-nilpotent multiplier of G, and denote it by $M^{(c)}(G)$ (see [8,10] for further details).

Definition 1. A variety $\mathcal{V}$ is said to be a *Schur-Baer variety* if for any group G for which the marginal factor group $G/V^*(G)$ is finite, then the verbal subgroup $V(G)$ is also finite and $|V(G)|$ divides a power of $|G/V^*(G)|$. Schur [8] proved that the variety of abelian groups, $\mathcal{A}$, is a Schur-Baer variety. Also, Baer [1] proved that if u and v have Schur-Baer property, then the variety defined by the word $[u,v]$ has the above property.

The following theorem gives a very important property of Schur-Baer varieties.

Theorem 2. *([10]) The following conditions on the variety $\mathcal{V}$ are equivalent:*
(i) $\mathcal{V}$ is a Schur-Baer variety.
(ii) For every finite group G, its Baer invariant, $\mathcal{V}M(G)$, is of order dividing a power of $|G|$.

Definition 3. ([4]) The basic commutators on letters $x_1, x_2, ..., x_n, ...$ are defined as follows:
(i) The letters $x_1, x_2, ..., x_n, ...$ are basic commutators of weight one, ordered by setting $x_i < x_j$ if $i < j$.
(ii) If basic commutators c_i of weight $w(c_i) < k$ are defined and ordered, then define basic commutators of weight k by the following rules. $[c_i, c_j]$ is a basic commutator of weight k if and only if
1. $w(c_i) + w(c_j) = k$;
2. $c_i > c_j$;
3. If $c_i = [c_s, c_t]$, then $c_j \geq c_t$.
Then we will continue the order by setting $c \geq c_i$, whenever $w(c) \geq w(c_i)$, fixing any order among those of weight k, and finally numbering them in order.

Theorem 4. *(P. Hall [4]) Let F be the free group on $\{x_1, x_2, ..., x_d\}$, then for all $1 \le i \le n$,*

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)}$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, ..., n+i-1$ on the letters $\{x_1, x_2, ..., x_d\}$.

Theorem 5. *(Witt formula [4]) The number of basic commutators of weight n on d generators is given by the following formula,*

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{\frac{n}{m}},$$

where $\mu(m)$ is the Mobious function, which defined to be

$$\mu(m) = \begin{cases} 1 & ; m = 1, \\ 0 & ; m = p_1^{\alpha_1} ... p_k^{\alpha_k} \quad , \exists \alpha_i > 1 \\ (-1)^s & ; m = p_1 ... p_s, \end{cases}$$

where the p_i are distinct prime numbers.

M.R. Jones in a series of three papers [5,6,7] studied on the order, the exponent and the number of generators of the Schur multiplier of finite groups, specially finite p-groups and presented some interesting inequalities about them. Also M.R.R. Moghaddam [13,14] generalized some of his results. The following are some of them which we deal with in this article.

Theorem 6. *(M.R. Jones [5]) Let G be a p-group of order p^n with center of exponent p^k. Then $|G'||M(G)|$ is no more than $p^{\frac{1}{2}(n-k)(n+k-1)}$. In particular*

$$|G'||M(G)| \le p^{\frac{1}{2}n(n-1)}.$$

Theorem 7. *(M.R. Jones [6]) Let G be a finite group and K any normal subgroup of it. Set $H = G/K$, then*
(i) $|M(H)|$ divides $|M(G)||G' \cap K|$;
(ii) $exp(M(H))$ divides $exp(M(G))$ $exp(G' \cap K)$;
(iii) $d(M(H)) \le d(M(G)) + d(G' \cap K)$.

Corollary 8. *(M.R. Jones [6]) Let G be a finite d-generator group of order p^n. Then*

$$p^{\frac{1}{2}d(d-1)} \le |G'||M(G)| \le p^{\frac{1}{2}n(n-1)}.$$

In 1981 M.R.R. Moghaddam [13,14] gave varietal generalizations of Theorem 7 and corollary 8. He presented the following theorem without the condition of being Schur-Baer on the variety. But this condition seems to be necessary, because $\mathcal{V}M(G)$ is finite if and only if variety $\mathcal{V}$ is a Schur-Baer variety.

Theorem 9. *(M.R.R. Moghaddam [13]) Let $\mathcal{V}$ be a Schur-Baer variety and G be a finite group with a normal subgroup K. Let $H = G/K$, then*
(i) $|\mathcal{V}M(H)|$ *divides* $|\mathcal{V}M(G)||V(G) \cap K|$;
(ii) $exp(\mathcal{V}M(H))$ *divides* $exp(\mathcal{V}M(G))\ exp(V(G) \cap K)$;
(iii) $d(\mathcal{V}M(H)) \leq d(\mathcal{V}M(G)) + d(V(G) \cap K)$.

Corollary 10. *(M.R.R. Moghaddam [13,14]) Let $\mathcal{V}$ be the variety of polynilpotent groups of a given class row. Let G be a finite d-generator group of order p^n. Then*

$$|\mathcal{V}M(\mathbf{Z}_p^{(d)})| \leq |\mathcal{V}M(G)||V(G)| \leq |\mathcal{V}M(\mathbf{Z}_p^{(n)})|,$$

where $\mathbf{Z}_n^{(m)}$ denotes the direct sum of m copies of $\mathbf{Z}_n$.

