MUTATION CLASSES OF SKEW-SYMMETRIZABLE 3×3 MATRICES

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Abstract. Mutation of skew-symmetrizable matrices is a fundamental operation that first arised in Fomin-Zelevinsky's theory of cluster algebras; it also appears naturally in many different areas of mathematics. In this paper, we study mutation classes of skew-symmetrizable 3×3 matrices and associated graphs. We determine representatives for these classes using a natural minimality condition, generalizing and strengthening results of Beineke-Brustle-Hille and Felikson-Shapiro-Tumarkin. Furthermore, we obtain a new numerical invariant for the mutation operation on skew-symmetrizable matrices of arbitrary size.

1. Introduction

Mutation of skew-symmetrizable matrices is a fundamental operation that first arised in Fomin-Zelevinsky's theory of cluster algebras; it also appears naturally in many different areas of mathematics. Mutation can also be naturally viewed as an operation on certain graphs, called diagrams. In this paper, we study mutation classes of skew-symmetrizable 3×3 matrices and their diagrams. We determine representatives for these classes using a natural minimality condition, generalizing and strengthening results of [3, 5]. Furthermore, we obtain a new numerical invariant for the mutation operation on skew-symmetrizable matrices of arbitrary size.

To state our results, we need some terminology. Let us recall that an integer matrix B is skew-symmetrizable if DB is skew-symmetric for some diagonal matrix D with positive diagonal entries. For any matrix index k, the mutation of a skew-symmetrizable matrix B at k is another skew-symmetrizable matrix $\mu_k(B) = B'$:

$$B' = \begin{cases} B'_{i,j} = -B_{i,j} & \text{if } i = k \text{ or } j = k \\ B'_{i,j} = B_{i,j} + sgn(B_{i,k})[B_{i,k}B_{k,j}]_+ & \text{else} \end{cases}$$

(where we use the notation $[x]_+ = max\{x, 0\}$ and $sgn(x) = x/|x|$ with $sgn(0) = 0$). Mutation is an involutive operation, so repeated mutations give rise to the *mutation-equivalence* relation on skew-symmetrizable matrices.

On the other hand, motivated by the Dynkin diagram construction in the thory of Kac-Moody algebras [8], for any skew-symmetrizable $n \times n$ matrix B, a directed graph $\Gamma(B)$, called diagram of B, is associated in [6] as follows: the vertices of $\Gamma(B)$ are the indices $1, 2, ..., n$ such that there is a directed edge from i to j if and only if $B_{ij} > 0$, and this edge is assigned the weight $|B_{ij}B_{ji}|$. Let us note that if B is not skew-symmetric, then the diagram $\Gamma(B)$ does not determine B as there could be several different skew-symmetrizable matrices whose diagrams are equal; however, if a skew-symmetrizing matrix D is fixed, then $\Gamma(B)$ determines B. In any case,

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we use the general term diagram to mean the diagram of a skew-symmetrizable matrix. Then the mutation μ_k can naturally be viewed as a transformation on diagrams (see Section 2 for a description). In the particular case where the vertex k is a source (resp. sink), i.e. all incident edges are oriented away (resp. towards) k, then μ_k acts by only reversing all edges incident to k; in that case we also call μ_k a reflection (as in classical Bernstein-Gelfand-Ponamarev reflection functors). Note also that if B is skew-symmetric then the diagram $\Gamma(B)$ may be viewed as a quiver and the corresponding mutation operation is also called quiver mutation . There are several categorical interpretations of the quiver mutation, we refer to [9] for a survey.

Given the appearance of the mutation operation in many different areas of mathematics, it is natural to study properties of the mutation classes of skew-symmetrizable matrices and the associated diagrams. Currently, a description of these classes are known for finite and affine types [2, 12], there is also a classification for the so called finite mutation type diagrams [5]. In this paper we consider the next basic case of size 3 skew-symmetrizable matrices, which is crucial to understand the mutation operation in general size. To be able to state our results, let us recall a little bit more terminology. By a subdiagram of Γ, we always mean a diagram obtained from Γ by taking an induced (full) directed subgraph on a subset of vertices and keeping all its edge weights the same as in Γ. By a cycle we mean a subdiagram whose vertices can be labeled by elements of $\mathbb{Z}/m\mathbb{Z}$ so that the edges betweeen them are precisely $\{i, i+1\}$ for $i \in \mathbb{Z}/m\mathbb{Z}$. We call a diagram Γ *mutation-acyclic* if it is mutation-equivalent to an acyclic diagram (i.e. a diagram which has no oriented cycles at all); otherwise we call it *mutation-cyclic*. Now we can state our first main result:

Theorem 1.1. *Suppose that M is a mutation class of diagrams with 3 vertices. For any Γ in M, let $s(\Gamma)$ denote the sum of the square roots of the weights in Γ. Then there is a diagram Γ_0 in M such that $s(\Gamma_0)$ is minimal. Furthermore, we have the following:*

(i) *If M is a mutation class of mutation-cyclic diagrams, then Γ_0 is unique upto a change of orientation which reverses all edges (and upto an enumeration of vertices).*

(ii) *If M is the mutation class of an acyclic diagram, then Γ_0 is acyclic and it is unique upto a reflection at a source or sink (and upto an enumeration of vertices).*

Note that in this theorem part (ii) generalizes and strengthens [5, Theorem 9.1] (which claims uniqueness upto an arbitrary change of orientation for quivers). In fact, part (ii) establishes a special case (rank three) of a standard conjecture of cluster algebra theory [7, Conjecture 4.14 (4)]. For quivers, the conjecture was obtained in [4, Corollary 4] using categorical methods. We use more elementary algebraic-combinatorial methods. Let us also note that a numerical criterion to check whether a given diagram is mutation-acyclic has been obtained by the author in [13]. (This criterion is recalled in Theorem 2.6).

