SILTING MUTATION FOR SELF-INJECTIVE ALGEBRAS

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ABSTRACT. We study 'silting mutaion' for self-injective algebras. In particular we focus on 'tilting mutation' and show that iterated irreducible 'silting mutation' transitively act on the set of silting objects for representation-finite symmetric algebras. Moreover we give some sufficient conditions for 'Bongartz-type Lemma' on silting objects.

1. Introduction

In representation theory of algebras the notion of 'mutation' often plays important roles. 'Mutation' is an operation for a certain class of objects in a fixed category to construct a new object from a given one by replacing a summand. An important case is 'tilting mutation' for tilting complexes and it was studied in modular representation theory. For example 'tilting mutation' appears in the study of Broué's abelian defect group conjecture [O, Ri2] and Brauer tree algebras [A, KZ, Z].

However 'tilting mutation' has a big disadvantage, that is, it is not always possible. It often occurs that an object which is constructed by replacing a summand of a tilting object is not tilting. The disadvantage is canceled by introducing 'silting mutation' which was studied in [AI]. The point is that 'silting mutation' is always possible in the sense that an object which is constructed by replacing any summand of a silting object is always silting. Hence we hope that 'silting mutation' gives us sufficiently many silting objects.

In this paper we consider two important questions.

The first is about 'silting transitivity'. We pose the following question:

Question 1.1. *Let A be a finite dimensional algebra A over a field and $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\text{-}A)$. Is the action of iterated irreducible 'silting mutation' on the set of basic silting objects in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\text{-}A)$ transitive?*

It was shown in [AI] that Question 1.1 is true for either a local algebra [AI, Corollary 2.42], a hereditary algebra or a canonical algebra [AI, Theorem 3.1].

In this paper we have the following partial answer (Theorem 5.2).

Theorem 1.2. *Question 1.1 has a positive answer if A is a representation-finite symmetric algebra.*

In the case of symmetric algebras any 'silting mutation' is always 'tilting mutation'. From the point of view, this is one of important cases for 'silting transitivity'.

To prove Theorem 1.2, in Section 4 we construct a 'pre-silting object' induced by a torsion pair (Lemma 4.9) and show that it is silting (tilting) under a nice condition (Theorem 4.10).

The second is about 'Bongartz-type Lemma' (cf. [B, Lemma 2.1]) in the sense that any module which is a self (1-) orthogonal module at most 1 in projective dimension is a direct summand of a (classical) tilting module. In general 'Bongartz-type Lemma' does not hold for tilting complexes, that is, there exists a 'pre-tilting object' which is not 'partial tilting' (cf. [Ri1, Section 8]). However the 'pre-tilting object' given in [Ri1, Section 8] (so it is 'pre-silting') is 'partial silting' (Remark 2.14). We hence hope that the following question has a partial answer.

Question 1.3. *Is any 'pre-silting object' always 'partial silting'?*

In this paper we give some sufficient conditions for the question mentioned above to have a partial answer (Theorem 2.12(a), Lemma 2.15).

Notations Let $\mathcal{T}$ be an additive category. For morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{T}$, we denote by $gf : X \to Z$ the composition. We say that a morphism $f : X \to Y$ is *right minimal* if any morphism $g : X \to X$ satisfying $fg = f$ is an isomorphism. Dually we define a *left minimal* morphism.

For a collection $\mathcal{X}$ of objects in $\mathcal{T}$, we denote by $\mathsf{add}\mathcal{X}$ the smallest full subcategory of $\mathcal{T}$ which is closed under finite direct sums, summands and isomorphisms and contains $\mathcal{X}$.

Let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$. We say that a morphism $f : X \to Y$ is a *right $\mathcal{X}$-approximation* of Y if $X \in \mathcal{X}$ and $\mathrm{Hom}_{\mathcal{T}}(X, f)$ is surjective for any $X \in \mathcal{X}$. We say that $\mathcal{X}$ is *contravariantly finite* if any object in $\mathcal{T}$ has a right $\mathcal{X}$-approximation. Dually, we define a *left $\mathcal{X}$-approximation* and a *covariantly finite subcategory.* We say that $\mathcal{X}$ is *functorially finite* if it is contravariantly and covariantly finite.

When $\mathcal{T}$ is a Krull-Schmidt category, we say that an object $X \in \mathcal{T}$ is *basic* if the endomorphism algebra of X is a basic algebra.

When $\mathcal{T}$ is a triangulated category, we denote by $\mathsf{thick}\mathcal{X}$ the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{X}$. For collections $\mathcal{X}$ and $\mathcal{Y}$ of objects in $\mathcal{T}$, we denote by $\mathcal{X} * \mathcal{Y}$ the collection of objects $Z \in \mathcal{T}$ appearing in a triangle $X \to Z \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

For an additive category $\mathcal{A}$, we denote by $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ the homotopy category of bounded complexes over $\mathcal{A}$.