The following theorem is useful in our investigation.

Theorem 11. *(B. Mashayekhy and M.R.R. Moghaddam [11]) Let $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus ... \oplus \mathbf{Z}_{n_k}$ be a finite abelian groups, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq k-1$. Then for all $c \geq 1$, the c-nilpotent multiplier of G is*

$$M^{(c)}(G) = \mathbf{Z}_{n_2}^{(\chi_{c+1}(2))} \oplus \mathbf{Z}_{n_3}^{(\chi_{c+1}(3)-\chi_{c+1}(2))} \oplus ... \oplus \mathbf{Z}_{n_k}^{(\chi_{c+1}(k)-\chi_{c+1}(k-1))}.$$

A useful corollary can be obtained by using Corollary 10 and Theorem 11.

Corollary 12. *Let G be a finite d-generator p-group of order p^n, then*

$$p^{\chi_{c+1}(d)} \leq |M^{(c)}(G)||\gamma_{c+1}(G)| \leq p^{\chi_{c+1}(n)}.$$

The next theorem gives some upper bounds in terms of normal subgroups and factor groups.

Theorem 13. *(M.R. Jones [6]) Let G be a finite group and K be a central subgroup of G. Set $H = G/K$, then*
(i) $|M(G)||G' \cap K|$ *divides* $|M(H)||M(K)||H \otimes K|$;
(ii) $exp(M(G))$ *divides* $exp(M(H))exp(M(K))exp(H \otimes K)$;
(iii) $d(M(G)) \leq d(M(H)) + d(M(K)) + d(H \otimes K)$.

The following theorem is a generalization of the above theorem, for which the condition of Schur-Baer variety seems to be necessary too.

Theorem 14. *(M.R.R. Moghaddam [13]) Let $\mathcal{V}$ be a Schur-Baer variety and G be a finite group with a marginal subgroup K. Let $G = F/R$ be a free presentation for G and $K = S/R$. Set $H = G/K \cong F/S$. Then*
(i) $|\mathcal{V}M(G)||V(G) \cap K| = |\mathcal{V}M(H)||[SV^*F]/[RV^*F]|$;
(ii) $exp(\mathcal{V}M(G))$ *divides* $exp(\mathcal{V}M(H))exp([SV^*F]/[RV^*F])$;
(iii) $d(\mathcal{V}M(G)) \leq d(\mathcal{V}M(H)) + d([SV^*F]/[RV^*F])$.

Theorem 15. *(M.R. Jones [6]) Let G be a finite nilpotent group of class c and let $Q_j = G/\gamma_j(G)$ for $2 \leq j \leq c$. Then*
(i) $|G'||M(G)|$ divides $|M(G/G')| \prod_{j=1}^{c-1} |\gamma_{j+1}(G) \otimes Q_{j+1}|$;
(ii) $exp(M(G))$ divides $exp(M(G/G')) \prod_{j=1}^{c-1} exp(\gamma_{j+1}(G) \otimes Q_{j+1})$;
(iii) $d(M(G)) \leq d(M(G/G')) + \sum_{j=1}^{c-1} d(\gamma_{j+1}(G) \otimes Q_{j+1})$.

The above theorem has an interesting corollary for which we need the following definition.

Definition 16. Let X be any group. We say that X has a special rank $r(X)$ if every subgroup of X may be generated by $r(X)$ elements and there is at least one subgroup of X that cannot be generated by fewer than $r(X)$ elements. It is easy to see that $r(X) = d(X)$ for every abelian group X. Also if N is a normal subgroup of X then $r(X) \leq r(X/N) + r(N)$.

Corollary 17. *If G is a finite p-group of class c and special rank r then*

$$d(M(G)) \leq \frac{1}{2} r((2c-1)r - 1).$$

Corollary 18. *If G is a finite p-group of class c and exponent p^e, then*

$$exp(M(G)) \leq p^{ec}$$

.

M. R. Jones in 1974 gave an improvement of Theorem 15 as follows.

Theorem 19. *(M.R. Jones [7]) Let G be a finite nilpotent group of class $c \geq 2$. Let $Z_j = Z_j(G)$ for all $1 \leq j \leq c$, then*
(i) $|\gamma_c(G)||M(G)|$ divides $|M(G/\gamma_c(G))||\gamma_c(G) \otimes (G/Z_{c-1}(G))|$;
(ii) $exp(M(G))$ divides $exp(M(G/\gamma_c(G)))exp(\gamma_c(G) \otimes (G/Z_{c-1}(G)))$;
(iii) $d(M(G)) \leq d(M(G/\gamma_c(G))) + d(\gamma_c(G) \otimes (G/Z_{c-1}(G)))$.