We also characterize Γ_0 using a "local" property, generalizing [3, Lemma 2.1] and [5, Theorem 9.1(3)]):

Theorem 1.2. *Suppose that M is a mutation class of diagrams with 3 vertices. Let Γ_0 be the diagram in M such that $s(\Gamma_0)$ is minimal as in the Theorem 1.1. Then we have the following:*

(i) *Γ_0 is the unique, upto the same conditions as in Theorem 1.1, diagram in M such that, for each vertex i, we have $s(\Gamma_0) \leq s(\mu_i(\Gamma_0))$.*

(ii) *For each Γ in M, there is a (possibly empty) sequence $\{\mu_i\}$ of mutations with $\Gamma_0 = \mu_1 ... \mu_n(\Gamma)$ such that for $\Gamma_{i-1} = \mu_i ... \mu_n(\Gamma)$ we have $s(\Gamma_{i-1}) < s(\Gamma_i)$, here $i = 1, ..., n$ with $\Gamma_n = \Gamma$. Furthermore, if Γ is mutation-cyclic then the sequence $\{\mu_i\}$ is uniquely determined: specifically, the vertex i is the vertex which is not incident to the edge with maximal [1] weight in Γ_{i+1}, $i = 0, 1, ..., n-1$.*

Conversely, for any maximal *sequence $\{\mu_i : i = 1, ..., n\}$ such that $s(\Gamma_{i-1}) < s(\Gamma_i)$ with $\Gamma_{i-1} = \mu_i ... \mu_n(\Gamma)$ and $\Gamma_n = \Gamma$, we have $\Gamma_0 = \mu_1 ... \mu_n(\Gamma)$.*

We also obtain the following result which gives a new numerical invariant for the mutation of diagrams with any number of vertices.

Theorem 1.3. *Suppose that Γ is a diagram with n vertices. For any vertex i in Γ, let $\delta_i = \delta_i(\Gamma)$ be the greatest common divisor of the weights of the edges which are incident to i. Let $\delta(\Gamma) = (\delta_1, \delta_2, ..., \delta_n)$ be the ordered sequence of these greatest common divisors such that $\delta_1 \geq \delta_2 \geq ... \geq \delta_n$. Then for any Γ' which is mutation-equivalent to Γ, we have $\delta(\Gamma) = \delta(\Gamma')$.*

Note that if Γ is the diagram of a skew-symmetric matrix, then the same conlusion holds if δ_i is defined as the greatest common divisor of the radicals of the weights of the edges which are incident to the vertex i. (Equivalently, in the quiver notation that represents skew-symmetric matrices, the conlusion of the theorem holds if δ_i is defined as the greatest common divisor of the number of arrows in the edges which are incident to the vertex i.)

We prove our results in Section 3 after some preparation in Section 2.

2. Preliminaries

In this section, we will recall some more terminology and prove some statements that we will use to prove our results. First, let us recall that the diagram of a skew-symmetrizable (integer) matrix has the following property:

(2.1) the product of weights along any cycle is a perfect square, i.e. the square of an integer.

Thus we can use the term diagram to mean a directed graph, with no loops or two-cycles, such that the edges are weighted with positive integers satisfying (2.1). Let us note that if an edge in a diagram has weight equal to one, then we do not specify its weight in the picture.

For any vertex k in a diagram Γ, the associated mutation μ_k changes Γ as follows [6]:

- The orientations of all edges incident to k are reversed, their weights intact.

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[1] We will show that Γ_{i+1} has a unique edge with maximal weight (Lemma 3.3).

- For any vertices i and j which are connected in Γ via a two-edge oriented path going through k (see Figure 1), the direction of the edge $\{i,j\}$ in $\mu_k(\Gamma)$ and its weight γ' are uniquely determined by the rule

$$\pm\sqrt{\gamma}\pm\sqrt{\gamma'}=\sqrt{\alpha\beta}\,, \tag{2.2}$$

where the sign before $\sqrt{\gamma}$ (resp., before $\sqrt{\gamma'}$) is "+" if i,j,k form an oriented cycle in Γ (resp., in $\mu_k(\Gamma)$), and is "$-$" otherwise. Here either γ or γ' can be equal to 0, which means that the corresponding edge is absent.
- The rest of the edges and their weights in Γ remain unchanged.

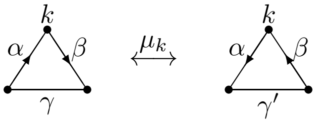

FIGURE 1. Diagram mutation

This operation is involutive, i.e. $\mu_k(\mu_k(\Gamma))=\Gamma$, so it defines an equivalence relation on the set of all diagrams. More precisely, two diagrams are called *mutation-equivalent* if they can be obtained from each other by applying a sequence of mutations. The *mutation class* of a diagram Γ is the set of all diagrams which are mutation-equivalent to Γ. If B is a skew-symmetrizable matrix, then $\Gamma(\mu_k(B))=\mu_k(\Gamma(B))$ (see Section 1 for the definition of $\mu_k(B)$). Let us note that if B is not skew-symmetric, then the diagram $\Gamma(B)$ does not determine B as there could be several different skew-symmetrizable matrices whose diagrams are equal; however, if a skew-symmetrizing matrix D is fixed, then $\Gamma(B)$ determines B, so mutation class of $\Gamma(B)$ determines that of B (the matrix $\mu_k(B)$ shares the same skew-symmetrizing matrix D with B [6]).