For a ring A, we denote by mod-A the category of all finitely generated right A-modules, by proj-A the category of finitely generated projective A-modules. When A is a finite dimensional algebra over a field k, we denote by $D := \mathrm{Hom}_k(-, k) : \text{mod-}A \leftrightarrow \text{mod-}A^{\mathrm{op}}$ the k-duality, by $\nu := D\mathrm{Hom}_A(-, A) : \text{mod-}A \to \text{mod-}A$ the Nakayama functor and by τ, τ^{-1} the Auslander-Reiten translations. For any $i \in \mathbb{Z}$, we denote by $H^i := \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{mod-}A)}(A, -[i]) : \mathsf{K}^{\mathrm{b}}(\text{mod-}A) \to \text{mod-}A$ the i-th cohomological functor. We denote by $\underline{\text{mod}}$-A the stable module category and by $\Omega : \underline{\text{mod}}\text{-}A \to \underline{\text{mod}}\text{-}A$ the syzygy functor.

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2. Silting objects and silting mutation

The aim of this section is to study silting mutation and a partial order in the sense of [AI] and to give some results which are necessary for this paper.

Let $\mathcal{T}$ be a triangulated category and we assume the following:

Assumption 2.1. $\mathcal{T}$ is Krull-Schmidt, k-linear for a field k and *Hom-finite*, that is, $\dim_k \mathrm{Hom}_{\mathcal{T}}(X, Y) < \infty$ for any $X, Y \in \mathcal{T}$.

Let us start with the definition of silting objects.

Definition 2.2. Let $T \in \mathcal{T}$.

(a) We say that T is *pre-silting* (respectively, *pre-tilting*) if $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any $i > 0$ (respectively, $i \neq 0$).

(b) We say that T is *silting* (respectively, *tilting*) if it is pre-silting (respectively, pre-tilting) and satisfies $\mathcal{T} = \mathsf{thick}T$. We denote by $\mathrm{silt}\,\mathcal{T}$ (respectively, $\mathrm{tilt}\,\mathcal{T}$) the isomorphism classes of basic silting (respectively, tilting) objects in $\mathcal{T}$.

(c) We say that T is *partial silting* if it is a direct summand of a silting object.

Now we define silting mutation.

Definition 2.3. Let $T \in \mathrm{silt}\,\mathcal{T}$. For a direct summand X of T, we define an object $\mu_X^+(T)$ as follows: We set $T = X \oplus M$ satisfying $\mathsf{add}X \cap \mathsf{add}M = 0$. Now we take a minimal left $\mathsf{add}M$-approximation $f : X \to M'$ and a triangle

$$X \xrightarrow{f} M' \longrightarrow Y \longrightarrow X[1].$$

It is possible by Assumption 2.1. We put

$$\mu_X^+(T) := Y \oplus M,$$

and call it a *left mutation* of T with respect to X. Dually we define a *right mutation* $\mu_X^-(T)$ of T with respect to X. *(Silting) mutation* is a left or right mutation.

We say that mutation is *tilting mutation* if both T and its mutation are tilting.

We say that mutation is *irreducible* if X is indecomposable.

Theorem 2.4. [AI, Theorem 2.30] *Any mutation of a silting object is again a silting object.*

Next we define a partial order on $\operatorname{silt}\mathcal{T}$.

Definition 2.5. For $T, U \in \operatorname{silt}\mathcal{T}$, we write $T \geq U$ if $\operatorname{Hom}_{\mathcal{T}}(T, U[i]) = 0$ for any $i > 0$.

Theorem 2.6. [AI, Theorem 2.10] $\geq$ *gives a partial order on* $\operatorname{silt}\mathcal{T}$.

The following subcategory is useful to understand a partial order on $\operatorname{silt}\mathcal{T}$.

Definition 2.7. For any $T \in \operatorname{silt}\mathcal{T}$, we define a subcategory of $\mathcal{T}$ by

$$\mathcal{T}_T^{\leq 0} := \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(T, X[i]) = 0 \text{ for any } i > 0\}.$$

We put $\mathcal{T}_T^{\leq \ell} = \mathcal{T}_T^{<\ell+1} := \mathcal{T}_T^{\leq 0}[-\ell]$ for $\ell \in \mathbb{Z}$.

We have the following useful property.

Proposition 2.8. [AI, Proposition 2.13] *Let* $T, U \in \operatorname{silt}\mathcal{T}$*. Then* $T \geq U$ *if and only if* $\mathcal{T}_T^{\leq 0} \supseteq \mathcal{T}_U^{\leq 0}$.

The following results play important roles in this paper.

Proposition 2.9. [AI, Proposition 2.23] *Let* $T \in \operatorname{silt}\mathcal{T}$*. For any* $U = U_0 \in \mathcal{T}_T^{\leq 0}$*, we have triangles*

$$\begin{aligned}
&U_1 \xrightarrow{g_1} T_0 \xrightarrow{f_0} U_0 \longrightarrow U_1[1],\\
&\cdots,\\
&U_\ell \xrightarrow{g_\ell} T_{\ell-1} \xrightarrow{f_{\ell-1}} U_{\ell-1} \longrightarrow U_\ell[1],\\
&0 \xrightarrow{g_{\ell+1}} T_\ell \xrightarrow{f_\ell} U_\ell \longrightarrow 0,
\end{aligned}$$

for some $\ell \geq 0$ *such that* f_i *is a minimal right* $\mathsf{add}T$*-approximation and* g_{i+1} *belongs to* $J_{\mathcal{T}}$ *for any* $0 \leq i \leq \ell$ *where* $J_{\mathcal{T}}$ *is the Jacobson radical of* $\mathcal{T}$.