Corollary 20. *Let G be a finite p-group of class $c \geq 2$ and exponent p^e. Then*

$$exp(M(G)) \leq p^{e(c-1)}.$$

G. Ellis [3], S. Kayvanfar and M.A. Sanati [9] generalized the result of M.R. Jones for the exponent of the Schur multiplier of G (Corollary 20). A result of G. Ellis [3] shows that if G is a p-group of class $k \geq 2$ and exponent p^e, then $exp(M^{(c)}(G)) \leq p^{e[\frac{k}{2}]}$, where $[\frac{k}{2}]$ denotes the smallest integer n such that $n \geq [\frac{k}{2}]$. Clearly the recent bound sharpens the bound obtained in 20.
S. Kayvanfar and M.A. Sanati [9] proved that $exp(M(G)) \leq exp(G)$ when G is a finite p-group of class 3, 4 or 5 and $exp(G)$ satisfies in some conditions.

Now, in this paper we are going to concentrate on the Jones results and trying to generalize some of them. We will give some upper bounds for the order, the exponent and the number of generators of the nilpotent multiplier of a finite group, specially of a finite p-group and compare some of them with previous results. We use the notation $\otimes^{c+1}(B,A)$ for the tensor product $B\otimes A\otimes ...\otimes A$ involving c copies of A, where A and B are arbitrary groups. The main results of this article are as follows

The following theorem is a generalization of Theorem 6.

Theorem A. Let G be a finite p-group of order p^n, B be a cyclic subgroup of $Z(G)$ of order p^k, where $p^k = exp(Z(G))$, and $A = G/B$ be a d-generator group. Then

$$|\gamma_{c+1}(G)||M^{(c)}(G)| \leq p^{\chi_{c+1}(n-k)+dk(1+d)^{c-1}}.$$

The next theorem extends Theorem 15 and has several interesting corollaries.

Theorem B. Let G be a finite nilpotent group of class $t \geq 1$ and let $Q_j = G/\gamma_j(G)$ for $2 \leq j \leq t$. Then
(i) $|\gamma_{c+1}(G)||M^{(c)}(G)|$ divides $|M^{(c)}(G/G')|\prod_{k=1}^{t-1}|\otimes^{c+1}(\gamma_{k+1}(G),Q_{k+1})|$;
(ii)$exp(M^{(c)}(G))$ divides $exp(M^{(c)}(G/G'))\prod_{k=1}^{t-1}exp(\otimes^{c+1}(\gamma_{k+1}(G),Q_{k+1}))$;
(iii)$d(M^{(c)}(G)) \leq d(M^{(c)}(G/G')) + \sum_{k=1}^{t-1}d(\otimes^{c+1}(\gamma_{k+1}(G),Q_{k+1}))$.

The following theorem generalizes Theorem 19 and gives some bounds in terms of the lower central series.

Theorem C. Let G be a finite nilpotent group of class $t \geq 2$. Let $Z_j = Z_j(G)$ for $1 \leq j \leq t$, then
(i)
a) If $c+1 \leq t$, then
$|\gamma_t(G)||M^{(c)}(G)|$ divides $|M^{(c)}(G/\gamma_t(G))||\otimes^{c+1}(\gamma_t(G),G/Z_{t-1}(G))$.
b) If $c+1 > t$, then
$|\gamma_{c+1}(G)||M^{(c)}(G)|$ divides $|M^{(c)}(G/\gamma_t(G))||\otimes^{c+1}(\gamma_t(G),G/Z_{t-1}(G))|$.
(ii) $exp(M^{(c)}(G))$ divides $exp(M^{(c)}(G/\gamma_t(G)))exp(\otimes^{c+1}(\gamma_t(G),G/Z_{t-1}(G)))$.
(iii) $d(M^{(c)}(G)) \leq d(M^{(c)}(G/\gamma_t(G))) + d(\otimes^{c+1}(\gamma_t(G),G/Z_{t-1}(G)))$.

2 Proofs of Main Results

In order to prove main results we need the following lemmas.

Lemma 21. *Let G be a group, then for any positive integer i and any* *normal subgroups M, N of G, we have*

$$[M, \gamma_i(N)] \subseteq [M, \ _iN].$$

Proof. We use induction on i. Suppose $[M, \gamma_i(N)] \subseteq [M, \ _iN]$, for any normal subgroups M, N of G. Then by the Three Subgroup Lemma we have

$$\begin{aligned}[\gamma_{i+1}(N), M] = [\gamma_i(N), N, M] &\subseteq [N, M, \gamma_i(N)][M, \gamma_i(N), N] \\ &\subseteq [M, \ _{i+1}N].\end{aligned}$$

□

Lemma 22. *Let F/R be a presentation of G as a factor group of a free group F. Let $B = S/R$ be a normal subgroup of G, so that $A = G/B \cong F/S$. Then there exists the following epimorphism*

$$\otimes^{c+1}(B, A) \longrightarrow \frac{[S, \ _cF]}{[R, \ _cF][S, \ _{c+1}F] \prod_{i=2}^{c+1} \gamma_{c+1}(S, F)_i},$$

where for all $2 \leq i \leq c$, $\gamma_{c+1}(S, F)_i = [D_1, D_2, ..., D_{c+1}]$ such that $D_1 = D_i = S$ and $D_j = F$, for all $j \neq 1, i$.