In this paper, we will mainly consider diagrams with exactly three vertices. Therefore it will be convenient for us to use special notation for these diagrams, generalizing the one used in [3] and [5]:

Definition 2.1. Suppose that Γ is a three-vertex diagram with weights α,β and γ. Let $a=\sqrt{\alpha}, b=\sqrt{\beta}$ and $c=\sqrt{\gamma}$. We call a,b,c the *radical weights* of Γ and use the following notation: if Γ is acyclic we write $\Gamma=(a,b,c)^-$; if Γ is cyclic we write $\Gamma=(a,b,c)$, without considering any particular ordering. We denote by $s(\Gamma)$ the sum of the radical weights of Γ. By the definition of a diagram, the product of the radical weights is an integer by (2.1).

Note that if B is a skew-symmetric matrix then the radical weights of $\Gamma(B)$ are equal to the positive entries of B. Also note that this notation does not uniquely determine Γ. Nevertheless it is convenient for us because it behaves well under the mutation operation:

Proposition 2.2. *Suppose that Γ is a diagram with three vertices. Then we have the following:*

(i) *If $\Gamma=(a,b,c)^-$ and k is a vertex which is a source or sink in Γ, then $\mu_k(\Gamma)=(a,b,c)^-$.*
(ii) *Suppose that $\Gamma=(a,b,c)^-$ and k is a vertex which is neither a source nor sink in Γ. Also assume that k is not incident to the edge with radical weight c. Then $\mu_k(\Gamma)=(a,b,c+ab)$*

(iii) *Suppose that* $\Gamma = (a, b, c)$ *and* k *is the vertex which is not incident to the edge with radical weight* c. *Then* $\mu_k(\Gamma) = (a, b, ab - c)$ *(resp.* $\mu_k(\Gamma) = (a, b, c - ab)^-$*) provided* $c < ab$ *(resp.* $c \geq ab$*).*

The proposition follows from the definition of the mutation operation and the following technical statement, providing skew-symmetrization by conjugation:

Lemma 2.3. [6, Proposition 8.1] *Let* B *be a skew-symmetrizable (integer) matrix Then there exists a diagonal matrix* H *with positive diagonal entries such that* HBH^{-1} *is skew-symmetric. Furthermore, the matrix* $S(B) = (S_{ij}) = HBH^{-1}$ *is uniquely determined by* B. *Specifically, the matrix entries of* $S(B)$ *are given by*

$$S_{ij} = \operatorname{sgn}(B_{ij})\sqrt{|B_{ij}B_{ji}|}\,. \tag{2.3}$$

Furthermore, for any matrix index k, *we have* $S(\mu_k(B)) = \mu_k(S(B))$.

The matrix H *can be taken as* $D^{1/2}$ *where* D *is a skew-symmetrizing matrix for* B.

Let us also record the following statements for convenience; they can be checked easily using the definition of the mutation operation:

Proposition 2.4. *Suppose that* Γ *is a diagram with three vertices. Then we have the following:*

(i) $s(\Gamma) = s(\mu_k(\Gamma))$ *if and only if* Γ *and* $\mu_k(\Gamma)$ *have the same weights; furthermore:*
  (a) *the diagrams* Γ *and* $\mu_k(\Gamma)$ *are both cyclic or both acyclic,*
  (b) *if* Γ *is acyclic, then the vertex* k *is a source or sink in* Γ.

(ii) $s(\Gamma) > s(\mu_k(\Gamma))$ *if and only if the edge which is not incident to the vertex* k *have smaller weight in* $\mu_k(\Gamma)$ *than in* Γ *(the weights of the remaining edges are equal).*

Proposition 2.5. *Suppose that* Γ *is a three-vertex diagram which has an edge whose weight is less than* 4. *Then* Γ *is mutation-acyclic.*

Proof. Suppose that $\Gamma = (a, b, c)$ is cyclic such that $c \leq a, b$ (so $c < 2$, thus $c = 1, \sqrt{2}$ or $\sqrt{3}$). Let us also assume, without loss of generality, that $a \leq b$. Let i (resp. j) be the vertex which is not incident to the edge with radical weight b (resp. a). If $c = 1$, then $\mu_i(\Gamma) = (a, b-a, c)^-$ is acyclic. Let us assume now that $c = \sqrt{2}$. If $ac \leq b$, then $\mu_i(\Gamma) = (a, b-ac, c)^-$ is acyclic, otherwise $\mu_j\mu_i(\Gamma) = (bc-a, ac-b, c)^-$ is acyclic (because $bc - a > 0$ for our assumption $a \leq b$ and $c = \sqrt{2} > 1$). For $c = \sqrt{3}$ we use a similar argument: if $ac \leq b$, then $\mu_i(\Gamma) = (a, ac - b, c)^-$ is acyclic, otherwise either $\mu_j\mu_i(\Gamma) = (bc - 2a, ac - b, c)^-$ is acyclic or (i.e. if $bc - 2a < 0$) $\mu_i\mu_j\mu_i(\Gamma) = (2a - bc, 2b - ac, c)^-$ is acyclic (note $2b - ac > 0$ because $b \geq a$ and $2 > c$). This completes the proof. □