Lemma 2.10. [AI, Lemma 2.24] *Let* $T \in \operatorname{silt}\mathcal{T}$ *and* $U_0, U_0' \in \mathcal{T}_T^{\leq 0}$*. For* U_0*, we take* $\ell \geq 0$ *and triangles in Proposition 2.9. Also for* U_0'*, we take triangles*

$$\begin{aligned}
&U_1' \xrightarrow{g_1'} T_0' \xrightarrow{f_0'} U_0' \longrightarrow U_1'[1],\\
&\cdots,\\
&U_{\ell'}' \xrightarrow{g_{\ell'}'} T_{\ell'-1}' \xrightarrow{f_{\ell'-1}'} U_{\ell'-1}' \longrightarrow U_{\ell'}'[1],\\
&0 \xrightarrow{g_{\ell'+1}'} T_{\ell'}' \xrightarrow{f_{\ell'}'} U_{\ell'}' \longrightarrow 0,
\end{aligned}$$

satisfying the same properties. If $\operatorname{Hom}_{\mathcal{T}}(U_0, U_0'[\ell]) = 0$ *holds, then we have* $(\mathsf{add}T_\ell) \cap (\mathsf{add}T_0') = 0$.

We now improve the result of [AI, Proposition 2.35].

Proposition 2.11. *Let* $T \in \operatorname{silt}\mathcal{T}$ *and* $U \in \mathcal{T}$*. If* U *is a pre-silting object with* $U \notin \mathsf{add}T$ *and* $U \in \mathcal{T}_T^{\leq 0}$*, then there exists an irreducible left mutation* P *of* T *such that* $T > P$ *and* $U \in \mathcal{T}_P^{\leq 0}$.

Proof. Since $U \notin \mathsf{add}T$ and $U \in \mathcal{T}_T^{\leq 0}$, we can take $U_0 \in \mathsf{add}U$ which does not belong to $\mathsf{add}T$ and triangles in Proposition 2.9 with $\ell > 0$. Now take an indecomposable object X of T_ℓ and put $P := \mu_X^+(T)$. By [AI, Theorem 2.34], we have $T > P$. To show $U \in \mathcal{T}_P^{\leq 0}$, we consider the triangle as in Definition 2.3. Since we have an exact sequence

$$\operatorname{Hom}_{\mathcal{T}}(X, U[i]) \to \operatorname{Hom}_{\mathcal{T}}(Y, U[i+1]) \to \operatorname{Hom}_{\mathcal{T}}(M', U[i+1]),$$

we obtain $\operatorname{Hom}_{\mathcal{T}}(P, U[i]) = 0$ for any $i > 1$. Thus it remains to prove $\operatorname{Hom}_{\mathcal{T}}(P, U[1]) = 0$. Since we have an exact sequence

$$\operatorname{Hom}_{\mathcal{T}}(M', U) \xrightarrow{\cdot f} \operatorname{Hom}_{\mathcal{T}}(X, U) \to \operatorname{Hom}_{\mathcal{T}}(Y, U[1]) \to \operatorname{Hom}_{\mathcal{T}}(M', U[1]) = 0,$$

we only have to show that $\mathrm{Hom}_{\mathcal{T}}(M',U) \xrightarrow{\cdot f} \mathrm{Hom}_{\mathcal{T}}(X,U)$ is surjective. Fix $a : X \to U$ and consider a diagram

$$\begin{array}{ccccccc} & & Y[-1] & \longrightarrow & X & \xrightarrow{f} & M' \\ & & & & \downarrow{\scriptstyle a} & & \\ U'_1 & \xrightarrow[g'_1]{} & T'_0 & \xrightarrow[f'_0]{} & U & \longrightarrow & U'_1[1] \end{array}$$

where the lower triangle is given in Lemma 2.10 as $U'_0 = U$. Since f'_0 is a right $\mathsf{add}T$-approximation, we get $b : X \to T'_0$ with $a = f'_0 b$. Since $\mathsf{add}T_\ell \cap \mathsf{add}T'_0 = 0$ by Lemma 2.10, we have $X \notin \mathsf{add}T'_0$ and so $T'_0 \in \mathsf{add}M$. Since f is a left $\mathsf{add}M$-approximation, we obtain $c : M' \to T'_0$ with $b = cf$. Thus we have $a = (f'_0 c)f$ and the assertion holds. □

Proposition 2.11 implies the following result, whose first assertion is 'Bongartz-type Lemma'.

Theorem 2.12. *Let $T \in \operatorname{silt}\mathcal{T}$ and $U \in \mathcal{T}$ be a pre-silting object with $U \in \mathcal{T}_T^{\leq 0}$. Assume that there exist only finitely many silting objects $P \in \operatorname{silt}\mathcal{T}$ satisfying $T \geq P$ and $U \in \mathcal{T}_P^{\leq 0}$. Then the following hold:*
(a) *U is a partial silting object;*
(b) *If U is a silting object, then it can be obtained from T by iterated irreducible (left) mutation.*

Proof. Assume $U \notin \mathsf{add}T$. By Proposition 2.11, we have a sequence of irreducible left mutation

$$T = T_0 > T_1 > \cdots > T_i > \cdots$$

satisfying $U \in \mathcal{T}_{T_i}^{\leq 0}$ for any $i \geq 0$. By our assumption, U is a direct summand of T_ℓ for some $\ell \geq 0$. The last assertion follows from [AI, Theorem 2.17]. □

The corollary below is an immediate consequence of Proposition 2.12.