Proof. Define

$$\theta : \otimes^{c+1}(\frac{S}{RS'}, \frac{F}{SF'}) \longrightarrow \frac{[S, \ _cF]}{[R, \ _cF][S, \ _{c+1}F] \prod_{i=2}^{c+1} \gamma_{c+1}(S, F)_i}$$

by $\theta(sRS', f_1SF', ..., f_cSF') = [s, f_1, ..., f_c][R, \ _cF][S, \ _{c+1}F] \prod_{i=2}^{c+1} \gamma_{c+1}(S, F)_i$, where $s \in S$ and $f_i \in F$ for all $i = 1, 2, ..., c$. The usual commutator calculations and Lemma 21 show that θ is well defined. Also for any $s_1, s_2 \in S$ and $f_1, ..., f_c, f'_1, ..., f'_c \in F$, we have
$[s_1s_2, f_1, ..., f_c] \equiv [s_1, f_1, ..., f_c][s_2, f_1, ..., f_c]$
$[s_1, f_1, ..., f_if'_i, ..., f_c] \equiv [s_1, f_1, ..., f_i, ..., f_c][s_1, f_1, ..., f'_i, ..., f_c](mod[S, \ _{(c+1)}F])$.
Then θ is a multilinear map. Therefore the universal property of the tensor product completes the proof. □

Now we are ready to prove Theorem A.

Proof of Theorem A.

Let F/R be a free presentation of G with $B = S/R$, so that $A = G/B = F/S$. Then

$$|\gamma_{c+1}(G)||M^{(c)}(G)| = |\frac{\gamma_{c+1}(F)R}{R}||\frac{\gamma_{c+1}(F) \cap R}{[R, \ _cF]}| = |\frac{\gamma_{c+1}(F)}{\gamma_{c+1}(F) \cap R}||\frac{\gamma_{c+1}(F) \cap R}{[R, \ _cF]}|$$

$$= |\frac{\gamma_{c+1}(F)/[R,\ _cF]}{(\gamma_{c+1}(F)\cap R)/[R,\ _cF]}||\frac{\gamma_{c+1}(F)\cap R}{[R,\ _cF]}| = |\frac{\gamma_{c+1}(F)}{[R,\ _cF]}| = |\frac{\gamma_{c+1}(F)}{[S,\ _cF]}||\frac{[S,\ _cF]}{[R,\ _cF]}|.$$

On the other hand by corollary 12 we have

$$|\frac{\gamma_{c+1}(F)}{[S,\ _cF]}| = |\gamma_{c+1}(A)||M^{(c)}(A)| \leq p^{\chi_{c+1}(n-k)}.$$

Now it is enough to show that $|[S,\ _cF]/[R,\ _cF]| \leq p^{dk(1+d)^{c-1}}$. We use the following notation for all $1 \leq j \leq c-1$

$$P_j = \prod_{(D_1,...,D_{c-1})\in Y_j} [S, F, D_1, ..., D_{c-1}],$$

where

$$Y_j = \{(D_1, ..., D_{c-1})|\ \ \exists\, i_1, i_2, ..., i_j\ s.t.\ D_k = S\ for\ all$$
$$k = i_s, 1 \leq s \leq j\ and\ D_k = F,\ otherwise\ \},$$

and $P_0 = [S,\ _cF]$, $P_c = \gamma_{c+1}(S)$. It is easy to see that
$|[S,\ _cF]/[R,\ _cF]| = |P_0/([R,\ _cF]P_1)||([R,\ _cF]P_1)/[R,\ _cF]| = ...$
$= |P_0/([R,\ _cF]P_1)||([R,\ _cF]P_1)/([R,\ _cF]P_2)|...|([R,\ _cF]P_{c-1})/[R,\ _cF]|$.
Since $B \leq Z(G)$, $[S,F] \leq R$ and thus $S' \leq R$. Also, since $M(B) = 1$, $S' \bigcap R = [R,S]$. So that $S' = [R,S]$. Hence $[S,S,\ _{c-1}F] \leq [R,\ _cF]$. Therefore by lemma 22 we have the following epimorphism

$$\otimes^{c+1}(B,A) \to \frac{P_0}{[R,\ _cF]P_1}$$

Also by an argument similar to the proof of lemma 22 one can deduce the following epimorphism

$$\oplus \sum_{(D_1,...,D_{c-1})\in Y'_j} B\otimes A\otimes D_1\otimes ...\otimes D_{c-1} \to \frac{[R,\ _cF]P_j}{[R,\ _cF]P_{j+1}},$$

where

$$Y'_j = \{(D_1, ..., D_{c-1})|\ \ \exists\, i_1, ..., i_j\ s.t.\ D_k = B\ for\ all$$
$$k = i_s, 1 \leq s \leq j\ and\ D_k = A,\ otherwise\ \},$$

for all $1 \leq j \leq c-1$. Now since $|A\otimes B| \leq \min\ \{|A_{ab}|^{d(B)}, |B_{ab}|^{d(A)}\}$, $|([R,\ _cF]P_j)/([R,\ _cF]P_{j+1})| \leq p^{\binom{c-1}{j-1}kd^{c-j+1}}$ for all $1 \leq j \leq c-1$. Therefore
$|[S,\ _cF]/[R,\ _cF]| \leq p^{\binom{c-1}{0}kd^c+\binom{c-1}{1}kd^{c-1}+...+\binom{c-1}{c-1}kd} = p^{kd(d+1)^{c-1}}$.