Determining whether a given diagram is mutation-acyclic or not is a natural problem in the theory of cluster algebras and related topics. For diagrams with three vertices, a numerical criterion for being mutation-acyclic has been obtained by the author in [13], using the notion of a *quasi-Cartan companion*. For the convenience of the reader, we will recall this criterion. First let us recall that a quasi-Cartan companion of a skew-symmetrizable matrix is a symmetrizable matrix whose diagonal entries are equal to 2 and whose off-diagonal entries differ only by signs [2]. A quasi-Cartan companion A of skew-symetrizable matrix B is called *admissible* if it satisfies the following sign condition: for any cycle Z in $\Gamma(B)$, the product $\prod_{\{i,j\}\in Z}(-A_{i,j})$ over all edges of Z is negative if Z is oriented and positive if Z is non-oriented [12]. The main examples of admissible companions are the generalized Cartan matrices: if $\Gamma(B)$ is acyclic, i.e. has no oriented cycles at all, then the quasi-Cartan companion A with $A_{i,j} = -|B_{i,j}|$, for all $i \neq j$, is admissible. However, for an arbitrary skew-symmetrizable matrix B, an admissible quasi-Cartan companion may not exist; if exists it is unique upto simultaneous sign changes in rows and columns. For any skew-symmetrizable matrix B of size 3, an admissible quasi-Cartan companion exists and it determines whether its diagram $\Gamma(B)$ is mutation-acyclic:

Theorem 2.6. *Suppose that B is a skew-symmetrizable matrix of size 3 and let A be an admissible quasi-Cartan companion of B. Then $\Gamma(B)$ is mutation-acyclic if and only if one of the following holds:*

(i) *$det(A) > 0$ and A is positive,*
(ii) *$det(A) = 0$ and A is semipositive*[2] *of corank 1,*
(iii) *$det(A) < 0$.*

Let us note that parts (i) and (ii) occur if and only if $\Gamma(B)$ is mutation-equivalent to a Dynkin and an extended Dynkin diagram respectively [2, 12] (here a Dynkin diagram is an orientation of a Dynkin graph). Let us also mention that the main ingredient in proving the theorem is an extension of the mutation operation to quasi-Cartan companions; we refer to [12, Section 2] for details.

The previous Theorem 2.6 was obtained as a non-trivial generalization of a characterization in [3] for *skew-symmetric* matrices of size 3, using a polynomial called the Markov constant. More explicitly, for a skew-symmetric B with $\Gamma(B) = (x, y, z)$, the associated Markov constant is defined as $C(B) = C(x, y, z) = x^2 + y^2 + z^2 - xyz$. Then skew-symmetric matrices with mutation-acyclic diagrams can be characterized as follows:

Theorem 2.7. [3, Theorem 1.1] *Suppose that B is a skew-symmetric (integer) matrix such that $\Gamma(B) = (x, y, z)$ is cyclic. Then the following are equivalent:*
(1) $\Gamma(B)$ is mutation-acyclic.
(2) The Markov constant satisfies $C(x, y, z) > 4$ or $\min\{x, y, z\} < 2$.
(3) The Markov constant satisfies $C(x, y, z) > 4$ or the triple (x, y, z) is in the following list (where we assume $x \geq y \geq z$):
a) $C(x, y, z) = 0 : (x, y, z) = (0, 0, 0)$,
b) $C(x, y, z) = 1 : (x, y, z) = (1, 0, 0)$,
c) $C(x, y, z) = 2 : (x, y, z) = (1, 1, 0)$ or $(1, 1, 1)$,
d) $C(x, y, z) = 4 : (x, y, z) = (2, 0, 0)$ or $(2, 1, 1)$.

Let us note that a generalization of this theorem to skew-symmetrizable matrices is not immediate because the Markov constant is not defined for non-skew-symmetric matrices; it is also not defined for skew-symmetric matrices whose diagrams are acyclic. It was observed in [13] that, for a skew-symmetric matrix B of size 3 and an admissible quasi-Cartan companion A of B, we have $detA = 2(4-C(B))$, leading to Theorem 2.6. For skew-symmetrizable matrices of arbitrary size, it seems that

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[2] i.e. DA is positive semidefinite, where D is a symmetrizing matrix of A.

one needs to consider the admissible quasi-Cartan companion itself rather than just its determinant, see [12] for a conjecture.

3. Proofs of Main Results

First we will prove some lemmas that we use to prove our theorems. In the proofs we assume, without loss of generality, that all diagrams are connected. The following statement generalizes [3, Lemma 2.1c].

Lemma 3.1. *Suppose that Γ is a three-vertex diagram. If there are vertices $i \neq j$ such that $s(\mu_i(\Gamma)) < s(\Gamma)$ and $s(\mu_j(\Gamma)) \leq s(\Gamma)$, then Γ has an edge whose weight is less than 4 and it is mutation-acyclic.*

Proof. Let us first note that Γ is not acyclic (otherwise for any vertex k, we have $s(\mu_k(\Gamma)) \geq s(\Gamma)$), so we suppose that $\Gamma = (a, b, c)$ is cyclic. We assume that i is not incident to the edge with radical weight a and j is not incident to the edge with radical weight b. Let us first assume that $\mu_i(\Gamma)$ and $\mu_j(\Gamma)$ are both cyclic. Then $\mu_i(\Gamma) = (-a + bc, b, c)$ and $\mu_j(\Gamma) = (a, -b + ac, c)$. The conditions of the lemma imply that $bc < 2a$ and $ac \leq 2b$ (Proposition 2.4(ii)). Multiplying the first inequality by c, we have $bc^2 < 2ac$; on the other hand, by the second inequality, we have $2ac \leq 4b$, implying that $c^2 < 4$. Then by Proposition 2.5, the diagram Γ is mutation-acyclic.