Corollary 2.13. *Assume that $\mathcal{T}$ has a silting object T such that for any $\ell \geq 0$, there exist only finitely many silting objects $P \in \operatorname{silt}\mathcal{T}$ satisfying $T \geq P \geq T[\ell]$. Then the action of iterated irreducible mutation on $\operatorname{silt}\mathcal{T}$ is transitive.*

Remark 2.14. Assume that $\mathcal{T}$ has a silting object. Then the assumption of Corollary 2.13 holds if $\mathcal{T} = \mathsf{add}\{M[i] \mid i \in \mathbb{Z}\}$ for some $M \in \mathcal{T}$. This implies that the action of iterated irreducible mutation on $\operatorname{silt}\mathcal{T}$ is transitive if the *dimension* of $\mathcal{T}$, which was defined in [Ro], is zero: For example it is satisfied if A is derived equivalent to a representation-finite hereditary algebra and $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\text{proj-}A)$.

We end this section by giving another 'Bongartz-type Lemma' for a condition different from that of Theorem 2.12, which is a modification of [AH, Lemma 3.1].

Lemma 2.15. *Let $T \in \mathcal{T}$ be a pre-silting object. Assume that there exists $P \in \operatorname{silt}\mathcal{T}$ satisfying $P \in \mathcal{T}_T^{\leq 0}$ and $T \in \mathcal{T}_P^{\leq 1}$. Then T is a partial silting object.*

Proof. We take a right $\mathsf{add}T$-approximation $f : T' \to P$ of P with $T' \in \mathsf{add}T$ and a triangle $Q \to T' \xrightarrow{f} P \to Q[1]$: it is possible by Assumption 2.1. Set $U = T \oplus Q$. Since P is a silting object, we can check $\mathcal{T} = \mathsf{thick}U$ easily.

(i) We show $\mathrm{Hom}_{\mathcal{T}}(T, Q[i]) = 0$ for any $i > 0$. Since we have an exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(T, P[i-1]) \xrightarrow{0} \mathrm{Hom}_{\mathcal{T}}(T, Q[i]) \to \mathrm{Hom}_{\mathcal{T}}(T, T'[i]) = 0,$$

we have $\mathrm{Hom}_{\mathcal{T}}(T, Q[i]) = 0$ for any $i > 0$.

(ii) We show $\mathrm{Hom}_{\mathcal{T}}(Q, U[i]) = 0$ for any $i > 0$. Since we have an exact sequence

$$0 \overset{\text{(i)}}{=} \mathrm{Hom}_{\mathcal{T}}(T', U[i]) \to \mathrm{Hom}_{\mathcal{T}}(Q, U[i]) \to \mathrm{Hom}_{\mathcal{T}}(P[-1], U[i]) \to \mathrm{Hom}_{\mathcal{T}}(T'[-1], U[i]) \overset{\text{(i)}}{=} 0,$$

we obtain $\mathrm{Hom}_{\mathcal{T}}(Q, U[i]) \simeq \mathrm{Hom}_{\mathcal{T}}(P[-1], U[i])$. Since we have an exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(P, P[i]) \to \mathrm{Hom}_{\mathcal{T}}(P, Q[i+1]) \to \mathrm{Hom}_{\mathcal{T}}(P, T'[i+1]) \overset{T \in \mathcal{T}_P^{\leq 1}}{=} 0,$$

we have $\mathrm{Hom}_{\mathcal{T}}(P, Q[i+1]) = 0$ for any $i > 0$. This implies $\mathrm{Hom}_{\mathcal{T}}(Q, U[i]) = 0$ for any $i > 0$.

By (i) and (ii), we have $\mathrm{Hom}_{\mathcal{T}}(U, U[i]) = 0$ for any $i > 0$. Thus the assertion holds. □

3. Derived invariances for self-injective algebras

Through this section, let A be a basic self-injective algebra and put $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\text{-}A)$. In this section we give useful applications of results studied in Section 2, which are a improvement of the proof given in [AH] and [AR]. In particular the assertion implies that the self-injectivity is preserved by derived equivalences.

Let us start with the following easy observations.

Lemma 3.1. $\mathrm{silt}\,\mathcal{T}$ *and* $\mathrm{tilt}\,\mathcal{T}$ *are closed under* ν.

Lemma 3.2. *For any* $T \in \mathrm{tilt}\,\mathcal{T}$*, we have* $T \geq \nu T$. *In particular,* $\mathcal{T}_T^{\leq 0}$ *is closed under* ν.