The following example compares the above bound and the upper bound of Corollary 12 .

Example 23. *If* $G = \mathbf{Z}_{p^k} \oplus \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus ... \oplus \mathbf{Z}_{p^{\alpha_d}}$ *such that* $\alpha_d \leq ... \leq \alpha_2 \leq \alpha_1 \leq k$ *and* $k > 1$, *then* $B = \mathbf{Z}_{p^k}$, $A \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus ... \oplus \mathbf{Z}_{p^{\alpha_d}}$ *and* $n = k + t$ *where* $t = \alpha_1 + \alpha_2 + \ldots + \alpha_d$. *Now for* $c = 2$, *by Corollary 12*

$$|\gamma_{c+1}(G)||M^{(c)}(G)| \leq p^{\chi_3(n)} = p^{1/3(t^3-t+3kt^2+k^3+3k^2-k)}.$$

But using Theorem A we have

$$|\gamma_{c+1}(G)||M^{(c)}(G)| \leq p^{\chi_3(t)+dk(1+d)} = p^{1/3(t^3-t+3dk+3d^2k)} \leq p^{1/3(t^3-t+3t^2k+3tk)}.$$

It is easy to see that $k^3 + 3k^2 - k > 3tk$. *Therefore the previous bound in Corollary 12 is larger than the bound of Theorem A for all finite abelian* p*-groups but elementary abelian* p*-groups.*

Now the following theorem is needed to prove Theorem B.

Theorem 24. *Let* G *be a finite nilpotent group of class* $t \geq 2$, *let* $G = F/R$ *be a free presentation of* G . *Then*
(i) $|\gamma_{c+1}(G)||M^{(c)}(G)| = |M^{(c)}(G/G')| \prod_{k=1}^{t-1} |[\gamma_{k+1}(F)R,\ _cF]/[\gamma_{k+2}(F)R,\ _cF]|$;
(ii) $exp(M^{(c)}(G))$ *divides* $exp(M^{(c)}(G/G')) \prod_{k=1}^{t-1} exp([\gamma_{k+1}(F)R,\ _cF]/[\gamma_{k+2}(F)R,\ _cF])$.
(iii) $d(M^{(c)}(G)) \leq d(M^{(c)}(G/G')) + \sum_{k=1}^{t-1} d([\gamma_{k+1}(F)R,\ _cF]/[\gamma_{k+2}(F)R,\ _cF])$;

Proof. With the previous notation, we have
(i)

$$\begin{aligned}
|\gamma_{c+1}(G)||M^{(c)}(G)| &= |\gamma_{c+1}(F)/[R,\ _cF]| \\
&= |\gamma_{c+1}(F)/([R\gamma_2(F),\ _cF])||[R\gamma_2(F),\ _cF]/[R,\ _cF]| \\
&= |M^{(c)}(G/G')||[R\gamma_3(F),\ _cF]/[R,\ _cF]| \\
&\quad |[R\gamma_2(F),\ _cF]/[R\gamma_3(F),\ _cF]| \\
&= ... \\
&= |M^{(c)}(G/G')||[R\gamma_{t+1}(F),\ _cF]/[R,\ _cF]| \\
&\quad \prod_{k=1}^{t-1} |[R\gamma_{k+1}(F),\ _cF]/[R\gamma_{k+2}(F),\ _cF]| \\
&= |M^{(c)}(G/G')| \prod_{k=1}^{t-1} |[R\gamma_{k+1}(F),\ _cF]/[R\gamma_{k+2}(F),\ _cF]|.
\end{aligned}$$

(ii)The proof is similar to (i).

$$\begin{array}{rcl} exp(M^{(c)}(G)) & = & exp(R\cap\gamma_{c+1}(F)/[R,\ _cF]) \\ & \mid & exp(\gamma_{c+1}(F)/[R,\ _cF]) \\ & \mid & exp(M^{(c)}(G/G'))exp([R\gamma_2(F),\ _cF]/[R,\ _cF]) \\ & \mid & exp(M^{(c)}(G/G'))exp([R\gamma_3(F),\ _cF]/[R,\ _cF]) \\ & & exp([R\gamma_2(F),\ _cF]/[R\gamma_3(F),\ _cF]) \\ & \mid & ... \\ & \mid & exp(M^{(c)}(G/G'))\prod_{k=1}^{t-1}exp([\gamma_{k+1}(F)R,\ _cF]/[\gamma_{k+2}(F)R,\ _cF]). \end{array}$$

(iii) We have

$$\begin{array}{rcl} d(M^{(c)}(G)) & = & r(M^{(c)}(G)) = r(R\cap\gamma_{c+1}(F)/[R,\ _cF]) \\ & \leq & r(\gamma_{c+1}(F)/[R,\ _cF]) \\ & \leq & r(M^{(c)}(G/G')) + r([R\gamma_2(F),\ _cF]/[R,\ _cF]) \\ & = & d(M^{(c)}(G/G')) + d([R\gamma_2(F),\ _cF]/[R,\ _cF]) \\ & \leq & d(M^{(c)}(G/G')) + d([R\gamma_3(F),\ _cF]/[R,\ _cF]) \\ & + & d([R\gamma_2(F),\ _cF]/[R\gamma_3(F),\ _cF]) \\ & \leq & ... \\ & \leq & d(M^{(c)}(G/G')) + \sum_{k=1}^{t-1} d([R\gamma_{k+1}(F),\ _cF]/[R\gamma_{k+2}(F),\ _cF]). \end{array}$$

□

Proof of Theorem B.