Let us now assume that one of $\mu_i(\Gamma)$ or $\mu_j(\Gamma)$ is acyclic; without loss of generality, suppose that $\mu_i(\Gamma)$ is acyclic. Then $a \geq bc$. If $\mu_j(\Gamma)$ is not acyclic, then by the condition of the lemma we have $ac \leq 2b$. Then $ac^2 \leq 2bc \leq 2a$, implying that $c^2 \leq 2$. Similarly if $\mu_j(\Gamma)$ is acyclic, then $ac \leq b$, implying $a \leq ac^2 \leq bc \leq a$, so $c = 1$ (and $a = b$). In any case, by Proposition 2.5, the diagram Γ is mutation-acyclic. This completes the proof. □

In view of the previos lemma, the following is a special case of Theorem 1.1(ii):

Lemma 3.2. *Suppose Γ be a three-vertex diagram which has an edge whose weight is less than 4. If Γ is cyclic, then it has a vertex k such that $s(\Gamma) > s(\mu_k(\Gamma))$; in particular, $s(\Gamma)$ is not minimal.*

Proof. Suppose that $\Gamma = (a, b, c)$ with $a^2 < 4$ (so $a = 1, \sqrt{2}, \sqrt{3}$). If the conclusion of the lemma is not satisfied then we have $bc \geq 2a$, $ac \geq 2b$ and $ab \geq 2c$ (here $\mu_i(\Gamma)$ is cyclic for any vertex i in Γ). Note that from the last inequality we have $b \geq 2c/a$, on the other hand, from the second inequality, we have $c \geq 2b/a$, implying $b \geq 4b/a^2$; however this is not possible because $a^2 < 4$. This completes the proof. □

Let us now show the following special case of Theorem 1.2(ii):

Lemma 3.3. *Suppose that Γ is mutation-cyclic and has a vertex i such that $s(\mu_i(\Gamma)) < s(\Gamma)$ (in particular $s(\Gamma)$ is not minimal). Then i is the unique vertex with this property. Furthermore Γ has a unique edge e with maximal weight: the vertex i is the vertex which is not incident to e.*

Note that the statement may not be true if mutation acyclic (e.g. for $\Gamma = (1, 3, 5)$).

Proof. Let j, k be the remaining vertices. Since Γ is mutation-cyclic, by Lemma3.1, we have $s(\mu_j(\Gamma)), s(\mu_k(\Gamma)) > s(\Gamma)$, which proves the first part of the statement. For the second part, suppose $\Gamma = (a, b, c)$ and assume that i is the vertex which is not incident to the edge with radical weight c. Then, by the first part, $ba < 2c$ but

$bc \geq 2a$ and $ac \geq 2b$. Multiplying the second inequality by a we have $abc \geq 2a^2$; since $2c^2 > abc$, we have $2c^2 > 2a^2$, thus $c > a$. Similarly $c > b$. This completes the proof. □

3.1. **Proof of Theorems 1.1 and 1.2.** We will first prove Theorem 1.1 and Theorem 1.2(i) both at the same time. For this, let us first note that the mutation class of Γ obviusly contains a diagram Γ_0 such that $s(\Gamma_0)$ is minimal. To show its uniqueness, note that if $s(\Gamma)$ is minimal, then it satisfies:

(*) $s(\Gamma) \leq s(\mu_i(\Gamma))$ for any vertex i.

We will show that, in the mutation class of Γ, there is a unique, upto a change of orientation as described in the statement of Theorem 1.1, diagram Γ_0 that satisfies (*). (This will prove Theorem 1.1 and Theorem 1.2(i)). For this purpose, let us suppose that Γ_0' is another diagram that satisfies (*) in the mutation class of Γ; say $\Gamma_0' = \mu_n ... \mu_1(\Gamma_0)$. We may assume without loss of generality that n is minimal (in particular $i \neq i+1$, i.e. a mutation is not applied consecutively). If $n = 1$, then, the effect of μ_1 on Γ_0 is to reverse all edges (because Γ_0 and $\Gamma_0' = \mu_1(\Gamma_0)$ have the same weights as they satisfy (*)), so Γ_0 and Γ_0' are equal as claimed in Theorem 1.1. Thus for the rest of the proof, we can assume $n \geq 2$. Let us denote $\Gamma_l = \mu_l ... \mu_1(\Gamma_0)$, $l = 1, 2, ..., n-1$. Note that, since n is minimal, Γ_i $i = 1, 2, ..., n-1$, does not satisfy (*) ; also $s(\Gamma_0) < s(\Gamma_1)$ and $s(\Gamma_0') < s(\Gamma_{n-1})$ (otherwise Γ_1 or Γ_{n-1} satisfies (*) by Proposition 2.4, contradicting the minimality of n; note that $\Gamma_{n-1} = \mu_n(\Gamma_0')$). Then there exist $1 < m_1 < ... < m_k < n$, $(k \geq 1)$, such that $s(\Gamma_0) < s(\Gamma_1) \leq ... \leq s(\Gamma_{m_1}) > s(\Gamma_{m_1+1}) \geq ... \geq s(\Gamma_{m_2}) < s(\Gamma_{m_2+1}) \leq ... \leq s(\Gamma_{m_3}) > s(\Gamma_{m_3+1}) ... s(\Gamma_{m_k}) > s(\Gamma_{m_k+1}) \geq ... \geq s(\Gamma_{n-1}) > s(\Gamma_0')$ (i.e. $m_1, ..., m_k$ are the end-points of the intervals of increase and decrease for $s(\Gamma_l)$). Since $s(\Gamma_{m_1-1}) \leq s(\Gamma_{m_1}) > s(\Gamma_{m_1+1})$, by Lemma 3.1, the diagram Γ is mutation-acyclic (note that $\mu_{m_1}(\Gamma_{m_1}) = \Gamma_{m_1-1}$ and $\mu_{m_1+1}(\Gamma_{m_1}) = \Gamma_{m_1+1}$), so we are done if Γ is mutation-cyclic.