Proof. Since we have $\mathrm{Hom}_{\mathcal{T}}(T, \nu T[i]) \simeq D\mathrm{Hom}_{\mathcal{T}}(T[i], T) = 0$ for any $i \neq 0$, we have $T \geq \nu T$. We can easily check $\nu \mathcal{T}_T^{\leq 0} \subseteq \mathcal{T}_{\nu T}^{\leq 0}$. Hence the last assertion follows from Proposition 2.8. □

The following result plays an important role.

Proposition 3.3. [AR, Theorem 1.1] *Let* Λ *be a finite dimensional algebra for a field, and* $\cdots, P^{-1}, P^0, P^1, \cdots$ *a sequence of finitely generated projective* Λ*-modules, such that* $P^i = 0$ *for all but finitely many* i. *Then up to isomorphism there are only finitely many tilting complexes*

$$P = \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots .$$

We have the following result.

Lemma 3.4. *For any* $T \in \mathrm{silt}\,\mathcal{T}$*, there exists a positive integer* n *such that* $\mathsf{add}T = \mathsf{add}\nu^n T$.

Proof. Since $\mathcal{T} = \mathsf{thick}A$, applying shifts to T, we can assume $T \leq A$. By Proposition 2.9, there exist $\ell \geq 0$ and $P_0, P_1, \cdots, P_\ell \in \mathrm{proj}\text{-}A$ such that $T \in P_0 * P_1[1] * \cdots * P_\ell[\ell]$. Since A is self-injective, we have a positive integer s such that $P_i \simeq \nu^s P_i$ for any $0 \leq i \leq \ell$. Therefore we also have $\nu^s T \in P_0 * P_1[1] * \cdots * P_\ell[\ell]$. By Proposition 3.3, there exists a multiple n of s such that $T \simeq \nu^n T$. □

Now we recover a result of [AH] and [AR].

Theorem 3.5. [AH, Lemma 1.7] [AR, Theorem 2.1] *Let* $T \in \mathrm{silt}\,\mathcal{T}$. *Then the following are equivalent:*

(1) *T is a tilting object;*
(2) $T \simeq \nu T$.

Proof. The implication (2) ⇒ (1) can be checked easily. Assume that T is a tilting object. Since $\nu^j T$ is tilting for any $j \in \mathbb{Z}$, by Lemma 3.2 we have a sequence of tilting objects

$$T \geq \nu T \geq \nu^2 T \geq \cdots .$$

By Lemma 3.4, we have $T \simeq \nu^n T$ for some $n > 0$. Hence the assertion follows from Theorem 2.6. □

The corollary below is an immediate consequence of Theorem 3.5.

Corollary 3.6. [AR, Theorem 2.1] *Self-injective algebras are closed under derived equivaleces.*

Let $\{P_1, \cdots, P_n\}$ be the set of non-isomorphic projective indecomposable A-modules. Then there exists a permutation ρ of the set $I := \{1, \cdots, n\}$, called the *Nakayama permutation* of A, such that $\nu P_i \simeq P_{\rho(i)}$ for any $i \in I$.

We can also recover a result of [AH].

Corollary 3.7. [AH, Proposition 2.14] *If the Nakayama permutation of* A *is transitive, then* $\mathrm{tilt}\,A = \{A[i] \mid i \in \mathbb{Z}\}$.

Proof. Let $T \in \mathcal{T}$ be a basic tilting object contained in $P_0 * P_1[1] * \cdots * P_\ell[\ell]$ for some $P_0, \cdots, P_\ell \in \mathsf{add}A$. By Theorem 3.5, we have $T \simeq \nu T$. This implies $P_i \simeq \nu P_i$ for any $0 \leq i \leq \ell$. Assume $\ell > 0$, i.e. $P_\ell \neq 0$. We take an indecomposable direct summand Q of P_ℓ. Since the action of ν is transitive, we have $\nu^j Q \in \mathsf{add}P_0$ for some integer j. Since $P_0 \simeq \nu P_0$, we obtain $Q \in \mathsf{add}P_0$. This is contradiction by Lemma 2.10. Thus we have $T \in \mathsf{add}A$, which implies $\mathsf{add}T = \mathsf{add}A$. □

4. Silting objects induced by torsion pairs

In this section we construct a silting object induced by a torsion pair satisfying a good condition. It is a modification of the construction in [AH]. The main result of this section (Theorem 4.10) essentially plays an important role to prove Theorem 1.2.

Through this section, let A be a finite dimensional algebra over a field and $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\text{-}A)$.

Definition 4.1. For any full subcategory $\mathcal{E}$ of mod-A, we define a full subcategory of mod-A by

$$^{\perp}\mathcal{E} = \{M \in \mathrm{mod}\text{-}A \mid \mathrm{Hom}_A(M, E) = 0 \text{ for any } E \in \mathcal{E}\},$$

$$\mathcal{E}^{\perp} = \{M \in \mathrm{mod}\text{-}A \mid \mathrm{Hom}_A(E, M) = 0 \text{ for any } E \in \mathcal{E}\}.$$

For any $X \in \mathrm{mod}\text{-}A$, we simply write $^{\perp}\mathsf{add}X$ (respectively, $\mathsf{add}X^{\perp}$) by $^{\perp}X$ (respectively, $X^{\perp}$).