Using the notation of lemma 22, put $B = (\gamma_{k+1}(F)R)/R$ and $A = G/\gamma_{k+1}(G) = Q_{k+1}$, then we have the following epimorphism

$$\otimes^{c+1}(\gamma_{k+1}(G), Q_{k+1}) \to \frac{[R\gamma_{k+1}(F),\ _cF]}{([R,\ _cF][R\gamma_{k+1}(F),\ _{c+1}F]\prod_{i=2}^{c+1}\gamma_{c+1}(R\gamma_{k+1}(F),F)_i)}.$$

On the other hand

$$[R,\ _cF][R\gamma_{k+1}(F),\ _{c+1}F]\prod_{i=2}^{c+1}\gamma_{c+1}(R\gamma_{k+1}(F),F)_i \leq [\gamma_{k+2}(F)R,\ _cF],$$

since $[R\gamma_{k+1}(F),\ _{c+1}F] = [R,\ _{c+1}F][\gamma_{k+1}(F),\ _{c+1}F] \leq [\gamma_{k+2}(F)R,\ _cF]$. Also, for all positive integers n, m such that $m+n = c-1$, we have $[R\gamma_{k+1}(F),\ _nF, R\gamma_{k+1}(F),\ _mF]$

$= [R, {}_nF, R\gamma_{k+1}(F), {}_mF][\gamma_{k+1}(F), {}_nF, R, {}_mF][\gamma_{k+1}(F), {}_nF, \gamma_{k+1}(F), {}_mF]$
$\leq [R, {}_cF][R, \gamma_{k+n+1}(F), {}_mF]\gamma_{2k+c+1}(F) \leq [R\gamma_{k+2}(F), {}_cF].$
Hence

$$\otimes^{c+1}(\gamma_{k+1}(G), Q_{k+1}) \rightarrow \frac{[R\gamma_{k+1}(F), {}_cF]}{[R\gamma_{k+2}(F), {}_cF]}$$

is an epimorphism. Now the results follows by theorem 24.

The following example shows that the above bound for the order of c-nilpotent multiplier of a finite p-group is sometimes smaller than the bound which is obtained in corollary 12.

Example 25. *If G is an extra special p-group of order p^3 then G is a nilpotent group of class 2. By Theorem B(i) we have*

$$\begin{aligned} |\gamma_3(G)||M^{(2)}(G)| &\leq |M^{(2)}(G/G')||\gamma_2(G) \otimes (G/\gamma_2(G)) \otimes (G/\gamma_2(G))| \\ &= |M^{(2)}(\mathbf{Z}_p \oplus \mathbf{Z}_p)||\mathbf{Z}_p \otimes (\mathbf{Z}_p \oplus \mathbf{Z}_p) \otimes (\mathbf{Z}_p \oplus \mathbf{Z}_p)| = p^6. \end{aligned}$$

But Corollary 12 implies that, $|\gamma_3(G)||M^{(2)}(G)| \leq p^{\chi_3(3)} = p^8$.

Corollary 26. *If G is a finite d-generator p-group of special rank r and nilpotency class t then $d(M^{(c)}(G)) \leq \chi_{c+1}(d) + r^{c+1}(t-1)$.*

Proof. Since G is a p-group and $d(G) = d(G/G') = d$, $d(M^{(c)}(G/G')) = \chi_{c+1}(d)$ by Theorem 11. In addition $d(\otimes^{c+1}(\gamma_{k+1}(G), Q_{k+1})) \leq d(\gamma_{k+1}(G))d(Q_{k+1})^c \leq rd^c \leq r^{c+1}$. Hence the required assertion follows by Theorem B for $t \geq 1$. □

Note that the inequality in corollary 26 is attained for all elementary abelian p-groups. Now as a final application of Theorem B we have the following result.

Corollary 27. *Further to the notation and assumptions of Theorem B, let $e_j = min\{exp(Q_{j+1}),\ exp(\gamma_{j+1}(G))\}$ for $1 \leq j \leq c-1$. Then $exp(M^{(c)}(G)) \leq exp(G/G') \prod_{j=1}^{t-1} e_j$. In particular, if G has exponent p^e then $exp(M^{(c)}(G)) \leq p^{et}$.*

Proof. Since G/G' is an abelian group, by Theorem 11 $exp(M^{(c)}(G/G')) \leq exp(G/G'))$. Also, by the properties of tensor products, we have $exp(\otimes^{c+1}(\gamma_{j+1}(G), Q_{j+1})) \leq e_j$. Now the result holds by Theorem B. □