We now assume that Γ is mutation-acyclic. Let us note that for each $l = 0, 1, 2, ..., n$, there is a sequence $\Gamma_l = \mu_{j_l} ... \mu_{j_1}(\Gamma_{m_l})$ such that $s(\Gamma_l) \leq s(\mu_{j_l-1} ... \mu_{j_1})(\Gamma_{m_l}) \leq s(\mu_{j_l-2} ... \mu_{j_1})(\Gamma_{m_l}) \leq ... \leq s(\mu_{j_1})(\Gamma_{m_l}) \leq (\Gamma_{m_l})$ and $s(\Gamma_{m_l-1}) < s(\Gamma_{m_l}) \geq s(\Gamma_{m_l+1})$ or $s(\Gamma_{m_l-1}) \leq s(\Gamma_{m_l}) > s(\Gamma_{m_l+1})$ (so Γ_{m_l} is the local maximum which is closest to Γ_l). By Lemma 3.1, the diagram Γ_m has an edge whose weight is less than 4, then by Proposition 2.4 the diagram Γ_l has an edge whose weight is less than 4. Then, by Lemma 3.2, the diagrams Γ_0 and Γ_0' are acyclic. Note that Γ_i, $1 \leq i \leq n-1$, is not acyclic because any acyclic diagram satisfies (*); in particular, Γ_1 is obtained from Γ_0 by mutating at the vertex which is not a source nor sink (similarly the vertex n is neither a source nor a sink in $\Gamma_0' = \Gamma_n$).

Case 1. Γ_0 *is skew-symmetric, so has an edge of weight one in* Γ_0*.* Let us first assume that Γ_0 is a tree, say $\Gamma_0 = (a, 1, 0)^-$. Then $\Gamma_1 = \mu_1(\Gamma_0) = (a, a, 1)$ (note that, since $s(\Gamma_0) < s(\Gamma_1)$, the vertex 1 is neither source nor sink in Γ_0). Then the mutation at any vertex $k \neq 1$ gives a tree which is equal to Γ_0 upto an enumeration of vertices, so the conclusion of the theorem holds.

Suppose now that Γ_0 is not a tree, say $\Gamma_0 = (1, a, b)^-$. Note that if the vertex 1 is not incident to any edge whose weight is 1, then $\Gamma_1 = \mu_1(\Gamma_0) = (a, b, 1+ab)$ with $a, b \geq 2$; contradicting our assumption that Γ_1 has an edge whose weight is less than 4. Thus, in Γ_0, we can assume that the vertex 1 is incident to an edge of weight 1, the remaining radical weights are a, b (so Γ_0 has a source or sink whose incident edges have radical weights a, b).

To proceed, let us first consider the subcase where a or b is equal to one; without loss of generality, say $a = 1$. Then it can be checked easily that, for $i = 1, ..., n-1$, we have $\Gamma_i = (1, 1, a+1)$ or $\Gamma_i = (1, a, a+1)$ (so $\Gamma_n = \Gamma_0' = (1, 1, a)^-$) and Γ_0 is equal to Γ_0' upto an enumeration of vertices possibly after a reflection at a source or sink.

Let us now assume that a and b are greater than one. We may assume without loss of generality that (in Γ_0) the vertex 1 is not incident to the edge with radical weight a (otherwise it is incident to the edge with radical weight b, then we exchange the letters a and b). Let us also assume that $\Gamma_0 = (1, a, b)^-$ such that the edge $2 \longrightarrow 1$ has radical weight 1, the edge $1 \longrightarrow 3$ has radical weight b and the edge $2 \longrightarrow 3$ has radical weight a. Then $\Gamma_1 = \mu_1(\Gamma_0) = (1, a+b, b)$. Now, the diagram Γ_2 is obtained by mutating Γ_1 at a vertex $v \neq 1$ such that v is incident to an edge e with weight one (otherwise $\mu_v(\Gamma_1)$ does not have any edge whose weight is less than 4); so e is the edge $1 \longrightarrow 2$ in Γ_1, thus $v = 2$. Then $\Gamma_2 = \mu_2(\Gamma_1) = (1, a+b, a)$. Similarly the diagram Γ_3 is obtained by mutating Γ_2 at a vertex $w \neq 2$ such that w is incident to an edge with weight one, then $w = 1$ (because the vertex 3 is incident to the edges with radical weights $a+b, a$, which are greater than or equal to 2). Then $\Gamma_3 = \mu_1(\Gamma_1) = (1, b, a)^-$ where the edge $1 \longrightarrow 2$ has radical weight 1, the edge $3 \longrightarrow 1$ has radical weight a and the edge $3 \longrightarrow 2$ has radical weight b. Thus $\Gamma_0' = \Gamma_3 = (1, b, a)^-$. Note then that Γ_0' can be obtained from Γ_0 by first applying the reflection at the vertex 3 then exchanging (renumbering) the vertices 1 and 2. This completes the case.