Definition 4.2. A pair $(\mathcal{C}, \mathcal{D})$ of full subcategories $\mathcal{C}, \mathcal{D}$ in mod-A is called a *torsion pair* for mod-A if $\mathcal{C} = {}^{\perp}\mathcal{D}$ and $\mathcal{D} = \mathcal{C}^{\perp}$.

Let $(\mathcal{C}, \mathcal{D})$ be a torsion pair for mod-A. Then for any $X \in \mathrm{mod}\text{-}A$, there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{C}$ and $X'' \in \mathcal{D}$. We call X' (respectively, X'') a *torsion part* (respectively, a *torsion-free part*) of X, and denote it by $\gamma(X)$.

We know a fundamental example of torsion pairs for mod-A.

Lemma 4.3. [ASS, Example VI.1.2(a)] *For any $X \in \mathrm{mod}\text{-}A$, the pair $\left({}^{\perp}X, ({}^{\perp}X)^{\perp}\right)$ is a torsion pair for* mod-A.

Definition 4.4. Let $\mathcal{E}$ be a full subcategory of mod-A closed under extensions. An A-module X is called *Ext-projective* (respectively, *Ext-injective*) in $\mathcal{E}$ if $\mathrm{Ext}^1_A(X, E) = 0$ (respectively, $\mathrm{Ext}^1_A(E, X) = 0$) for any $E \in \mathcal{E}$.

The following result plays an important role for a vanishing condition of silting objects.

Lemma 4.5. [AH, Lemma 2.6, 2.12] *Let $(\mathcal{C}, \mathcal{D})$ be a torsion pair for* mod-A *and* $X \in \mathrm{mod}\text{-}A$.

(1) *Assume $X \in \mathcal{C}$. Then the following hold:*
  - (*i*) *X is Ext-injective in $\mathcal{C}$ if and only if it is a torsion part of an injective A-module;*
  - (*ii*) *X is Ext-projective in $\mathcal{C}$ if and only if $\tau(X) \in \mathcal{D}$.*

(2) *Assume $X \in \mathcal{D}$. Then the following hold:*
  - (*i*) *X is Ext-projective in $\mathcal{D}$ if and only if it is a torsion-free part of a projective A-module;*
  - (*ii*) *X is Ext-injective in $\mathcal{D}$ if and only if $\tau^{-1}(X) \in \mathcal{C}$.*

For a full subcategory $\mathcal{E}$ of mod-A, we define the annihilator of $\mathcal{E}$ to be the ideal $\mathsf{ann}\mathcal{E}$ of A by

$$\mathsf{ann}\mathcal{E} = \{a \in A \mid Ea = 0 \text{ for any } E \in \mathcal{E}\}.$$

Let $(\mathcal{C}, \mathcal{D})$ be a torsion pair for mod-A and B the factor algebra $A/\mathsf{ann}\mathcal{C}$ of A by $\mathsf{ann}\mathcal{C}$. Then it is easy to see that any A-module in $\mathcal{C}$ is in a natural way a B-module and the torsion part of DA is isomorphic to DB as a right B-module (cf. [S]).

We recall a (classical) tilting module. For any $X \in \mathrm{mod}\text{-}A$, we denote by $\delta(X)$ the number of non-isomorphic indecomposable direct summands of X. An A-module T is called a *tilting module* if it satisfies $\mathrm{Ext}^1_A(T, T) = 0$, the projective dimension of T is at most one, and $\delta(T) = \delta(A)$.

The following result plays an important role for a generating condition of silting objects.

Lemma 4.6. [S] *Let $(\mathcal{C}, \mathcal{D})$ be a torsion pair for* mod-A. *Then the following are equivalent:*

(1) *A direct sum of non-isomorphic indecomposable Ext-projective modules in $\mathcal{C}$ is a tilting module as a right $A/\mathsf{ann}\mathcal{C}$-module;*
(2) *$\mathcal{C}$ is covariantly finite in* mod-A;

Remark 4.7. (See [AH] and [H].) For any $M \in \mathrm{mod}\text{-}A$, $\mathsf{add}M$ is covariantly finite in mod-A. Hence if A is representation-finite, then the condition (2) in Lemma 4.6 automatically holds.

Now we construct a complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\text{-}A)$ induced by a torsion pair.

Definition 4.8. Let $(\mathcal{C},\mathcal{D})$ be a torsion pair for mod-A. We define a complex $T \in \mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ as follows: We put X as a direct sum of non-isomorphic indecomposable Ext-projective modules in $\mathcal{C}$. Set U as the (-1)-shift of a projective presentation of X and V as a direct sum of non-isomorphic indecomposable injective modules in $\mathcal{D}$. Now we define $T := T_{(\mathcal{C},\mathcal{D})} := U \oplus \nu^{-1}V$ where $\nu^{-1} := \mathrm{Hom}_A(DA,-)$.

We have the following result.