Example 28. *It can be seen that the inequality of Corollary 27 is attained. Let G be a dihedral group of order 8, D_8. By a theorem of M. R. R. Moghaddam [15] we have*

$$M^{(c)}(D_8) \cong \mathbf{Z}_4 \oplus \underbrace{\mathbf{Z}_2 \oplus ... \oplus \mathbf{Z}_2}_{(\chi_{c+1}(2)-1)-times} \quad .$$

Then $exp(M^{(c)}(G)) = 4$. *On the other hand Corollary 27 implies that,* $exp(M^{(c)}(G)) \leq exp(\mathbf{Z}_2 \oplus \mathbf{Z}_2)\ (min\{exp(\mathbf{Z}_2 \oplus \mathbf{Z}_2),\ exp(\mathbf{Z}_2)\}) = 4$.

The following theorem helps us to proof Theorem C.

Theorem 29. *Let* G *be a finite nilpotent group of class* $t \geq 2$ *and let* $G = F/R$ *be a free presentation for* G. *Then*
(i)
a) If $c+1 \leq t$, *then*
$|\gamma_t(G)||M^{(c)}(G)| = |M^{(c)}(G/\gamma_t(G))||[R\gamma_t(F),\ _cF]/[R,\ _cF]|$.
b) If $c+1 > t$, *then*
$|\gamma_{c+1}(G)||M^{(c)}(G)| = |M^{(c)}(G/\gamma_t(G))||[R\gamma_t(F),\ _cF]/[R,\ _cF]|$;
(ii) $exp(M^{(c)}(G))$ *divides* $exp(M^{(c)}(G/\gamma_t(G)))exp([R\gamma_t(F),\ _cF]/[R,\ _cF])$;
(iii) $d(M^{(c)}(G)) \leq d(M^{(c)}(G/\gamma_t(G))) + d([R\gamma_t(F),\ _cF]/[R,\ _cF])$.

Proof. Since $\gamma_t(G) = (\gamma_t(F)R)/R \cong \gamma_t(F)/(R \cap \gamma_t(F))$ and $G/\gamma_t(G) \cong F/(\gamma_t(F)R)$, we have $(M^{(c)}(G/\gamma_t(G))) = (\gamma_{c+1}(F) \cap \gamma_t(F)R)/[R\gamma_t(F),\ _cF]$.
(i) We consider two cases.
Case One: If $c+1 \leq t$,then

$$\frac{(\gamma_{c+1}(F) \cap R)\gamma_t(F)}{[R\gamma_t(F),\ _cF]} \cong \frac{((\gamma_{c+1}(F) \cap R)\gamma_t(F))/[R,\ _cF]}{[R\gamma_t(F),\ _cF]/[R,\ _cF]}.$$

Hence
$$|\frac{(\gamma_{c+1}(F) \cap R)\gamma_t(F)}{[R,\ _cF]}| = |\frac{[R\gamma_t(F),\ _cF]}{[R,\ _cF]}||M^{(c)}(\frac{G}{\gamma_t(G)})|.$$

But
$$\begin{aligned}\frac{((\gamma_{c+1}(F) \cap R)\gamma_t(F))/[R,\ _cF]}{(\gamma_{c+1}(F) \cap R)/[R,\ _cF]} &\cong \frac{(\gamma_{c+1}(F) \cap R)\gamma_t(F)}{\gamma_{c+1}(F) \cap R} \\ &\cong \frac{\gamma_t(F)}{\gamma_t(F) \cap R} \cong \gamma_t(G),\end{aligned}$$

so that

$$\begin{aligned}|\frac{(\gamma_{c+1}(F) \cap R)\gamma_t(F)}{[R,\ _cF]}| &= |\gamma_t(G)||\frac{\gamma_{c+1}(F) \cap R}{[R,\ _cF]}| \\ &= |\gamma_t(G)||M^{(c)}(G)|.\end{aligned}$$

Therefore
$$|\gamma_t(G)||M^{(c)}(G)| = |M^{(c)}(\frac{G}{\gamma_t(G)})||\frac{[R\gamma_t(F),\ _cF]}{[R,\ _cF]}|.$$

Case Two: If $c+1 > t$, then we have

$$M^{(c)}(\frac{G}{\gamma_t(G)}) \cong \frac{\gamma_{c+1}(F)}{[R\gamma_t(F),\ _cF]} \cong \frac{\gamma_{c+1}(F)/[R,\ _cF]}{[R\gamma_t(F),\ _cF]/[R,\ _cF]}.$$

Thus

$$|\frac{\gamma_{c+1}(F)}{[R,\ _cF]}| = |M^{(c)}(\frac{G}{\gamma_t(G)})||\frac{[R\gamma_t(F),\ _cF]}{[R,\ _cF]}|.$$

But

$$\frac{\gamma_{c+1}(F)/[R,\ _cF]}{(\gamma_{c+1}(F)\cap R)/[R,\ _cF]} \cong \frac{\gamma_{c+1}(F)}{\gamma_{c+1}(F)\cap R} \cong \frac{\gamma_{c+1}(F)R}{R} \cong \gamma_{c+1}(G).$$

Hence the result follows as for case one.