Case 2. *Γ_0 is not skew-symmetric and has an edge of weight one in Γ_0.* Let us denote this edge by e. As in the previous case, if $\Gamma_0 = (a, 1, 0)^-$ is a tree, then $\Gamma_1 = \mu_1(\Gamma_0) = (a, a, 1)$ (recall that 1 is the vertex which is neither source nor sink in Γ_0). Then mutation at any vertex $k \neq 1$ gives a tree with the same weights, so the uniqueness conclusion of the theorems holds. Let us now suppose that $\Gamma_0 = (1, a, b)^-$ is not a tree. Note that since Γ_0 is not skew-symmetric, the numbers a, b are not integers (but square-roots of integers). For convenience, we consider in subcases:

Subcase 2.1. *a or b is equal to $\sqrt{2}$.* Let us suppose without loss of generality that $a = \sqrt{2}$. (Note then that $b = m\sqrt{2}$ where m is integer). Then by similar arguments as in Case 1 above it follows that Γ_i, $1 \leq i \leq n-1$, (in fact $n \leq 5$), belongs to one of the following types (in the notation of Definition 2.1): $(\sqrt{2}, b, \sqrt{2}b + 1)$; $(\sqrt{2}, b + \sqrt{2}, \sqrt{2}b + 1)$; $(\sqrt{2} + b, \sqrt{2}, 1)$; $(\sqrt{2} + b, 1, b)$ such that Γ_0' can be obtained from Γ_0 possibly after enumerating the vertices and reflecting at a source or sink.

Subcase 2.2. *a or b is equal to $\sqrt{3}$.* Let us suppose without loss of generality that $a = \sqrt{3}$: Then by similar arguments as in Case 1 above it follows that Γ_i, $1 \leq i \leq n-1$, (in fact $n \leq 6$), is of one of the following types: $(\sqrt{3}, b, 1 + \sqrt{3}b)$; $(\sqrt{3}, \sqrt{3} + 2b, 1 + \sqrt{3}b)$; $(\sqrt{3}, \sqrt{3} + 2b, 2 + \sqrt{3}b)$; $(\sqrt{3}, \sqrt{3} + b, 2 + \sqrt{3}b)$; $(\sqrt{3}, \sqrt{3} + b, 1)$; $(b, \sqrt{3} + b, 1)$ such that Γ_0' can be obtained from Γ_0 possibly after renumbering the vertices and reflecting at a source or sink.

Subcase 2.3. *$a, b \geq 2$.* Note that if the vertex 1 is not incident to e, then $\Gamma_1 = \mu_1(\Gamma_0) = (a, b, ab + 1)$, so Γ_1 does not have any edge whose weight is less than four, contradicting our assumption. Thus in this case the vertex 1 is incident to e. Then Γ_i, $1 \leq i \leq n-1$, (in fact $n = 3$), is of type $(1, a+b, b)$ or $(1, a+b, a)$ such that Γ_0'

can be obtained from Γ_0 by a reflection at a source or sink and renumbering the vertices if necessary.
Case 3. Γ_0 *is not skew-symmetric and minimal edge weight is equal to* 2. Let us write $\Gamma_0 = (\sqrt{2}, a, b)^-$, where $a, b \geq \sqrt{2}$; if Γ_0 is a tree, then we take $b = 0$.
Subcase 3.1. *a or b is equal to* $\sqrt{2}$. Let us assume, without loss of generality, that $a = \sqrt{2}$. If $\Gamma_0 = (\sqrt{2}, \sqrt{2}, 0)^-$ is a tree, then it is easily checked that $\Gamma_i = (\sqrt{2}, \sqrt{2}, 2)$, $i = 1, ...n-1$, and Γ_0' is as required in the conclusion of the uniqueness claims in the theorems. Let us now assume that Γ_0 is not a tree. (Note then that b is an integer). Then Γ_i, $1 \leq i \leq n-1$, (in fact $n = 4$), belongs to one of the following types: $(\sqrt{2}, \sqrt{2} + \sqrt{2}b, b)$; $(\sqrt{2}, \sqrt{2} + \sqrt{2}b, b+2)$; $(\sqrt{2}, \sqrt{2}, b+2)$ such that Γ_0' can be obtained from Γ_0 by a reflection at a source or sink and renumbering the vertices if necessary.
Subcase 3.2. *a or b is equal to* $\sqrt{3}$. Let us assume, without loss of generality, that $a = \sqrt{3}$. If Γ_0 is a tree, then it is easily checked that, for $1 \leq i \leq n-1$, $\Gamma_i = (\sqrt{2}, \sqrt{6}, \sqrt{3})$; $(2\sqrt{2}, \sqrt{6}, \sqrt{3})$ and the uniqueness conlusion of the theorems is satisfied. Let us now assume that Γ_0 is not a tree. (Note then that $b = \sqrt{2}\sqrt{3}m$ where m is integer). Then Γ_i, for $1 \leq i \leq n-1$, (in fact $n \leq 6$), is of one of the following types: $(\sqrt{3}, \sqrt{2} + \sqrt{3}b, b)$; $(\sqrt{3}, \sqrt{2} + \sqrt{3}b, \sqrt{6} + 2b)$; $(\sqrt{3}, 2\sqrt{2} + b\sqrt{3}, \sqrt{6} + 2b)$; $(\sqrt{3}, 2\sqrt{2} + b\sqrt{3}, \sqrt{6} + b)$; $(\sqrt{3}, \sqrt{2}, \sqrt{6} + b)$; $(\sqrt{2}, \sqrt{3} + \sqrt{2}b, b)$; $(\sqrt{2}, \sqrt{3} + \sqrt{2}b, \sqrt{6} + b)$ such that Γ_0' can be obtained from Γ_0 by a reflection at a source or sink and renumbering the vertices if necessary.
Subcase 3.3. $a, b \geq 2$. Note that if the vertex 1 is not incident to the edge with radical weight $\sqrt{2}$, because otherwise $\Gamma_1 = \mu_1(\Gamma_0) = (\sqrt{2} + ab, a, b)$ does not have any edge whose weight is less than four, contradicting our assumption. Then, by similar arguments as in Case 1, it follows that Γ_i, $1 \leq i \leq n-1$, (in fact $n = 4$), belongs to one of the following types: $(\sqrt{2}, a, b + \sqrt{2}a)$; $(\sqrt{2}, a + \sqrt{2}b, b + \sqrt{2}a)$; $(\sqrt{2}, a + \sqrt{2}b, b)$ such that Γ_0' can be obtained from Γ_0 by a reflection at a source or sink and renumbering the vertices if necessary.
Case 4. Γ_0 *is not skew-symmetric and minimal edge weight is equal to* 3. Let us write $\Gamma_0 = (\sqrt{3}, a, b)^-$, where $a, b \geq \sqrt{3}$; if Γ_0 is a tree, then we take $b = 0$. If Γ_0 is a tree, then for $1 \leq i \leq n-1$, (in fact $n = 5$), $\Gamma_i = (\sqrt{3}, a, \sqrt{3}a)$ or $\Gamma_i = (\sqrt{3}, 2a, \sqrt{3}a)$ and the uniqueness conlusion of the theorems is satisfied. We now assume that Γ_0 is a cycle (triangle). Suppose first that one of the radical weights a, b is less than two; without loss of generality, say $a = \sqrt{3}$. Then, by similar arguments as in Case 1, it follows that, for $1 \leq i \leq n-1$, (in fact $n = 6$), the diagram Γ_i belongs to one of the following types: $(\sqrt{3}, \sqrt{3} + \sqrt{3}b, b)$; $(\sqrt{3}, \sqrt{3} + \sqrt{3}b, 3 + 2b)$; $(\sqrt{3}, 2\sqrt{3} + \sqrt{3}b, 3 + 2b)$; $(\sqrt{3}, 2\sqrt{3} + \sqrt{3}b, 3 + b)$; $(\sqrt{3}, \sqrt{3}, 3 + b)$ such that Γ_0' can be obtained from Γ_0 by a reflection at a source or sink and renumbering the vertices if necessary.