Lemma 4.9. *Let $(\mathcal{C},\mathcal{D})$ be a torsion pair for* mod-A*. Then $T := T_{(\mathcal{C},\mathcal{D})}$ is a pre-silting object.*

Proof. We use the notation as in Definition 4.8. We only have to show $\mathrm{Hom}_{\mathcal{T}}(T,T[1]) = 0$. By Lemma 4.5, we have $H^0(\nu T) \simeq \tau(X) \oplus V \in \mathcal{D}$. Hence we obtain

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}}(T,T[1]) &\simeq D\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{mod-}A)}(T[1],\nu T)\\ &\simeq D\mathrm{Hom}_A(H^1(T),H^0(\nu T))\\ &\simeq D\mathrm{Hom}_A(X,\tau(X)\oplus V)\\ &= 0.\end{aligned}$$

Thus the assertion holds. □

We now state the main result of this section.

Theorem 4.10. *Let $(\mathcal{C},\mathcal{D})$ be a torsion pair for* mod-A*. If $\mathcal{C}$ is covariantly finite in* mod-A*, then $T := T_{(\mathcal{C},\mathcal{D})}$ is a silting object. In particular if $\mathcal{C} \supseteq \nu\mathcal{C}$, then T is a tilting object.*

Proof. Let X, V be as in Definition 4.8. We can check $A \in \mathcal{T}_T^{\leq 0}$ and $T \in \mathcal{T}_A^{\leq 1}$ easily. By Lemma 4.9 and Lemma 2.15, T is a partial silting object. Since $\mathcal{C}$ is covariantly finite in mod-A, X is a tilting $A/\mathsf{ann}\mathcal{C}$-module by Lemma 4.6. Hence $\delta(X)$ is equal to the number of non-isomorphic indecomposable direct summands of a torsion part of DA, and it is equal to the number of non-isomorphic injective indecomposable modules which does not belong to $\mathcal{D}$. This implies $\delta(A) = \delta(X) + \delta(V) = \delta(T)$. By [AI, Corollary 2.27], T must be a silting object.

Assume $\mathcal{C} \supseteq \nu\mathcal{C}$. Then we also have $\mathcal{D} \supseteq \nu^{-1}\mathcal{D}$. Since $\mathcal{D} \supseteq \nu^{-1}\mathcal{D}$ and $\tau(X) \in \mathcal{D}$ by Lemma 4.5, we have $\nu^{-1}V \in \mathcal{D}$ and $\Omega^2(X) \simeq \nu^{-1}\tau(X) \in \mathcal{D}$. Hence we obtain

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}}(T,T[-1]) &\simeq \mathrm{Hom}_A(H^1(T),H^0(T))\\ &\simeq \mathrm{Hom}_A(X,\Omega^2(X)\oplus\nu^{-1}V)\\ &= 0.\end{aligned}$$

Thus the second assertion holds. □

5. Silting transitivity for representation-finite symmetric algebras

We show the silting transitivity for representation-finite symmetric algebras. In this case any silting mutation is always tilting mutation, since any silting object is tilting (e.g. [AI, Example 2.8]). The assumption to be a symmetric algebra but not only a self-injective one plays a role finally.

In this section let A be a finite dimensional algebra over a field and $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\text{proj-}A)$.

For any non-zero complex $X \in \mathcal{T}$ with $X^i = 0$ unless $a \leq i \leq b$, we denote by $\mathsf{length}(X) = b - a + 1$ the *length* of X. When A is a self-injective algebra, the length of a complex is useful to understand a partial order on $\mathrm{silt}\,\mathcal{T}$ as follows.

Proposition 5.1. *Let $T \in \mathrm{silt}\,\mathcal{T}$, $P \in \mathrm{tilt}\,\mathcal{T}$ and $\ell \geq 0$. Assume that A is a self-injective algebra. Then the following are equivalent:*

(1) $P[-\ell] \geq T \geq P$;

(2) $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(P,T[i]) = 0$ *unless* $0 \leq i \leq \ell$.

In particular, the following are equivalent:

(1) $\mathsf{length}(T) \leq \ell + 1$;

(2) *There exists an integer* n *such that* $A[-\ell+n] \geq T \geq A[n]$.

Proof. The implications follow from Theorem 3.5. □

The aim of this section is to prove the following result.

Theorem 5.2. *If* A *is a representation-finite symmetric algebra, then the action of iterated irreducible silting mutation on* $\operatorname{silt}\mathcal{T}$ *is transitive.*

To prove Theorem 5.2, we consider the following conditions:
For any derived equivalent algebra Λ to A,

(A1) Any silting object T with $\Lambda[-1] \geq T \geq \Lambda$ can be obtained from $\Lambda[-1]$ and Λ by iterated irreducible silting mutation on $\mathsf{K}^{\mathrm{b}}(\text{proj-}\Lambda)$;
(A2) Let $P \in \mathsf{K}^{\mathrm{b}}(\text{proj-}\Lambda)$ be a tilting object with $\Lambda[-\ell] \geq P \geq \Lambda$ for a positive integer $\ell > 0$. Then there exists a tilting object $T \in \mathsf{K}^{\mathrm{b}}(\text{proj-}\Lambda)$ with $\Lambda[-1] \geq T \geq \Lambda$ such that $T[-\ell+1] \geq P \geq T$;
(A3) For any silting object $T \in \mathsf{K}^{\mathrm{b}}(\text{proj-}\Lambda)$, there exists a tilting object which can be obtained from T by iterated irreducible silting mutation on $\mathsf{K}^{\mathrm{b}}(\text{proj-}\Lambda)$.

We have the following result.