(ii),(iii) Since

$$M^{(c)}(\frac{G}{\gamma_t(G)}) \cong \frac{(\gamma_{c+1}(F)\cap\gamma_t(F)R)/[R,\ _cF]}{[R\gamma_t(F),\ _cF]/[R,\ _cF]}$$

and

$$M^{(c)}(G) \cong \frac{\gamma_{c+1}(F)\cap R}{[R,\ _cF]} \leq \frac{\gamma_{c+1}(F)\cap\gamma_t(F)R}{[R,\ _cF]},$$

we have

$$exp(M^{(c)}(G)) \mid exp(\frac{\gamma_{c+1}(F)\cap R\gamma_t(F)}{[R,\ _cF]}) \mid exp(M^{(c)}(\frac{G}{\gamma_t(G)}))exp(\frac{[R\gamma_t(F),\ _cF]}{[R,\ _cF]}).$$

On the other hand by Lemma 21, $((\gamma_{c+1}(F)\cap R)\gamma_t(F))/[R,\ _cF]$ is an abelian group, therefore

$$d(M^{(c)}(G)) \leq r(\frac{\gamma_{c+1}(F)\cap R\gamma_t(F)}{[R,\ _cF]}) = d(\frac{\gamma_{c+1}(F)\cap R\gamma_t(F)}{[R,\ _cF]})$$

$$\leq d(M^{(c)}(\frac{G}{\gamma_t(G)})) + d(\frac{[R\gamma_t(F),\ _cF]}{[R,\ _cF]}).$$

This completes the proof. □

Proof of Theorem C

Let F/R be a free presentation for G with $Z_j = Y_j/R$ for $1 \leq j \leq t$. Then we have $\gamma_t(G) = \gamma_t(F)R/R$ and $G/Z_j \cong F/Y_j$ for $1 \leq j \leq t$. Define

$$\theta : \frac{\gamma_t(F)R}{R} \times \frac{F}{Y_{t-1}} \times ... \times \frac{F}{Y_{t-1}} \longrightarrow \frac{[R\gamma_t(F),\ _cF]}{[R,\ _cF]}$$

by $\theta(gR, f_1Y_{t-1}, ..., f_cY_{t-1}) = [g, f_1, ..., f_c][R,\ _cF]$ for $f_1, ..., f_c$ in F and g in $\gamma_t(F)$. Suppose $g' = gr$ and $f'_i = f_iy_i$ for r in R and y_i in Y_{t-1} for $1 \leq i \leq c$. Then the commutator calculations and Lemma 21 show that $[g, f_1, ..., f_c] \equiv [g', f'_1, ..., f'_c]$ (mod $[R,\ _cF]$) and θ is well defined. Moreover for $g,\ g'$ in $\gamma_t(G)$ and f_i, f'_i in F , $[gg', f_1, ..., f_c] \equiv [g, f_1, ..., f_c][g', f_1, ..., f_c]$

and $[g,f_1,...,f_if_i',...,f_c] \equiv [g,f_1,...,f_i,..,f_c][g,f_1,...,f_i',..,f_c] \pmod{[R,\ _cF]}$ for $1 \le i \le c$. Hence θ is multilinear map and therefore $([R\gamma_t(F),\ _cF])/[R,\ _cF]$ is a homomorphic image of $\otimes^{c+1}(\gamma_t(G), G/Z_{t-1}(G))$, by the universal property of tensor product. Now the result follows from previous theorem.

Remarks

(i) The inequality in Theorem C (i) is attained for extra special p-group of order p^3 of exponent p and $c = 1$. Also equality holds in (ii), for dihedral group of order 8. In addition, Example 25 helps us to see that the bound is some times better than the bound obtained by Corollary 12 for the order of c-nilpotent multiplier of a finite p-group.

(ii) Note that a result similar to Theorem C has been proved in a different method by J. Burns and G. Ellis [2, Proposition 5]. Their proof is based on nonabelian tensor product argument. Note that when we consider the exterior product of abelian groups by the canonical homomorphisms $\gamma_t(G) \to G/Z_{t-1}(G)$ and $G/Z_{t-1}(G) \to G/Z_{t-1}(G)$ as crossed modules, then the rule of θ in the proof of the Theorem C gives the following epimorphism:

$$\hat{\theta} : \gamma_t(G) \wedge \frac{G}{Z_{t-1}(G)} \wedge ... \wedge \frac{G}{Z_{t-1}(G)} \to \frac{[R\gamma_t(F),\ _cF]}{[R,\ _cF]}.$$

Hence we can replace $\gamma_t(G) \otimes G/Z_{t-1}(G) \otimes ... \otimes G/Z_{t-1}(G)$ by $\gamma_t(G) \wedge G/Z_{t-1}(G) \wedge ... \wedge G/Z_{t-1(G)}$ in Theorem C. Also it seems that there is a missing point in the proof of the similar result of J. Burns and G. Ellis [2, Proposition 5] for the right exactness of the sequence

$$M^{(c)}(G) \to M^{(c)}(\frac{G}{\gamma_t(G)}) \to \gamma_t(G) \to 1.$$

So we should state part (i) of Theorem C in two cases.

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