Suppose now that $a, b \geq 2$. Note that the vertex 1 is not incident to the edge with radical weight $\sqrt{3}$, because otherwise $\Gamma_1 = \mu_1(\Gamma_0) = (\sqrt{3} + ab, a, b)$ does not have any edge whose weight is less than four, contradicting our assumption. Then by similar arguments as in Case 1 it follows that Γ_i, $1 \leq i \leq n-1$, (in fact $n = 6$), belongs to one of the following types: $(\sqrt{3}, a, b + \sqrt{3}a)$; $(\sqrt{3}, 2a + \sqrt{3}b, b + \sqrt{3}a)$; $(\sqrt{3}, 2a + \sqrt{3}b, 2b + \sqrt{3}a)$; $(\sqrt{3}, a + \sqrt{3}b, 2b + \sqrt{3}a)$;$(\sqrt{3}, a + \sqrt{3}b, b)$ such that Γ_0' can be obtained from Γ_0 by a reflection at a source or sink and renumbering the vertices if necessary. This completes the case.

We have completed the proof of Theorem 1.1 and Theorem 1.2(i). Then Theorem 1.2(ii) follows from these statements and Lemma 3.3. This completes the proofs of the theorems.

3.2. **Proof of Theorem 1.3.** It is enough to show the theorem for $\Gamma' = \mu_k(\Gamma)$, where k is a vertex in Γ. In the proof we will use the following notation: for any two vertices i and j, we denote by $\omega_{i,j}$ (resp. $\omega'_{i,j}$) the corresponding weight in Γ (resp. Γ') (note that $\omega_{i,j} = \omega_{j,i}$, also if the vertices i, j are not connected in Γ, then $\omega_{i,j} = 0$, similarly in Γ'). We will show that, for any vertex i, we have $\delta_i(\Gamma') = \delta_i(\Gamma)$. For this purpose, let us first note that $\delta_k(\Gamma') = \delta_k(\Gamma)$ by the definition of the mutation (because the weights of the edges which are incident to k are not affected). Similarly for any vertex i which is not adjacent to k, we have $\delta_i(\Gamma') = \delta_i(\Gamma)$. To complete the proof, let us now assume that i is a vertex which is adjacent to k. Then, for any vertex j, the weight $\omega'_{i,j}$ is equal to one of the following: $\omega_{i,j}$ or $w_{i,j} + 2\sqrt{w_{i,j}w_{j,k}w_{k,i}} + w_{j,k}w_{k,i}$ or $w_{i,j} - 2\sqrt{w_{i,j}w_{j,k}w_{k,i}} + w_{j,k}w_{k,i}$. Here note that $\sqrt{w_{i,j}w_{j,k}w_{k,i}}$ is an integer by the definition of a diagram, furthermore it is divided by δ_i (because δ_i divides both $\omega_{i,j}$ and $\omega_{i,k}$). Thus δ_i divides $\omega'_{i,j}$ for any j, so it divides $\delta'_i = \delta_i(\Gamma')$. Since μ_k is involutive, δ'_i divides $\omega_{i,j}$ for any j, so δ'_i divides δ_i as well, consequently $\delta_i = \delta'_i$. (For diagrams of skew-symmetric matrices, the same arguments work if δ_i is defined as the greatest common divisor of the radicals of the weights of the edges which are incident to the vertex i). This completes the proof.

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