Lemma 5.3. *The conditions* (A1)(A2) *and* (A3) *follow a positive answer for Question 1.1.*

Proof. By (A3), we shall show that any tilting object can be obtained from A by iterated silting mutation on $\mathsf{K}^{\mathrm{b}}(\text{proj-}A)$. Let $P \in \mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ be a tilting object. Note that all shifts of A can be obtained from A by (A1). Therefore we can assume $A[-\ell] \geq P \geq A$ for some $\ell > 0$. Let $T \in \mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ be a tilting object as in (A2). Put $B = \operatorname{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T)$ and an equivalence of triangulated categories $F : \mathsf{K}^{\mathrm{b}}(\text{proj-}A) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\text{proj-}B)$ which sends T to B. By (A2), we have $B[-\ell+1] \geq F(P) \geq B$. This implies that P can be obtained from A by the induction on ℓ and (A1). □

In the rest of this section we assume that A is a self-injective algebra.
Let us start with an observation of satisfying the condition (A1).

Lemma 5.4. *The condition* (A1) *holds if* A *is representation-finite.*

Proof. Note first that every complex of length 2 in $\mathcal{T}$ is a shift of a projective presentation of an A-module up to projective direct summand. Since A is representation-finite, there exist only finitely many basic silting objects $T \in \mathcal{T}$ satisfying $A[-1] \geq T \geq A$ by Proposition 5.1. Thus the assertion follows from Theorem 2.12. □

Next we show satisfying the condition (A2).

Lemma 5.5. *The condition* (A2) *holds if* A *is representation-finite.*

To prove Lemma 5.5, we need the following important result.

Lemma 5.6. *Let* $P \in \mathcal{T}$ *be a pre-silting object with* $H^i(P) = 0$ *unless* $0 \leq i \leq \ell$ *for some* $\ell > 0$. *If* ${}^{\perp}H^0(\nu P)$ *is covariantly finite in* mod-A, *there exists* $T \in \operatorname{silt}\mathcal{T}$ *with* $A[-1] \geq T \geq A$ *satisfying* $T \in \mathcal{T}_P^{\leq 0}$ *and* $P \in \mathcal{T}_T^{\leq \ell-1}$. *In particular if* $\mathsf{add}P = \mathsf{add}\nu P$, *then* T *is a tilting object.*

Proof. Set $X := H^0(\nu P)$, $\mathcal{C} := {}^{\perp}X$ and $\mathcal{D} := \mathcal{C}^{\perp}$. By Lemma 4.3, the pair $(\mathcal{C}, \mathcal{D})$ is a torsion pair for mod-A. Since $\mathcal{C}$ is covariantly finite in mod-A, $T := T_{(\mathcal{C},\mathcal{D})}$ is a silting object by Theorem 4.10.

(i) We show $T \in \mathcal{T}_P^{\leq 0}$. It is enough to prove $\operatorname{Hom}_{\mathcal{T}}(P, T[1]) = 0$. Since we have $H^1(T) \in \mathcal{C}$, we obtain

$$\begin{aligned}\operatorname{Hom}_{\mathcal{T}}(P, T[1]) &\simeq D\operatorname{Hom}_{\mathcal{T}}(T[1], \nu P)\\ &\simeq D\operatorname{Hom}_{\mathcal{T}}(H^1(T), X)\\ &= 0.\end{aligned}$$

(ii) We show $P \in \mathcal{T}_T^{\leq \ell-1}$. We only have to prove $\operatorname{Hom}_{\mathcal{T}}(T, P[\ell]) = 0$. Since we have

$$\operatorname{Hom}_A(H^{\ell}(P), X) \simeq \operatorname{Hom}_{\mathcal{T}}(P[\ell], \nu P) \simeq D\operatorname{Hom}_{\mathcal{T}}(P, P[\ell]) = 0,$$

we have $H^\ell(P) \in \mathcal{C}$. Since $H^0(\nu T) \in \mathcal{D}$ by Lemma 4.5, we obtain

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}}(T, P[\ell]) &\simeq D\mathrm{Hom}_{\mathcal{T}}(P[\ell], \nu T) \\ &\simeq D\mathrm{Hom}_A(H^\ell(P), H^0(\nu T)) \\ &= 0.\end{aligned}$$

Thus the first assertion holds. Since A is self-injective, ν and ν^{-1} commute with H^0. This implies the second assertion by Theorem 4.10. □

Now we are ready to prove Lemma 5.5.

Let P be a tilting object with $A[-\ell] \geq P \geq A$ for some $\ell > 0$. Take a torsion pair $(\mathcal{C}, \mathcal{D})$ as in the proof of Lemma 5.6. By Theorem 3.5, one has $P \simeq \nu P$. Since $\mathcal{C}$ is covariantly finite by Remark 4.7, we can get a tilting object T with $A[-1] \geq T \geq A$ and $T[-\ell+1] \geq P \geq T$ by Lemma 5.6. □

Now we are ready to prove Theorem 5.2.

We have (A1) by Lemma 5.4, (A2) by Lemma 5.5. Since A is symmetric, any silting object is tilting by [AI, Example 2.8]. This immediately implies (A3). Thus the assertion follows from Lemma 5.3. □

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